Andrey Grishkevich

WSB University in Chorzów
ORCID. 0000-0002-2008-0672

doi:10.15199/48.2021.12.20

Search for the optimal band of a given width in a simply connected region

***Abstract**. An original method for finding the optimal band of a given width based on the algorithms for the maximum flow / minimum cut construction is proposed. An example of the presented algorithm work is given*

***Streszczenie.** Zaproponowano oryginalną metodę znajdowania optymalnego pasa o danej szerokości w oparciu o algorytmy konstruowania maksymalnego przepływu / minimalnego przekroju. Podano przykład działania prezentowanego algorytmu. (**Znalezienie optymalnego pasa o danej szerokości w obszarze jednospójnym**).*

Keywords: graph, network flow algorithms, maximum flow, minimum cut, cut vertices.
Słowa kluczowe: graf, algorytmy przepływu w sieci, maksymalny przepływ, minimalny przekrój, przekrój wierzchołkowy w grafie.

Introduction

Algorithms for finding the minimum cuts (maximum flows), being one of the classical problems of combinatorial analysis, are well studied theoretically [1]. The development of computer programs that are highly efficient on high-dimensional graphs [2–4] has stimulated application of these algorithms in vision and imaging technology such as image and video segmentation, co-segmentation, stereo vision, and multi-view reconstruction [5,6]. Parallel versions of similar algorithms [7] are being investigated with the aim of increasing their speed. The increase of effectively solved problems has resulted in the expansion of the practical application of such algorithms. This paper proposes an original method for finding the optimal band (band of variation) based on the algorithms for the maximum flow / minimum cut construction. It is the requirement to take the width into account (to search for a band, not a curve) in the model that dictates the need to find an alternative to the shortest path search algorithm (Dijkstra's algorithm [8]).

In the process of designing extended structures (power lines, roads and railways, canals, etc.), the problem of choosing the optimal terrain (corridor) band is solved. The structure, together with the exclusion band, fits into the selected corridor and is characterised by the usage of certain types of circular transition curves, and also satisfies the specified requirements for slopes, minimum curvature radii, etc. By the optimal terrain band for such a structure, we mean the terrain band of a certain fixed width on which the global (absolute) minimum of the used quality criterion is achieved. Similar problems arise in imaging. A certain linear structure (power line, road, skin [9], glade, channel) needs to be identified in an image, or the image needs to be segmented into parts, assuming that the border between the segments is blurred, i.e. characterised by some width.

The problem of the optimal band choice

The problem of choosing the optimal band can be formulated as a variation problem

$$P(A) = \iint\limits_{A(s,t,r)\subset\Omega} \omega(x,y)\,ds \tag{1}$$

where Ω is the area under consideration (part of the terrain / image in which the linear structure is being designed / searched); s, t are the start and end points of the strip (of the designed structure); $\omega(x, y)$ is a certain function defined on Ω which characterizes the efficiency (quality criterion of a design solution); A is a certain region including the points s, t ($s, t \in A$) which is characterized by the width r. An illustration of the problem (1) is shown in Fig. 1.

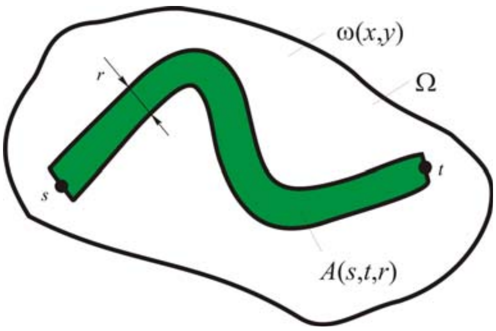

Fig.1. The variational problem of choosing the optimal band

Problem (1) relates to multidimensional problems of variation calculus. The direct analytical solution of problem (1) which determines the optimal corridor is difficult to find, since there are no general methods for its solution. The reduction of this problem to a problem of discrete optimisation can be done due to the fact that the discrete model most adequately reflects the knowledge of the domain Ω. Images, such as satellite imagery, form an array of pixels. The result of topographic, geological, hydrological and other conditions surveyed in a field environment is the information about specific points (linked to geographic coordinates) of the area, which is then summarised in the form of digital maps, diagrams, etc.

Reduction of the optimal band determination problem to the minimum cut in the network problem

Suppose that the simply connected domain Ω, in which the optimal corridor is to be found, chosen as follows (Fig. 2). The points s, t lie on the border of a rectangular area (this almost does not reduce the scope of the developed method, since there are special techniques that allow this restriction to be removed).

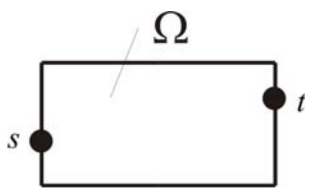

Fig.2. Selection of a rectangular area Ω

An undirected graph $G=(P,U)$ is considered as a model of the domain Ω, where P is the set of vertices of the graph, and U is the set of arcs of the graph.

The set of vertices P of the graph G forms the centres of the elementary squares into which the domain Ω is divided by the generators of some regular grid superimposed on Ω (Fig. 3). The step of this grid is considered equal to h. Without loss of generality, it can be assumed that the set of vertices is

$$P=\{p_{ij}=(x_i,y_j): i=0,1,2,...,M-1,$$
$$j=0,1,2,...,N-1\},$$
$$s=(x_{i_s},y_0)\in P,\ t=(x_{i_t},y_{N-1})\in P.$$

In the set of vertices P of the graph G, we select two subsets S and T of the vertices (Fig. 3)

$$S=S_1\cup S_2\cup S_3,$$
$$S_1=\{(x_0,y_j): j=1,2,...,N-2;\},$$
$$S_2=\{(x_i,y_0): i=0,2,...,k, \rho(x_i,s)=|x_i-x_s|>r/2\},$$
$$S_3=\{(x_i,y_{N-1}): i=0,2,...k, \rho(x_i,t)=|x_i-x_t|>r/2\},$$
$$T=T_1\cup T_2\cup T_3,$$
$$T_1=\{(x_{M-1},y_j): j=1,2,...,N-2;\},$$
$$T_2=\{(x_i,y_0): i=k,k+1,...,M-1,$$
$$\rho(x_i,s)=|x_i-x_s|>r/2\},$$
$$T_3=\{(x_i,y_{N-1}): i=k,k+1,...,M-1,$$
$$\rho(x_i,t)=|x_i-x_t|>r/2\}.$$

These two disjoint sets are formed by removing vertices $s=(x_{i_s},y_0)$ and $t=(x_{i_t},y_{N-1})$, together with adjacent the vertices (lying in a neighbourhood of $r/2$), from the boundary vertices of Ω.

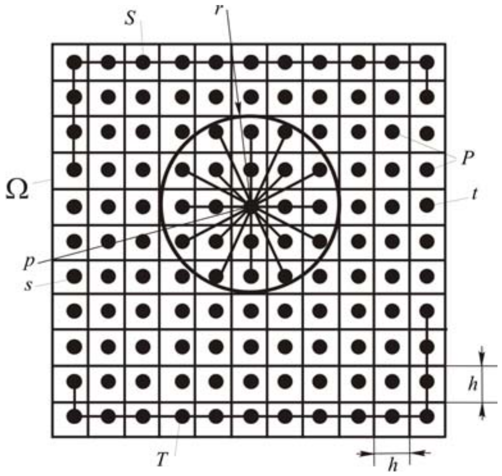

Fig.3. An Model of a variational problem in the form of a graph

The set of arcs U of the graph G is defined by the following expression

$$U=\{(p_1,p_2)=(p_{i_1,j_1},p_{i_2,j_2}):$$
$$p_1,p_2\in P,$$
$$0<\rho^2(p_1,p_2)=(x_{i_1}-x_{i_2})^2+$$
$$+(y_{i_1}-y_{i_2})^2<r^2\}.$$

This means that for any vertices p_1,p_2 of the graph G, the arc (p_1,p_2) connecting these vertices belongs to U if and only if the distance between the corresponding nodes $\rho(p_1,p_2)$ in the domain Ω does not exceed r. In Fig. 3. for the sake of clarity, the set of arcs is given only for one vertex of the graph G (it is assumed that $r>h$).

The vertex cut γ separating the sets I and S in the graph G represents, the set of graph nodes in the strip of width r in the domain Ω.

In the domain Ω, we select the subdomain R (Fig. 4), which forms a union of the elementary squares corresponding to the nodes of the cut γ. Obviously, the smaller the h/r ratio, the more the region R looks like a strip. In the limit, under the condition

$$h\to 0, \frac{h}{r}\to 0$$

the area R turns into a strip of width r.

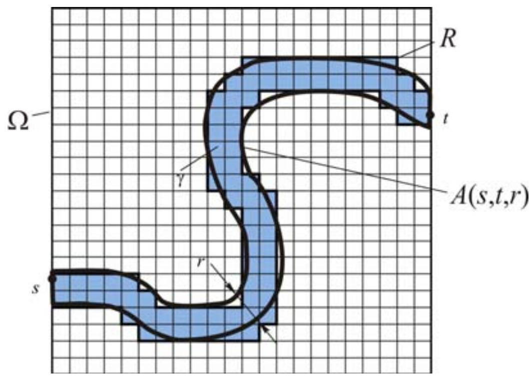

Fig.4. Approximation of a band by a vertex cut of a graph

Thus, in the selected discrete model (graph G), the concept of a vertex cut is the analogue of the corridor concept, and the task of constructing an optimal corridor is reduced to the problem of choosing a certain “optimal cut”.

The most natural quality criterion (when designing extended structures) is weight (cost).

Each node p of the graph G is assigned a number $c(p)$, which is equal to the specific reduced costs within the elementary square corresponding to the node p. This number is called the weight of the node p.

Then $c(\gamma)$, obtained by the formula

$$c(\gamma)=\sum_{p\in\gamma} c(p) \tag{2}$$

serves as a numerical expression of the quality criterion on some section γ. In other words, $c(\gamma)$ will express the cost of transforming an area R into the designed structure.

Under certain conditions (in particular, $h\to 0$), the weight of the region R tends to the weight of a band of width r.

As a result of the introduced criterion (2), the “optimal cut” is a cut with minimum weight, and the task of constructing an optimal corridor can be formulated as the problem of choosing a vertex cut γ^* with a minimum weight in the graph G (minimum cut)

$$\gamma^*=\arg\min_{\gamma} c(\gamma) \tag{3}$$

If we interpret the weight $c(p)$ of each node p of the graph G as the bandwidth of the node, the capacities of the arcs are considered unbounded, the nodes of the set S are considered sources, and the nodes of the set T are sinks, then for the constructed graph G with the given capacities of the nodes, the minimum cut search is based on the maximum flow algorithms.

Construction of the calculation graph

Problem (3) is one of the varieties of the known maximum flow [5] and can be effectively solved with the help of a computer. To solve this problem, you can use the program for searching the maximum flow (minimum cut) in the transformed oriented graph $Q=(V,E)$, since the standard programs for constructing the maximum flow are designed for graphs with known throughput capacities of arcs.

Each node $p \in P$ of the graph $G=(P,U)$ should be divided into two parts, the “left” p and “right” p, and an oriented arc leading from the “left” part p to the “right” part p should be added.

Assume $V=\{v_l, v_{l+1} : l=0,2,4,...,2MN-2\}$ ($V=\{v_{2k}, v_{2k+1} : k=0,1,2,...,MN-1\}$).

The correspondence between vertices $p_{ij} \in P$ and $v_l \in V$ ($v_{2k} \in V$) can be described using relations $l=2(iN+j)$ ($k=iN+j$). Accordingly, the inverse transform is $i=l/2/N$, $j=l/2\%N$ ($i=k/N$, $j=k\%N$).

The throughput for vertices $p_{ij} \in V \setminus (S \cup T)$ is set equal to the throughput of the corresponding node $c(p_{ij})=c(v_l,v_{l+1})=\omega(i,j)$, and the capacity of the edges for vertices $p_{ij} \in S \cup T$ is $c(p_{ij})=c(v_l,v_{l+1})=MAX$, where MAX is some large number (numerical analogue to ∞), which excludes the inclusion of the corresponding element in the minimum cut. The last condition ensures that the minimum cut will contain vertices $v_s=v_{2i_sN} \in V$, $v_t=v_{2(i_tN+N-1)+1} \in V$, corresponding to the vertices $s=(x_{i_s}, y_0) \in P$, $t=(x_{i_t}, y_{N-1}) \in P$ of the graph G.

Each undirected arc $u=(p_1,p_2)=(p_{i_1,j_1}, p_{i_2,j_2}) \in U$, $p_1, p_2 \in P$ should be replaced by two oriented ones, one of which leads from the “right” part $p_2=p_{i_2,j_2}$ to the “left” part $p_1=p_{i_1,j_1}$, and the other – from the “right” part $p_1=p_{i_1,j_1}$ to the “left” part $p_2=p_{i_2,j_2}$. The capacity of these arcs is assumed to be equal to infinity $c(e)=c(v_{l_1},v_{l_2})=MAX$, $0<\rho(p_1,p_2)<r$. This is effectively implemented by the following code

```
for (l1 = 1; l1 < 2*M*N; l1+=2)
{
   i1 = l1 / 2 / N;
   j1 = l1 / 2 % N;
   for (l2 = 0; l2 < 2*M*N; l2+=2)
   {
       i2 = l2 / 2 / N;
       j2 = l2 / 2 % N;
if (l1!=l2 &&
l1 != l2 + 1 &&
// ρ²(p1,p2) < r² &&
(i1 - i2)^2 + (j1 - j2)^2 < r^2)
       {
add edge ("v"l1, "v"l2) in graph Q;
set c("v"l1, "v"l2) = MAX;
       } } }
```

Basically, an adjacency list is constructed for a graph describing a strip by means of minimal cuts. For a regular grid, it is taken that $x_i=i$, $y_j=j$.

It is sufficient to choose an arbitrary node from the sets S (T) as a source (sink) in the network Q, for example v_0 (v_{2MN-1}).

Computer calculation

Example 1. The initial data for choosing the optimal band is shown in Fig. 5. Numbers represent function values $\omega(i,j)$ $i=0,1,2,...,M-1$, $j=0,1,2,..., N-1$, $M=N=22$.

Fig.5. The function ω(x,y) (quality criterion) in the domain Ω

With a computer band search $r^2=7$, MAX = 9 is taken. The calculation results are shown in Fig. 6.

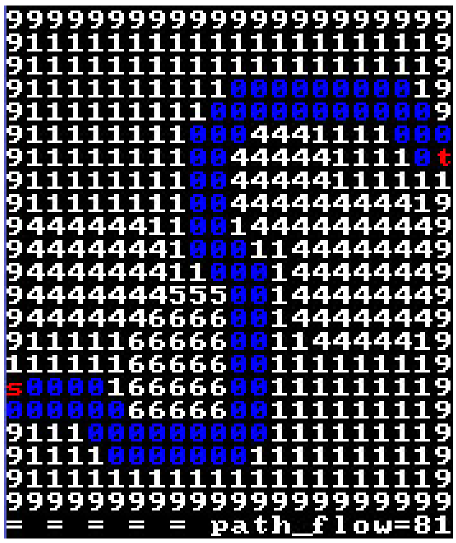

Fig.6. Computer calculation of the optimal band

Example 2. Changing the width of the strip leads to a change in its position (Fig. 7, $\omega_1 < \omega_2 < \omega_3 < \omega_4$, $r_1 < r_2$).

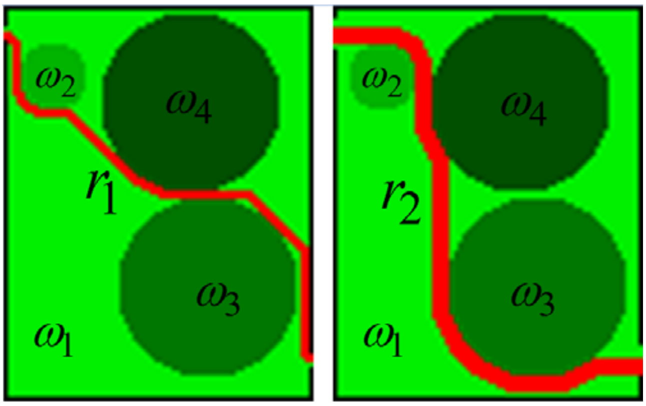

Fig.7. Changing the width of the strip

Conclusions

A mathematical model for finding the optimal band of a given width between arbitrary points of a simply connected domain is presented in the form of a combinatorial optimisation problem that allows for an efficient algorithm to obtain an optimal solution.

The task of building an optimal corridor of a given width is reduced to the problem of finding the minimum cut in a graph. An algorithm for the formation of the calculation graph is presented, which allows the width of the optimal band (corridor) to be set and standard programs to be used for the maximum flow / minimum cut search to find it.

Author: *prof. dr hab. inż. Andrey Grishkevich, Wyższa Szkoła Bankowa w Poznaniu, Wydział Zamiejscowy w Chorzowie, ul. Sportowa 29, 41-506 Chorzów, E-mail: andrey.grishkevich@chorzow.wsb.pl, grishkev.gmai@gmail.com.*

REFERENCES

[1] Galil Z., On the theoretical efficiency of various network flow algorithms", *Theoretical Computer Science,* 14 (1981), 103-111

[2] Boykov Y., Kolmogorov V., An experimental comparison of min-cut/max-flow algorithms for energy minimization in vision, *IEEE Trans Pattern Anal Mach Intell*, 26(9) (2004), 1124-1137

[3] Fishbain B., Hochbaum D.S., Mueller S., A competitive study of the pseudoflow algorithm for the minimum s–t cut problem in vision applications, *J Real-Time Image Proc*, 1 (2016)

[4] Otsukia K., Kobayashib Y., Murota K., Improved max-flow min-cut algorithms in a Circular Disk Failure Model with application to a road network, *European Journal of Operational Research*, 248 (2016), 396–403

[5] Sun W., Dong E., Kullback-Leibler distance and graph cuts based active contour model for local segmentation, *Biomedical Signal Processing and Control*, 52 (2019), 120–127

[6] Liu Z., Song Y.-Q., Sheng V.S. et al., Liver CT sequence segmentation based with improved U-Net and graph cut, *Expert Systems With Applications*, 126 (2019), 54–63

[7] Gianinazzi L., Kalvoda P., Palma A., Besta M., Hoefler T., Communication-Avoiding Parallel Minimum Cuts and Connected Components, *ACM Conference Principles and Practice of Parallel Programming* 2018 (PPoPP'18) (http://www.unixer.de/~htor/publications), (2018)

[8] Hu T.C., Integer programming and network flows, Addison-Wesley, 1969

[9] Radlak K., Frackiewicz M., Palus H., Smolka B., Finger joint synovitis detection in ultrasound images, *Bulletin of the Polish Academy of Sciences: Technical Sciences*, 66(2) (2018), 235-245