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Modelling a solution of a homogeneous parabolic equation with random initial condition from $L_p(\Omega)$

Abstract. *In this paper, a model of solutions to the heat equation with initial conditions from the Orlicz space $L_p(\Omega)$ of random variables is built. The constructed model approximates the solution of a homogeneous parabolic equation with given reliability and accuracy in some Orlicz space.*

Streszczenie. *W pracy zbudowano model rozwiązań równania ciepła z warunkami początkowymi z przestrzeni Orlicza $L_p(\Omega)$ zmiennych losowych. Skonstruowany model przybliża rozwiązanie jednorodnego równania parabolicznego z zadaną niezawodnością i dokładnością w pewnej przestrzeni Orlicza.* ***(Modelowanie rozwiązania jednorodnego równania parabolicznego z losowym warunkiem początkowym z $L_p(\Omega)$)***

Keywords: simulation, Orlicz space, parabolic equation, stochastic processes
Słowa kluczowe: symulacja, przestrzeń Orlicza, równanie paraboliczne

Introduction

When solving second-order partial differential equations describing physical processes, the properties and convergence criteria of random series are often investigated. The question arises under which conditions the Fourier method can be used to sloved some type equation of mathematical physics with randm factors. Based on the method developed by Kozachenko Yu.V. and Buldigin V.V., the solution of such equation can be approximated using partial sums of convergent series over different function spaces. We also used this method during the research.

The subject of this paper is the construction of a model that approximates a solution to the heat equation with random Orlicz initial conditions. In [11] we found the conditions under which the solution of the boundary value problem for the homogeneous parabolic equation exists with probability one and can be represented in the form of a certain continuously differentiable series. Some methods of simulation of random processes and random fields can be found the [1], [13], [6], [5], [8], [4] by Yu. V. Kozachenko and his students. In these works the authors not only construct the models of stochastic processes and fields, but also investigate the accuracy and reliability of these models. The main result of this study is the construction of a model of solution to parabolic equation with given reliability and accuracy. In addition, we consider the problem in some special cases. In particular, we bulid the model for the problem

$$\frac{\partial^2 V(t,x)}{\partial x^2} - \frac{\partial V(t,x)}{\partial t} = 0$$

$$V(t,0) = 0 \quad V(t,\pi) = 0$$

$$V(0,x) = \xi(x);$$

where $\xi(x) \in L_p(\Omega)$.

Random processes from the Orlicz space

Defintion 1 *[3] A continuous even convex function $U(x)$ such that $U(0) = 0$ and $U(x) \neq 0$ for $x \neq 0$ is called C-function.*

Let $\{\Omega, \Im, P\}$ be the probability space.

Defintion 2 *[10] The set of random variables $\xi(\omega) = \xi$, $\omega \in \Omega$ is called the Orlicz space $L_U(\Omega)$ of random variables generated by the C-function $U(x)$ if for any $\xi \in L_U(\Omega)$ there exists a constant r_ξ such that $EU(\frac{\xi}{r_\xi}) < \infty$ (where $E(\cdot)$ – expected value).*

The Orlicz space $L_U(\Omega)$ is a Banach space with respect to the norm

(1) $$\| \xi \|_{L_U} = inf\{r > 0 : EU(\frac{\xi}{r}) \leq 1\}.$$

Defintion 3 *[3] We say that the g-condition holds for a C-function $U(x)$ if there are some constants $z_0 \geq 0,\ K > 0,$ and $A > 0$ such that*

(2) $$U(x)U(y) \leq AU(Kxy).$$

for all $x \geq z_0$ and $y > z_0$.

Defintion 4 *[3] We say that a C-function $U(x)$ is subordinate to a C-function $V(x)$ and denote $U \prec V$ if there exist two numbers $x_0 \geq 0$ and $m > 0$ such that $U(x) \leq V(mx)$ for all x such that $|x| > x_0$.*

Defintion 5 *[2] Let $U(x)$ be a C-function such that $V(x) = x^2$ is subordinate to the function $U(x)$. A family Δ of centered random variables ($E\xi = 0$, $\xi \in \Delta$) from the Orlicz space $L_U(\Omega)$ is called a strictly Orlicz family if there exists a constant C_Δ, such that*

(3) $$\| \sum_{i \in I} \lambda_i \xi_i \|_{L_U} \leq C_\Delta (E(\sum_{i \in I} \lambda_i \xi_i)^2)^{1/2}$$

for all finite collections of random variables $\xi_i \in \Delta$, $i \in I$, and for all $\lambda_i \in R^1$, $i \in I$.

Remark 6 *For Orlicz space $L_p(\Omega)$, $p \geq 2$, $(u(x) = |x|^p)$ the constant C_Δ can be represented in the form*

$$C_\Delta = 2\sqrt{2}\Big(\Gamma\Big(\frac{p+1}{2\sqrt{2}}\Big)\Big)^{\frac{1}{p}}$$

where $\Gamma(\cdot)$ – is a gamma function.

Defintion 7 *A stochastic process $X(t) = \{X(t), t \in T\}$ belongs to the Orlicz space $L_U(\Omega)$ if the random variable $X(t)$ belongs to $L_U(\Omega)$ for all $t \in T$.*

The main result

Consider a boundary value problem for a parabolic equation with two independent variables

(4) $$\frac{\partial}{\partial x}\Big(p(x)\frac{\partial V(t,x)}{\partial x}\Big) - q(x)V(t,x) - \rho(x)\frac{\partial V(t,x)}{\partial x} = 0$$

(5) $$V(t,0) = 0, V(t,\pi) = 0$$

(6) $$V(0,x) = \xi(x),$$

where $0 \leq x \leq \pi$ and $t \geq 0$, $\xi(x)$ is a continuous stochastic process with probability one belonging to the Orlicz space

$L_U(\Omega)$, such that $E\xi(x) = 0$. The functions $p = (p(x), x \in [0, \pi]), q = (q(x), x \in [0, \pi])$, $\rho = (\rho(x), x \in [0, \pi])$ in equation (4) satisfy the conditions

1. $p(x) > 0, q(x) \geq 0, \rho(x) > 0, x \in [0, \pi]$
2. $\rho(x)$, $p(x)$ are twice continuously differentiable for all $x \in [0, \pi]$
3. $q(x)$ is continuously differentiable for all $x \in [0, \pi]$.

Let

(7) $$V(t, x) = \sum_{k=1}^{\infty} \xi_k e^{-\lambda_k t} X_k(x),$$

where $X_k(x)$ are eigenfunctions and λ_k are eigenvalues of the Sturm–Liouville problem

(8)
$$L(X) = \frac{d}{dx}\left(p(x)\frac{dX(x)}{dx}\right) - q(x)V(x) + \lambda\rho(x)X(x) = 0$$

(9) $$X(0) = 0, \qquad X(\pi) = 0$$

and

(10) $$\xi_k = \int_0^{\pi} \rho(x) X_k(x) \xi(x)\, dx.$$

The next Theorem follows from the Theorem 3.2 ([7]).

Theorem 8 ([7]) *Let the initial condition $\xi = \{\xi(x), x \in [0, \pi]\}$ on the right hand side of (6) be a strictly Orlicz stochastic process belonging to the Orlicz space $L_U(\Omega)$ of random variables, where $U(x)$ is a C-function such that the function $V(x) = x^2$ is subordinate to $U(x)$ and condition g holds for $U(x)$. Assume that the stochastic process ξ is separable and mean square continuous, $E\xi(x) = 0$. Let there exist a continuous increasing function $\varphi = \{\varphi(x), x > 0\}$ such that $\varphi(x) > 0$, for $x > 0$, and $\Psi(x) = \frac{x}{\varphi(x)}$, $x > 0$ increases for $x \geq v_0$, where v_0 is a certain constant. We also assume that*

(11) $$\sup_{|x-y|\leq h} (E(\xi(x) - \xi(y))^2)^{\frac{1}{2}} \leq C\left(\varphi\left(\frac{1}{h} + v_0\right)\right)^{-1}.$$

Moreover, let

(12) $$\int_0^{\varepsilon} U^{(-1)}\left(\left(\frac{\pi}{2}\left(\varphi^{(-1)}\left(\frac{C}{v}\right) - v_0\right) + 1\right)^2\right) dv < \infty$$

for all $\varepsilon > 0$ and $C > 0$. If the series

(13) $$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} |E\xi_k\xi_l| \varphi(\lambda_k + v_0)\varphi(\lambda_l + v_0) < \infty$$

converges, for all $\varepsilon > 0$,

(14) $$P\{\sup_{x,t\in A_T} |\sum_{k=N}^{\infty} \xi_k e^{-\lambda_k t} X_k(x)| > \varepsilon\} \leq \left(U\Big(\frac{\varepsilon}{\hat{B}_N(\theta)}\Big)\right)^{-1},$$

where $A_T = \{0 \leq x \leq \pi, \sigma \leq t \leq T\}, T > 0; \sigma > 0$

(15)
$$\hat{B}_N(\theta) = \frac{1}{\theta(1-\theta)} \int_0^{\omega_{0N}\cdot\theta} \chi_U\left(\left(\Big(\frac{\pi}{2}\varphi^{(-1)}\big(\frac{R_N}{v}\big) - v_0\Big) + 1\right) \cdot \left(\Big(\frac{T}{2}\varphi^{(-1)}\big(\frac{R_N}{v}\big) - v_0\Big) + 1\right)\right) dv,$$

(16) $$\chi_U(n) = \begin{cases} n, & n < U(z_0) \\ C_U U^{(-1)}(n), & n \geq U(z_0), \end{cases}$$

$C_U = K(1 + U(z_0)) \max(1, A)$, $z_0, K, A,$ *- constants from Definition 3, and* $0 < \theta < 1$,
$R_N = 2C_\Delta\sqrt{W_N} \cdot \max(2C_x, L)$,
$W_N = \sum_{k=N}^{\infty}\sum_{l=N}^{\infty} |E\xi_k\xi_l|\varphi(\lambda_k + v_0)\varphi(\lambda_l + v_0) < \infty$, $C_\Delta-$ *is a constant from Definition 5, C_x is a constant such that $|X_k(x)| \leq C_x$, $\omega_{0N} = \frac{R_N}{\varphi(\frac{1}{max(T,\pi)+v_0})}$, for L the following conditions hold $|X_k(x) - X_k(y)| \leq L|x - y|\lambda_k$.*

Theorem 9 *Let the initial condition $\xi = \{\xi(x), x \in [0, \pi]\}$ on the right hand side of (6) be a strictly Orlicz stochastic process belonging to the Orlicz space $L_p(\Omega)$ $(p \geq 2)$ of random variables. Assume that the stochastic process ξ is separable and mean square continuous, $E\xi(x) = 0$. Let*

(17) $$\sup_{|x-y|\leq h} (E(\xi(x) - \xi(y))^2)^{\frac{1}{2}} \leq C|h|^{\beta}$$

if $\beta > \frac{2}{p}$ and the series

(18) $$\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} \Big|E\xi_k\xi_l\Big| \lambda_k^{\beta}\lambda_l^{\beta} < \infty$$

converge, then

(19)
$$P\{\sup_{x,t\in A_T} |\sum_{k=N}^{\infty} \xi_k e^{-\lambda_k t} X_k(x)| > \varepsilon\} \leq C_{Tp\beta}\frac{(W_N)^{\frac{p}{2}}}{\varepsilon^p},$$

where

(20)
$$C_{Tp\beta} = \frac{(p\beta+2)^{\frac{2+p\beta}{\beta}}}{(p\beta-2)^p} T^{p\beta} C_\Delta^p \cdot 2^{p-\frac{2}{\beta}}(\max(2C_x, L))^p,$$

C_Δ from Defintion 5 , T is a certain constant: $T > \pi$.
and $W_N = \sum_{l=N}^{\infty}\sum_{k=N}^{\infty} |E\xi_k\xi_l|\lambda_k^{\beta}\lambda_l^{\beta} < \infty$.

Proof As we know, the Orlicz space $L_p(\Omega)$ is generated by the function $U(x) = |x|^p$. In our case $\varphi(\lambda) = |\lambda|^{\beta}$, $\beta > \frac{2}{p}$, and the statement of Theorem 8 holds true for $\varphi(\cdot)$. Check each condition of the Theorem 8 . Let consider the condition (12), if $v_0 = 0$

$$\int_0^{\varepsilon} \left(\frac{\pi}{2}\Big(\frac{C}{v}\Big)^{\frac{1}{\beta}} + 1\right)^{\frac{2}{p}} dv$$

$$\le \int_0^{\varepsilon} \left(\frac{\pi}{2}\Big(\frac{C}{v}\Big)^{\frac{1}{\beta}}\right)^{\frac{2}{p}} dv + \int_0^{\varepsilon} 1^{\frac{2}{p}} dv$$
$$= \left(\frac{\pi}{2}\right)^{\frac{2}{p}} C^{\frac{2}{p\beta}} \frac{\beta p}{\beta p - 2} \varepsilon^{1-\frac{2}{\beta p}} + \varepsilon \cdot 1 < \infty.$$

for $\beta > \frac{2}{p}$. Condition (13) implies the convergence of (18).

And finally in (14): if $\chi_n(u) = n^{\frac{1}{p}}$, then function $B_N(\theta)$ can be estimated as:

$$\hat{B}_N(\theta) = \frac{1}{\theta(1-\theta)} \cdot$$
$$\cdot \int_0^{\omega_{0N}\cdot\theta} \left(\left(\frac{\pi}{2}\Big(\frac{R_N}{v}\Big)^{\frac{1}{\beta}} + 1\right)\left(\frac{T}{2}\Big(\frac{R_N}{v}\Big)^{\frac{1}{\beta}} + 1\right)\right)^{\frac{1}{p}} dv \le$$
$$\frac{1}{\theta(1-\theta)} \int_0^{\omega_{0N}\cdot\theta} \left(\left(\frac{T}{2}\Big(\frac{R_N}{v}\Big)^{\frac{1}{\beta}} + 1\right)\right)^{\frac{2}{p}} dv$$

for $T > \pi$; we can derive that

$$\omega_{0N} = R_N T^{\beta} < \frac{R_N T^{\beta}}{2^{\beta}},$$

then

$$\hat{B}_N(\theta) \le \frac{1}{\theta(1-\theta)} \int_0^{\omega_{0N}\cdot\theta} \left(T \frac{R_N^{\frac{1}{\beta}}}{v^{\frac{1}{\beta}}}\right)^{\frac{2}{p}} dv =$$
$$\frac{1}{\theta(1-\theta)} T^{\frac{2}{p}} R_N^{\frac{2}{p\beta}} \int_0^{\omega_{0N}\cdot\theta} \frac{1}{v^{\frac{2}{p\beta}}} dv =$$
$$\frac{1}{\theta^{\frac{2}{p\beta}}(1-\theta)} T^{\beta} R_N \frac{1}{1-\frac{2}{p\beta}},$$

consequently

$$\min \hat{B}_N(\theta) = \frac{(p\beta+2)^{\frac{2+p\beta}{p\beta}}}{2^{\frac{2}{p\beta}}(p\beta-2)} T^{\beta} R_N.$$

If we have (18) then (14) holds true.

□

Let η_k be independent random centred variables $E\eta_k = 0$, $E\eta_k^2 < \infty$ and $\xi = \{\xi(t), t \in T\}$ $(E(\xi(t))^2 < \infty)$

$$\xi(s) = \sum_{k=1}^{\infty} \eta_k f_k(s),$$

then the series in the expansion converges in the mean square.

As a model for the process $\xi(x)$ we consider

$$\hat{\xi}(s) = \sum_{k=1}^{M-1} \eta_k f_k(s), \tag{21}$$

Consider a Banach space $A(T)$ with norm $\|\cdot\|_A$, let $\xi(t) \in A(T)$ and $\hat{\xi}(t) \in A(T)$; $N = 1, 2, \ldots$.

Defintion 10 *[13] A stochastic process $\hat{\xi}(t)$ approximates the process $\xi(t)$ with reliability $1-\alpha$, $0<\alpha<1$ and accuracy ε if*

$$P\{\|\xi(t) - \xi_N(t)\| > \varepsilon\} \le \alpha$$

for $\xi(t) \in A(T)$, $\hat{\xi}(t) \in A(T)$, $N = 1, 2, \ldots$.

Let $\xi(x)$ be a stochastic process from (6) and his model be $\hat{\xi}(x)$, then as a model for the process ξ_k we consider

$$\hat{\xi}_k = \int_0^{\pi} \rho(x) X_k(x) \hat{\xi}(x)\, dx.$$

Let us denote:

$$V_N(t,x) \doteq \sum_{k=1}^{N-1} \xi_k e^{-\lambda_k t} X_k(x);$$

and

$$\hat{V}_N(t,x) \doteq \sum_{k=1}^{N-1} \hat{\xi}_k e^{-\lambda_k t} X_k(x). \tag{22}$$

$$A_T \doteq \{0 \le x \le \pi, \delta \le t \le T\},$$

where $\delta > 0$ is a constant.

Theorem 11 *Let the initial condition $\xi = \{\xi(x), x \in [0,\pi]\}$ on the right hand side of (6) be a strictly Orlicz stochastic process belonging to the Orlicz space $L_p(\Omega)$ $(p \ge 2)$ of random variables. Assume that the stochastic process ξ is separable and mean square continuous, $E\xi(x) = 0$. If the conclusion of Theorem 9 holds for $\xi(x)$ and we have*

$$\int_0^{\pi} |\big(E|\xi(x) - \hat{\xi}(x)|^p\big)^{\frac{1}{p}} dx < \Lambda. \tag{23}$$

for the $\hat{\xi}(x)$, where $\hat{\xi}(x)$ is the model for a process $\xi(x)$, then $\hat{V}_N(t,x)$ is the model for the process $V(t,x)$ with reliability $1-\alpha$ and accuracy ε in uniform metric of A_T, if for the N the following conditions hold :

$$\sqrt{W_N} < \Big(\frac{\alpha}{2C_{Tp\beta}}\Big)^{1/p} \varepsilon \cdot l, \tag{24}$$

$$K_{N-1} < \frac{\alpha^{\frac{1}{p}}(1-l)\varepsilon}{2^{\frac{1}{p}}\rho(x) C_x^2 \Lambda} \tag{25}$$

where W_N and $C_{Tp\beta}$ are defined in Theorem 9, $0 < l < 1$, $\rho = \max \rho(x)$, $|X_k(x)| < C_x$, $K_{N-1} = \sup_{\delta \le t \le T} |\sum_{k=1}^{N-1} e^{-\lambda_k t}|$, $\delta > 0$ *some constant.*

Proof We need to estimate the expression:

$$P\{\sup_{t\in A_T} |V(t,x) - \widehat{V}_N(t,x)| > \varepsilon\}$$

(26)
$$P\{\sup_{t\in A_T} |V(t,x) - \hat{V}_N(t,x)| > \varepsilon\} =$$
$$P\{\|V(t,x) - \hat{V}_N(t,x)\|_C > \varepsilon\} =$$
$$P\{\|V(t,x) - V_N(t,x) + V_N(t,x) - \hat{V}_N(t,x)\| > \varepsilon\} \le$$
$$P\{\|V(t,x) - V_N(t,x)\| + \|V_N(t,x) - \hat{V}_N(t,x)\| > \varepsilon\} \le$$
$$P\{\|V(t,x) - V_N(t,x)\| > \varepsilon \cdot l\} +$$
$$P\{\|V_N(t,x) - \hat{V}_N(t,x)\| > (1-l)\varepsilon\} = A_1 + A_2$$

where $0 < l < 1$. Let us denote:

(27) $$A_2 \doteq P\{\|V_N(t,x) - \widehat{V}_N(t,x)\| > (1-l)\varepsilon\}$$

and consider

(28) $$\begin{aligned} &\left(E|\xi_k - \hat{\xi_k}|^p\right)^{\frac{1}{p}} = \\ &\Big(E|\int_0^\pi \rho(x)X_k(x)\xi(x)dx - \\ &\int_0^\pi \rho(x)X_k(x)\hat{\xi}(x)dx|^p\Big)^{\frac{1}{p}} = \\ &\left(E|\int_0^\pi \rho(x)X_k(x)|\xi(x)-\hat{\xi}(x)|dx|^p\right)^{\frac{1}{p}} \le \\ &\int_0^\pi |\rho(x)|\cdot|X_k(x)|\big(E|\xi(x)-\hat{\xi}(x)|^p\big)^{\frac{1}{p}}dx \le \\ &\rho\cdot C_X\int_0^\pi |\big(E|\xi(x)-\hat{\xi}(x)|^p\big)^{\frac{1}{p}}dx. \end{aligned}$$

where $\rho = \max\rho(x)$, $|X_k(x)| < C_X$. Therefore

(29) $$\begin{aligned} &E\Big|\sum_{k=1}^{N-1}|\xi_k - \hat{\xi}_k|e^{-\lambda_k t}\Big|^p \le \\ &\left(\sum_{k=1}^{N-1}\left(E|\xi_k-\hat{\xi}_k|^p\right)^{\frac{1}{p}}e^{-\lambda_k t}\right)^p \le \\ &\left(\rho\cdot C_X\cdot\Lambda\sum_{k=1}^{N-1}e^{-\lambda_k t}\right)^p = \\ &(\rho C_X)^p\cdot\Lambda^p\big(K_{N-1}\big)^p, \end{aligned}$$

where $\int_0^\pi |\big(E|\xi(x)-\hat{\xi}(x)|^p\big)^{\frac{1}{p}}dx < \Lambda$

Taking into account that

(30) $$\begin{aligned} &\|V_N(t,x) - \widehat{V}_N(t,x)\| = \\ &\|\sum_{k=1}^{N-1}\xi_k e^{-\lambda_k t}X_k(x) - \sum_{k=1}^{N-1}\hat{\xi}_k e^{-\lambda_k t}X_k(x)\| = \\ &\|\sum_{k=1}^{N-1}\big(\xi_k - \hat{\xi}_k\big)e^{-\lambda_k t}X_k(x)\| \le \\ &\sum_{k=1}^{N-1}|\xi_k - \hat{\xi}_k|\cdot e^{-\lambda_k t}\cdot C_X, \end{aligned}$$

where $|X_k(x)| \le C_X$,
we have for A_2 :

(31) $$\begin{aligned} A_2 =& P\{\|V_N(t,x) - \widehat{V}_N(t,x)\| > (1-l)\varepsilon\} \le \\ &P\{C_X\sum_{k=1}^{N-1}|\xi_k-\hat{\xi}_k|\cdot e^{-\lambda_k t} > (1-l)\varepsilon\} = \\ &P\left\{\sum_{k=1}^{N-1}|\xi_k-\hat{\xi}_k|\cdot e^{-\lambda_k t} > \frac{(1-l)\varepsilon}{C_X}\right\} \le \\ &\frac{E\Big|\sum\limits_{k=1}^{N-1}|\xi_k-\hat{\xi}_k|\cdot e^{-\lambda_k t}\Big|^p}{\left(\frac{(1-l)\varepsilon}{C_X}\right)^p} \le \\ &\left(\frac{\rho C_X^2\Lambda K_{N-1}}{(1-l)\varepsilon}\right)^p < \frac{\alpha}{2} \end{aligned}$$

From Theorem 9 we have the approximations of A_1

(32) $$\begin{aligned} A_1 \doteq& P\{\|V(t,x) - V_N(t,x)\| > \varepsilon\cdot l\} \le \\ &C_{Tp\beta}\frac{(\sqrt{W_N})^p}{(l\varepsilon)^p} < \frac{\alpha}{2} \end{aligned}$$

Using the approximations (31), (32), we get the expression

(33) $$\begin{aligned} &P\{\sup_{t\in A_T}|V(t,x) - \hat{V}_N(t,x)| > \varepsilon\} \le \\ &C_{Tp\beta}\frac{\sqrt{W_N}^p}{(l\varepsilon)^p} + \left(\frac{\rho C_X^2\Lambda\Phi_{N-1}(t)}{(1-l)\varepsilon}\right)^p < \alpha. \end{aligned}$$

We can make the following conclusion: the stochastic process $\hat{V}_N(t,x)$ is the model for a process $V(t,x)$ with reliability $1-\alpha$ and accuracy ε in uniform metric of A_T.

□

Remark 12 *[9] The following equality holds true*

(34) $$W_N \le \pi^2 C_x^2 \sup_{\substack{0\le x\le\pi\\0\le y\le\pi}}(W(x,y))\sum_{k=N}^{\infty}\sum_{l=N}^{\infty}\frac{1}{\lambda_k\lambda_l},$$

where λ_k *are eigenvalues of Sturm Liouville problems.*

Example

Consider a boundary value problem for a parabolic equation with two independent variables in domain
$A_T \doteq \{0 \le x \le \pi, \delta \le t \le T\}$, ($\delta > 0, T > 0$):

(35) $$\frac{\partial^2 V(t,x)}{\partial x^2} - \frac{\partial V(t,x)}{\partial t} = 0$$

(36) $$V(t,0) = 0 \quad V(t,\pi) = 0;$$

(37) $$V(0,x) = \xi(x);$$

where $\xi(x)$ is a continuous with probability one stochastic process belonging to the Orlicz space $L_U(\Omega)$, such that $E\xi(x) = 0$. In this case we have:

$$L(x) = X''(x) + \lambda X(x) = 0,$$

$$X(0) = 0 \quad X(\pi) = 0.$$

From [14] it is well known that

(38) $$X_k(x) = \sin kx, \quad \lambda_k = k^2,$$

and

(39) $$V(t,x)=\sum_{k=1}^{\infty}\xi_k e^{-k^2t}\sin kx;$$

$$\xi_k=\int_0^{\pi}\xi(x)\sin kx\,dx,$$

where $\xi(x)\in L_p(\Omega)$.
Let's bulid a model of stochastic process (39). From [12] (85.p.) we know if stochastic process $\xi(x)$ admits the expansion

(40) $$\xi(s)=\sum_{l=1}^{\infty}\eta_l\frac{\sin ls}{l^{\kappa}}$$

then $\xi(x)$ where $\kappa>\frac{1}{2}$, η_k - are independent, have the same distribution $E\eta_k=0$, $E\eta_k^2=\sigma^2$, $\eta_k\in L_p(\Omega)$, where $p\geq 2$

Remark 13 *If the stochastic process admits the expansion (40), then $\xi(x)$ is a strictly Orlicz stochastic process belonging to the Orlicz space $L_p(\Omega)$ of random variables.*

Proof In [2], it is proved, that stochastic process (40) is a strictly Orlicz stochastic process, because the series in the expansions of the process converge in the mean square.

□

From [12] we have, if stochastic process $\xi(x)$ admits the expansion (40) then model of (40) can be represented by the following scheme:

(41) $$\hat{\xi}(s)=\sum_{l=1}^{M-1}\eta_l\frac{\sin ls}{l^{\kappa}},$$

where $s\in[0;\pi]$.

From (22), (21) and (40) we have

(42)
$$\hat{V}_N(t,x)=\sum_{k=1}^{N-1}e^{-k^2t}\sin kx\sum_{l=1}^{M-1}\eta_l\int_0^{\pi}\frac{\sin ls}{l^{\kappa}}\sin(ks)\,ds,$$

where $\hat{V}_N(t,x)$ is a model for a process $V(t,x)$ in uniform metric A_T.

Corollary 14 *If for M we have*

(43) $$M>\left(\frac{\tilde{C}\sigma\pi}{\Lambda\sqrt{2\kappa-1}}\right)^{\frac{2}{2\kappa-1}}+1,$$

where $\tilde{C}=\frac{\sqrt[p]{E|\eta_k|^p}}{\sqrt{E|\eta_k|^2}}$, $p\geq 2$, $\sigma^2=E(\eta_k)^2$, then for the model $\hat{\xi}(s)$ from (41) of the process $\xi(s)$ from (40) hold true (23).

Proof Let us consider

$$\left(E|\xi(s)-\hat{\xi}(s)|^p\right)^{\frac{1}{p}}=\left(E\Big|\sum_{k=M}^{\infty}\eta_k\frac{\sin ks}{k^{\kappa}}\Big|^p\right)^{\frac{1}{p}}\leq$$

$$\tilde{C}\left(E\Big|\sum_{k=M}^{\infty}\eta_k\frac{\sin ks}{k^{\kappa}}\Big|^2\right)^{\frac{1}{2}}\leq\tilde{C}\Big(\sum_{k=M}^{\infty}E\eta_k^2\frac{\sin^2 ks}{k^{2\kappa}}\Big)^{\frac{1}{2}}=$$

$$\tilde{C}\sigma\Big(\sum_{k=M}^{\infty}\frac{\sin^2 ks}{k^{2\kappa}}\Big)^{\frac{1}{2}}\leq\tilde{C}\sigma\Big(\sum_{k=M}^{\infty}\frac{1}{k^{2\kappa}}\Big)^{\frac{1}{2}}$$

$$\leq\tilde{C}\sigma\Big(\sum_{k=M}^{\infty}\int_{k-1}^{k}\frac{1}{x^{2\kappa}}dx\Big)^{\frac{1}{2}}=\frac{\tilde{C}\sigma}{\sqrt{2\kappa-1}(M-1)^{\kappa-0,5}}$$

$$\int_0^{\pi}\left(E|\xi(s)-\hat{\xi}(s)|^p\right)^{\frac{1}{p}}ds\leq\frac{\tilde{C}\sigma\pi}{\sqrt{2\kappa-1}(M-1)^{\kappa-0,5}},$$

the enquality (23) holds if:

$$\frac{\tilde{C}\sigma\pi}{\sqrt{2\kappa-1}(M-1)^{\kappa-0,5}}<\Lambda$$

i.e.

$$M>\left(\frac{\tilde{C}\sigma\pi}{\Lambda\sqrt{2\kappa-1}}\right)^{\frac{2}{2\kappa-1}}+1.$$

□

Corollary 15 *Let the stochastic process $\xi(x)\in L_p(\Omega)$ from (37) admits the expansion (40) and model of $\xi(x)$ admits the expansion (41). If the following inequalities hold true:*

(44) $$\Lambda<\left(\frac{\alpha}{2}\right)^{\frac{1}{p}}\cdot\frac{(1-l)\cdot\varepsilon\cdot\delta}{2}$$

$$N\geq\left(\frac{2^{\frac{2}{p}}\pi^2C_{tp\beta}^{\frac{2}{p}}}{(2\kappa-5)\alpha^{\frac{2}{p}}\varepsilon^2l^2}\right)^{\frac{1}{2\kappa-3}}+1$$

where $\kappa>\frac{1}{2}$, $0<l<1$, $C_{tp\beta}^{\frac{2}{p}}$ is constant from (20) then $\hat{V}_N(t,x)$ is the model for a process $V(t,x)$ with reliability $1-\alpha$ and accuracy ε in uniform metric of $A_T\doteq\{0\leq x\leq\pi,\delta\leq t\leq T\}$, (where $\delta>0$, $T>0$ const).

Proof The statement of corollary follows from the Theorem 11. First we check the (23).

From(12) we have

$$W_N\leq\pi^2\sup_{\substack{0\leq x\leq\pi\\0\leq y\leq\pi}}(W(x,y))\sum_{k=N}^{\infty}\sum_{l=N}^{\infty}\frac{1}{k^2l^2}.$$

From the proof of theorem 3.3 [11] it is easy to see that $W(x,y)$ admits the expansion (in the case of (35)-(37))

$$W(x,y)=\frac{\partial^4B(x,y)}{\partial x^2\partial y^2}-\frac{\partial^2B(x,y)}{\partial x\partial y}$$

from (40), we have

$$\sup_{\substack{0\leq x\leq\pi\\0\leq y\leq\pi}}W(x,y)\leq$$

$$\Big|\sum_{k=1}^{\infty}\Big(\frac{k^4\sin kx\sin ky-k^2\cos kx\cos ky}{k^{2\kappa}}\Big)\Big|\leq$$

$$\sum_{k=N}^{\infty}\frac{k^4}{k^{2\kappa}}\leq\int_{N-1}^{\infty}\frac{1}{x^{2\kappa-4}}dx=$$

$$=\frac{1}{(2\kappa-5)(N-1)^{2\kappa-5}}.$$

And, consequently from (34) we have got:

$$W_n\leq\frac{\pi^2}{(2\kappa-5)(N-1)^{2\kappa-3}}$$

Taking the (38) into account we can claim that (24) holds true if

$$N \geq \left(\frac{2^{\frac{2}{p}} \pi^2 C_{tp\beta}^{\frac{2}{p}}}{(2\kappa - 5)\alpha^{\frac{2}{p}} \varepsilon^2 l^2} \right)^{\frac{1}{2\kappa-3}} + 1$$

Finally we want to prove, in our case, that the (25) holds true. Considering the (38) we can estimate the series:

$$|\sum_{k=1}^{N-1} e^{-\lambda_k t}| = |\sum_{k=1}^{N-1} e^{-k^2 t}| \leq |\sum_{k=1}^{N-1} \frac{1}{k^2 t}| \leq \frac{1}{t}(2 - \frac{1}{N-1}),$$

for K_{N-1} from Theorem 11 we have:

$$K_{N-1} = \sup_{\delta \leq t \leq T} |\sum_{k=1}^{N-1} e^{-\lambda_k t}| \leq$$

(45) $$\sup_{\delta \leq t \leq T} \frac{1}{t}\left(2 - \frac{1}{N-1}\right) = \frac{1}{\delta}\left(2 - \frac{1}{N-1}\right) \leq \frac{1}{\delta} \cdot 2.$$

Using (45) and (44) we have that (25) holds true if $C_X = 1$, $\rho(x) = 1$. Thus, from the Theorem 11 we get the assertion of this corollary.

□

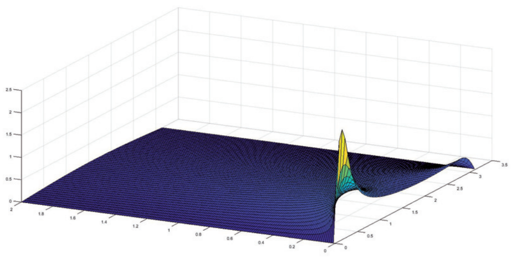

Fig. 1. The model that approximates a solution to the heat equation with Orlicz initial conditions

Let $p = 6$, $\alpha = 0,01$ $\varepsilon = 0,01$, $l = 0,5$.
Then: $\Lambda = 0,00001$, if η_k is uniformly distributed on the interval $[-1;1]$ and $\kappa = 4$ then M=27. In our case we have:
(46)

$$\hat{V}_N(t,x) = \sum_{k=1}^{N-1} e^{-k^2 t} \sin kx \sum_{l=1}^{M-1} \eta_l \int_0^{\pi} \frac{\sin ls}{l^\kappa} \sin(ks)\, ds,$$

then $\hat{V}_N(t,x)$ is the model for the process (39) with reliability $1-\alpha$ and accuracy ε in uniform metric of
$A_T = \{0 \leq x \leq \pi, 0,01 \leq t \leq 2\}$ where $\beta = 0,5$, $N = 114$.

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