Tadeusz KACZOREK
Bialystok University of Technology

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Generalized Frobenius matrices and angles between them in analysis of linear electrical circuits

***Abstract**. Generalized Frobenius matrices and their inverses are applied in analysis of the linear electrical circuits. The basic properties of generalized Frobenius matrices are analyzed. It is shown that if the state matrix of electrical circuit has generalized Frobenius form then its inverse system matrix has also generalized Frobenius form. The notion of an angle between state matrices of linear electrical circuits is proposed and its basic properties are investigated.*

***Streszczenie.** Zaproponowane w tej pracy uogólnione macierze Frobeniusa oraz ich odwrotności zostały zastosowane w analizie liniowych obwodów elektrycznych. Zostały zbadane podstawowe własności tych macierzy. Wykazano między innymi, że macierze odwrotne uogólnionych macierzy Frobeniusa mają również postać uogólnionych macierzy Frobeniusa .Wprowadzono pojęcie kąta między macierzami stanu liniowych obwodów elektrycznych oraz zbadano ich podstawowe własności. (**Uogólnione macierze Frobeniusa i kąty między nimi w analizie liniowych obwodów elektrycznych**).*

Keywords: angle between state matrices, generalized Frobenius matrix, linear, electrical circuit.
Słowa kluczowe: kąt między macierzami, uogólniona postać Frobeniusa macierzy, macierz stanu, liniowy, obwód elektryczny.

Introduction

The linear electrical circuits have been analyzed in many papers and books [4-6, 8-14]. The constructability and observability of standard and positive electrical circuits have been addressed in [5], the decoupling zeros in [6] and minimal-phase positive electrical circuits in [8]. A new class of normal positive linear electrical circuits has been introduced in [9]. Positive fractional linear electrical circuits have been investigated in [12] and positive unstable electrical circuits in [13]. Infinite eigenvalue assignment by output-feedback for singular systems has been analyzed in [7]. Zeroing of state variables in descriptor electrical circuits has been addressed in [15]. Controller synthesis for positive linear systems with bounded controls has been investigated in [1]. Stability of continuous-time and discrete-time linear systems with inverse state matrices has been analyzed in [16].

In this paper the generalized Frobenius matrices and the angles between state matrices of linear electrical circuits will be investigated.

The paper is organized as follows. In section 2 the basic properties of generalized Frobenius matrices are analyzed. The linear electrical circuits with state matrices in general Frobenius forms are investigated in section 3 and the inverse matrices of electrical circuits with generalized Frobenius forms in section 4. The angles between state matrices of linear electrical circuits are analyzed in section 5. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n=\Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

Generalized Frobenius matrices

Definition 1. [14] The following matrices

(1)
$$A_1=\begin{bmatrix}0 & b_1 & 0 & \cdots & 0\\ 0 & 0 & b_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & b_{n-1}\\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}\end{bmatrix},$$

$$A_2=A_1^T=\begin{bmatrix}0 & 0 & \cdots & 0 & -a_0\\ b_1 & 0 & \cdots & 0 & -a_1\\ 0 & b_2 & \cdots & 0 & -a_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & b_{n-1} & -a_{n-1}\end{bmatrix},$$

$$A_3=\begin{bmatrix}-a_{n-1} & -a_{n-2} & \cdots & -a_1 & -a_0\\ b_1 & 0 & \cdots & 0 & 0\\ 0 & b_2 & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & b_{n-1} & 0\end{bmatrix},$$

$$A_4=A_3^T=\begin{bmatrix}-a_{n-1} & b_1 & 0 & \cdots & 0\\ -a_{n-2} & 0 & b_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -a_1 & 0 & 0 & \cdots & b_{n-1}\\ -a_0 & 0 & 0 & \cdots & 0\end{bmatrix},$$

$$b_k>0,\ k=1,\dots,n-1$$

are called the matrices in generalized Frobenius form.

It is easy to verify that

(2)
$$\det[I_ns-A_j]=s^n+a_{n-1}s^{n-1}+\dots+a_1b_2\dots b_{n-1}s+a_0b_1\dots b_{n-1}\quad \text{for } j=1,\dots,4$$

and the coefficients of the polynomial are positive if and only if $a_k>0$ and $b_k>0$ for $k=1,\dots,n-1$.

Theorem 1. The inverse matrix of the generalized Frobenius matrix is also the generalized Frobenius matrix.

Proof. The proof will be given only for the matrix A_1. The proof for the remaining matrices (1) is similar.

It is easy to verify that

(3)
$$A_1^{-1}=\begin{bmatrix}0 & b_1 & 0 & \cdots & 0\\ 0 & 0 & b_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & b_{n-1}\\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1}\end{bmatrix}^{-1}=\begin{bmatrix}-a_0^{-1}b_1^{-1}a_1 & -a_0^{-1}b_2^{-1}a_2 & \cdots & -a_0^{-1}b_{n-1}^{-1}a_{n-1} & -a_0^{-1}\\ b_1^{-1} & 0 & \cdots & 0 & 0\\ 0 & b_2^{-1} & \cdots & 0 & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & b_{n-1}^{-1} & 0\end{bmatrix}.$$

Theorem 2.

1) If s_{ji}, $j=1,...,4$, $i=1,...,n$ are the nonzero eigenvalues of the generalized Frobenius matrix A_j then s_{ji}^{-1} are the eigenvalues of the inverse matrix A_j^{-1}, $j=1,...,4$.
2) The inverse matrix of the generalized Frobenius matrix is asymptotically stable if and only if the generalized Frobenius matrix is also asymptotically stable.

Proof. Let s_{ji}, $j=1,...,4$, $i=1,...,n$ be the zeros of the characteristic equation of the matrix A_j

(4) $\det[I_n s_j - A_j]=0$ for $j=1,...,4$.

Then multiplying (4) by $\det[-s_j^{-1}A_j^{-1}]$ we obtain

$$\det[I_n s_j - A_j]\det[-s_j^{-1}A_j^{-1}]$$

(5) $$=\det[(I_n s_j - A_j)(-s_j^{-1}A_j^{-1})].$$

$$=\det[I_n s_j^{-1} - A_j^{-1}]=0$$

Therefore, if s_{ji}, $j=1,...,4$, $i=1,...,n$ are the eigenvalues of the matrix A_j then s_{ji}^{-1} are the eigenvalues of the matrix A_j^{-1}, $j=1,...,4$. The proof of 2) follows from the fact that $\mathrm{Re}\, s = \mathrm{Re}(\alpha_k + j\beta_k) = \alpha_k < 0$, $k=1,...,n$ if and only if

$$\mathrm{Re}\, s_k^{-1} = \mathrm{Re}\frac{1}{\alpha_k + j\beta_k} = \mathrm{Re}\left(\frac{\alpha_k}{\alpha_k^2+\beta_k^2} - j\frac{\beta_k}{\alpha_k^2+\beta_k^2}\right)$$

$$=\frac{\alpha_k}{\alpha_k^2+\beta_k^2}<0,\quad k=1,...,n.$$

Theorem 3. The characteristic polynomial of the inverse matrices in the generalized Frobenius forms (1) is given by

(6) $$\det[I_n s - A_j^{-1}] = s^n + a_0^{-1}b_1^{-1}a_1 s^{n-1} + ... + a_0^{-1}b_1^{-1}...b_{n-1}^{-1}a_{n-1}s + a_0^{-1}b_1^{-1}...b_{n-1}^{-1}$$

for $j=1,...,4$ where I_n is the $n\times n$ identity matrix.

Proof. Using (3) and developing the determinant with respect to the first row we obtain

$$\det[I_n s - A_1^{-1}]$$

(7) $$=\det\begin{bmatrix} s+a_0^{-1}b_1^{-1}a_1 & a_0^{-1}b_2^{-1}a_2 & \cdots & a_0^{-1}b_{n-1}^{-1}a_{n-1} & a_0^{-1} \\ -b_1^{-1} & s & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & s & 0 \\ 0 & 0 & \cdots & -b_{n-1}^{-1} & s \end{bmatrix}$$

$$=s^n + a_0^{-1}b_1^{-1}a_1 s^{n-1} + ... + a_0^{-1}b_1^{-1}..b_{n-1}^{-1}a_{n-1}s + a_0^{-1}b_1^{-1}..b_{n-1}^{-1}.$$

Similar results we obtain for $j=2,3,4$. □

Example 1. The characteristic polynomial of the generalized Frobenius matrix

(8) $$A=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ -2 & -4 & -5 \end{bmatrix}$$

has the form

(9) $$\det[I_3 s - A] = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -2 \\ 2 & 4 & s+5 \end{vmatrix} = s^3+5s^2+8s+4$$

and its zeros are $s_1=-1$, $s_2=s_3=-2$.

The inverse matrix of (8) has the form

(10) $$A^{-1}=\begin{bmatrix} -2 & -1.25 & -0.5 \\ 1 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}$$

and its characteristic polynomial

(11) $$\det[I_3 s - A^{-1}] = \begin{vmatrix} s+2 & 1.25 & 0.5 \\ -1 & s & 0 \\ 0 & -0.5 & s \end{vmatrix}$$

$$=s^3+2s^2+1.25s+0.25$$

with zeros $s_1^{-1}=-1$, $s_2^{-1}=s_3^{-1}=-0.5$.

Electrical circuits with state matrices in general Frobenius form

Consider the electrical circuit shown in Fig. 1 with known resistance R, inductance L, capacitance C and source voltage e.

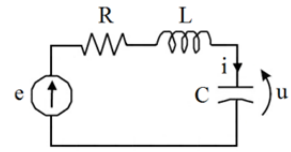

Fig. 1. Electrical circuit.

As the state variable we choose the voltage u on the capacitor with given capacitance C and the current i in the coil with given inductance L. Using the Kirchhoff's laws we obtain the equations

(12a) $$e = Ri + L\frac{di}{dt} + u,$$

(12b) $$i = C\frac{du}{dt},$$

which can be written in the form

(13a) $$\frac{d}{dt}\begin{bmatrix} u \\ i \end{bmatrix} = A_{11}\begin{bmatrix} u \\ i \end{bmatrix} + B_{11}e,$$

where

(13b) $$A_{11}=\begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix},\quad B_{11}=\begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$$

or

(13c) $$\frac{d}{dt}\begin{bmatrix} i \\ u \end{bmatrix} = A_{12}\begin{bmatrix} i \\ u \end{bmatrix} + B_{12}e,$$

where

(13d) $$A_{12}=\begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix},\quad B_{12}=\begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}.$$

Note that the matrices A_{11} and A_{12} have the generalized Frobenius form and

(14) $$\det[I_2 s - A_{11}] = \det[I_2 s - A_{12}] = s^2 + \frac{R}{L}s + \frac{1}{LC}$$

Therefore, the electrical circuit is asymptotically stable for all values $R>0$, $L>0$ and $C>0$.

Now let us consider the electrical circuits shown in Figure 2 with known resistance R, inductance L, capacitance C and source voltage e.

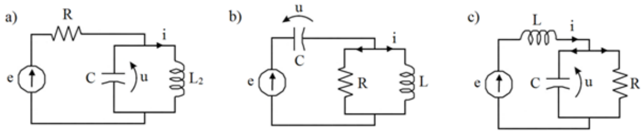

Fig 2. Electrical circuits

Fig. 2a we obtain the equations

(15a) $e = R\left(i + C\frac{du}{dt}\right) + u$,

(15b) $u = L\frac{di}{dt}$,

Using the Kirchhoff's laws for the electrical circuit shown in which can be written in the form

(16a) $\frac{d}{dt}\begin{bmatrix} u \\ i \end{bmatrix} = A_{21}\begin{bmatrix} u \\ i \end{bmatrix} + B_{21}e$,

where

(16b) $A_{21} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}$, $B_{21} = \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix}$

or

(16c) $\frac{d}{dt}\begin{bmatrix} i \\ u \end{bmatrix} = A_{22}\begin{bmatrix} i \\ u \end{bmatrix} + B_{22}e$,

where

(16d) $A_{22} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\frac{1}{RC} \end{bmatrix}$, $B_{22} = \begin{bmatrix} 0 \\ \frac{1}{RC} \end{bmatrix}$.

Note that the matrices A_{21} and A_{22} have the generalized Frobenius form and

(17) $\det[I_2 s - A_{21}] = \det[I_2 s - A_{22}] = s^2 + \frac{1}{RC}s + \frac{1}{LC}$

Therefore, the electrical circuit shown in Fig. 2a is asymptotically stable for all values $R > 0$, $L > 0$ and $C > 0$. In a similar way we may shown that the state equation of the electrical circuit shown in Fig. 2b has the form

(18a) $\frac{d}{dt}\begin{bmatrix} u \\ i \end{bmatrix} = A_{31}\begin{bmatrix} u \\ i \end{bmatrix} + B_{31}e$,

where

(18b) $A_{31} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}$, $B_{31} = \begin{bmatrix} \frac{1}{RC} \\ \frac{1}{L} \end{bmatrix}$

or

(18c) $\frac{d}{dt}\begin{bmatrix} i \\ u \end{bmatrix} = A_{32}\begin{bmatrix} i \\ u \end{bmatrix} + B_{32}e$,

where

(18d) $A_{32} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}$, $B_{32} = \begin{bmatrix} \frac{1}{L} \\ \frac{1}{RC} \end{bmatrix}$.

The state equation of the electrical circuit shown in Fig. 2c has the form

(19a) $\frac{d}{dt}\begin{bmatrix} u \\ i \end{bmatrix} = A_{41}\begin{bmatrix} u \\ i \end{bmatrix} + B_{41}e$,

where

(19b) $A_{41} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}$, $B_{41} = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}$

or

(19c) $\frac{d}{dt}\begin{bmatrix} i \\ u \end{bmatrix} = A_{42}\begin{bmatrix} i \\ u \end{bmatrix} + B_{42}e$,

where

(19d) $A_{42} = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{bmatrix}$, $B_{42} = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}$.

Note that the matrices A_{31}, A_{32}, A_{41} and A_{42} have the generalized Frobenius form. It is easy to see that

(20) $$\det[I_2 s - A_{31}] = \det[I_2 s - A_{32}] = \det[I_2 s - A_{41}] = \det[I_2 s - A_{42}] = s^2 + \frac{1}{RC}s + \frac{1}{LC}$$

Therefore, the electrical circuits shown in Fig. 2 are asymptotically stable for all values $R > 0$, $L > 0$ and $C > 0$.
Conisder the electrical circuits shown in Fig. 3 with known resistance R, inductance L, capacitance C and two source voltages e_1 and e_2.

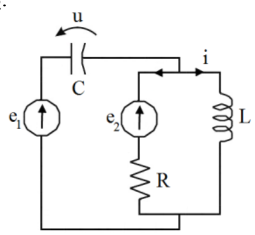

Fig. 3. Electrical circuits

Using the Kirchhoff's laws we obtain the equations

(21a) $e_1 = L\frac{di}{dt} + u$,

(21b) $C\frac{du}{dt} = i + (e_1 - u - e_2)\frac{1}{R}$,

which can be written in the form

(22a) $\frac{d}{dt}\begin{bmatrix} u \\ i \end{bmatrix} = A_{51}\begin{bmatrix} u \\ i \end{bmatrix} + B_{51}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$,

where

(22b) $A_{51} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}$, $B_{51} = \begin{bmatrix} \frac{1}{RC} & -\frac{1}{RC} \\ \frac{1}{L} & 0 \end{bmatrix}$.

Note that the matrix A_{51} has the generalized Frobenius form as the matrix A_{41}.
Therefore, we have the following collorary.

Collorary 1. The state matrix A of the electrical circuit is independent of its source voltage.
Combining the electrical circuit shown in Fig. 2a and 2b we obtain the electrical circuit presented in Fig. 4 with given resistances R_1, R_2, inductances L_1, L_2, capacitances C_1, C_2 and source voltage e.
Taking into account (16a), (16b) and (18a), (18b) we obtain

(23) $$\frac{d}{dt}\begin{bmatrix} u_1 \\ i_1 \\ u_2 \\ i_2 \end{bmatrix} = \begin{bmatrix} A_{21} & 0 \\ 0 & A_{31} \end{bmatrix}\begin{bmatrix} u_1 \\ i_1 \\ u_2 \\ i_2 \end{bmatrix} + \begin{bmatrix} B_{21} \\ B_{31} \end{bmatrix} e,$$

where A_{21}, B_{21} are given by (16b) and A_{31}, B_{31} are given by (18b).

Continuing this procedure we may obtain in general case an electrical circuit with $n \times n$ state matrix in generalized Frobenius form.

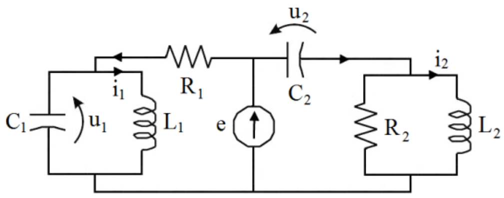

Fig. 4. Electrical circuit.

Inverse matrices of electrical circuits with generalized Frobenius forms

Consider the electrical circuit shown in Fig. 2a with given parameters R, L, C and e. The inverse matrix of the state matrix A_{21} (given by (15b)) has the generalized Frobenius form

(24) $$A_{21}^{-1} = \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & L \\ -C & -\frac{L}{R} \end{bmatrix}$$

and its characteristic equation is given by

(25) $$\det[I_2 s - A_{21}^{-1}] = \begin{vmatrix} s & -L \\ C & s+\frac{L}{R} \end{vmatrix} = s^2 + \frac{L}{R}s + LC = 0.$$

The inverse matrix of the electrical circuit shown in Fig. 2b with given parameters R, L, C and e has the generalized Frobenius form

(26) $$A_{31}^{-1} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -L \\ C & -\frac{L}{R} \end{bmatrix}$$

and its characteristic polynomial is given by

(27) $$\det[I_2 s - A_{31}^{-1}] = \begin{vmatrix} s & L \\ -C & s+\frac{L}{R} \end{vmatrix} = s^2 + \frac{L}{R}s + LC.$$

Similarly, the inverse matrix of the electrical circuit shown in Fig. 2c with given parameters R, L, C and e has the generalized Frobenius form

(28) $$A_{41}^{-1} = \begin{bmatrix} -\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & -L \\ C & -\frac{L}{R} \end{bmatrix}$$

and its characteristic polynomial is given by

(29) $$\det[I_2 s - A_{41}^{-1}] = \begin{vmatrix} s & L \\ -C & s+\frac{L}{R} \end{vmatrix} = s^2 + \frac{L}{R}s + LC.$$

Note that the characteristic polynomials (25), (27) and (29) have the same form and the inverse matrices A_{21}^{-1}, A_{31}^{-1} and A_{41}^{-1} are asymptotically stable for all $R > 0$, $L > 0$ and $C > 0$. Therefore, the asymptotic stability of the electrical circuits is independent how are connected the elements R, L and C.

The characteristic equation of the matrix A_{21} has the form

(30) $$\det[I_2 s - A_{21}^{-1}] = \begin{vmatrix} s+\frac{1}{RC} & \frac{1}{C} \\ -\frac{1}{L} & s \end{vmatrix} = s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

and after multiplication by LC we obtain

(31) $$LCs^2 + \frac{L}{R}s + 1 = 0.$$

Note that the characteristic polynomal (29) has the same coefficients but in reverse order. This confirms Theorem 2. These considerations can be easily extended to generalized Frobenius forms matrices for $n > 2$.

Angles between state matrices of linear electrical circuits

In this section the angle between two matrices will be defined and used in linear electrical circuits.

To any given matrix $A = [a_{ij}] \in \Re^{n \times m}$ the following two corresponding vectors can be defined

(32a) $$\overline{A} = [a_{11} \dots a_{1m} \;\; a_{21} \dots a_{2m} \;\; a_{31} \dots a_{nm}]^T \in \Re^{nm}$$

and

(32b) $$\hat{A} = [a_{11} \dots a_{n1} \;\; a_{12} \dots a_{n2} \;\; a_{13} \dots a_{nn}]^T \in \Re^{nm}$$

T denotes the transpose.

Using the vectors of the matrices $A \in \Re^{n \times m}$ and $B = [b_{ij}] \in \Re^{n \times m}$ we may defined the following scalar product of the two matrices.

Definition 1. The scalar

(33) $$(\overline{A}, \overline{B}) = (\hat{A}, \hat{B}) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} b_{ij}$$

is called the scalar product of the matrices A and B.

In particular case if $A = B$ then

(34) $$(\overline{A}, \overline{A}) = (\hat{A}, \hat{A}) = \left|\overline{A}\right|^2 = \left|\hat{A}\right|^2 = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij}^2 > 0$$

for any nonzero matrix $A \in \Re^{n \times m}$.

Using (33) and (34) we may defined the angle φ between two given matrices A and B of the same dimensions.

Definition 2. The angle defined by

(35a) $$\varphi = \varphi_{A,B} = \arccos\frac{(\overline{A}, \overline{B})}{\left|\overline{A}\right|\left|\overline{B}\right|} = \arccos\frac{(\hat{A}, \hat{B})}{\left|\hat{A}\right|\left|\hat{B}\right|}, \quad 0 < \varphi < \pi$$

is called the angle φ between the matrices A and B.

The relation (35a) can be equivalently written in the form

(35b) $$\cos\varphi = \cos\varphi_{A,B} = \frac{(\overline{A}, \overline{B})}{\left|\overline{A}\right|\left|\overline{B}\right|} = \frac{(\hat{A}, \hat{B})}{\left|\hat{A}\right|\left|\hat{B}\right|}.$$

From (35b) it follows $\cos\varphi_{A,B} = \cos\varphi_{B,A}$ and $\cos\varphi_{-A,-B} = \cos\varphi_{B,A}$.

In particular case if $\overline{B} = \overline{A}$ then from (35b) we have $\cos\varphi = 1$ and $\varphi = 0$.

Example 2. Find the $\cos\varphi$ between the following matrices

(36) $$A = \begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 2 \\ 1 & 0 \\ -1 & 1 \end{bmatrix}.$$

In this case

(37a) $\overline{A}=[-1\ -2\ 0\ 1\ 2\ 3]^T$, $\overline{B}=[0\ 2\ 1\ 0\ -1\ 1]^T$
and

(37b) $\hat{A}=[1\ 0\ 2\ -2\ 1\ 3]^T$, $\hat{B}=[0\ 1\ -1\ 2\ 0\ 1]^T$.
Using (33), (34), (35b) and (37) we obtain

(38a) $(\overline{A},\overline{B})=(\hat{A},\hat{B})=-3$, $\left|\overline{A}\right|^2=\left|\hat{A}\right|^2=19$, $\left|\overline{B}\right|^2=\left|\hat{B}\right|^2=7$

and

(38b) $\cos\varphi=\cos\varphi_{A,B}=\dfrac{(\overline{A},\overline{B})}{\left|\overline{A}\right|\left|\overline{B}\right|}=\dfrac{(\hat{A},\hat{B})}{\left|\hat{A}\right|\left|\hat{B}\right|}=-\dfrac{3}{\sqrt{19}\sqrt{7}}$.

Consider the following two matrices of the same dimensions

(39) $A=[a_{ij}]\in\Re^{n\times m}$, $B=[b_{ij}]\in\Re^{n\times m}$.

Definition 3. The matrix defined by

(40) $A\circ B=\begin{bmatrix} a_{11}b_{11} & \cdots & a_{1m}b_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1}b_{n1} & \cdots & a_{nm}b_{nm} \end{bmatrix}\in\Re^{n\times m}$

is called the Hadamard product of the matrices (39).

Theorem 4. The angle φ between the matrices (39) is equal to $\dfrac{\pi}{2}$ if the Hadamard product (40) of the matrices (39) is zero matrix.

Proof. From Definitions 1 and 3 it follows that $A\circ B=0$ implies

(41) $(\overline{A},\overline{B})=\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}b_{ij}=0$.

In this case from (35b) we obtain $\cos\varphi=0$ and $\varphi=\dfrac{\pi}{2}$. □

Example 3. Using (40) for the matrices

(42) $A=\begin{bmatrix} 1 & 0 & -2 \\ 0 & -1 & 3 \end{bmatrix}$, $B=\begin{bmatrix} 0 & 1 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

we obtain

(43) $A\circ B=\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

and

(44) $(\overline{A},\overline{B})=\sum_{i=1}^{2}\sum_{j=1}^{3}a_{ij}b_{ij}=0$.

Therefore, by Theorem 4 the angle between the matrices (42) is equal $\dfrac{\pi}{2}$.

Theorem 4. The angle φ between the matrices $A=[a_{ij}]\in\Re^{n\times m}$, $B=[b_{ij}]\in\Re^{n\times m}$ satisfies the condition $\cos\varphi\geq 0$ if and only if

(45a) $(\overline{A},\overline{B})=\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}b_{ij}\geq 0$

and $\cos\varphi<0$ if and only if

(45b) $(\overline{A},\overline{B})=\sum_{i=1}^{n}\sum_{j=1}^{m}a_{ij}b_{ij}<0$.

Proof. Note that if (45a) is satisfied then from (35b) it follows that $\cos\varphi\geq 0$ since $\left|\overline{A}\right|>0$ and $\left|\overline{B}\right|>0$.

Proof of (45b) is similar. □

A matrix $A=[a_{ij}]\in\Re^{n\times m}$ is called the Metzler matrix if $a_{ij}\geq 0$ for $i\neq j$, $i,j=1,...,n$. The Metzler matrix is asymptotically stable (Hurwitz) if and only if there exists a strictly positive vector $\lambda=[\lambda_1,...,\lambda_n]$, $\lambda_k>0,\ k=1...,n$ such that [17]

(46) $A\lambda<0$.

Examples of electrical circuits with Metzler state matrix A are given in [17].

Theorem 5. The angle φ between two asymptotically stable Metzler matrices $A=[a_{ij}]\in M_n$, $B=[b_{ij}]\in M_n$ satisfies the condition $0<\varphi<\dfrac{\pi}{2}$.

Proof. From (45) it follows that the diagonal entries a_{ii} and b_{ii} for $i=1...,n$ of asymptotically stable Metzler matrices A and B are negative. In this case the condition (45a) is satisfied and $0<\varphi<\dfrac{\pi}{2}$. □

Example 4. Consider the following two asymptotically stable Metzler matrices

(47) $A=\begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix}$, $B=\begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$.

Using (33), (35b) and (47) we obtain

(48) $\overline{A}=\begin{bmatrix} -2 \\ 1 \\ 0 \\ -3 \end{bmatrix}$, $\overline{B}=\begin{bmatrix} -1 \\ 1 \\ 0 \\ -1 \end{bmatrix}$, $(\overline{A},\overline{B})=6$, $\left|\overline{A}\right|^2=14$, $\left|\overline{B}\right|^2=3$

and

(49) $\cos\varphi=\dfrac{(\overline{A},\overline{B})}{\left|\overline{A}\right|\left|\overline{B}\right|}=\dfrac{6}{\sqrt{14}\sqrt{3}}=0.926$

This confirms the thesis of Theorem 5.

Example 5. Find the $\cos\varphi$ between asymptotically stable Metzler matrix A given by (47) and the unstable Metzler matrix

(50) $B=\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$

In this case we obtain

(51) $\overline{A}=\begin{bmatrix} -2 \\ 1 \\ 0 \\ -3 \end{bmatrix}$, $\overline{B}=\begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$, $(\overline{A},\overline{B})=-7$, $\left|\overline{A}\right|^2=14$, $\left|\overline{B}\right|^2=6$

and

(52) $\cos\varphi=\dfrac{(\overline{A},\overline{B})}{\left|\overline{A}\right|\left|\overline{B}\right|}=\dfrac{-7}{\sqrt{14}\sqrt{6}}=-0.764$.

Theorem 6. Consider the electrical circuits shown in Fig. 2 with given parameters R, L, C for two choices of the components of their state vectors. The angles between two matrices corresponding to different choice of the state variables in state vectors is the same.

(53) $\cos\varphi_k=\dfrac{(\overline{A}_{k1},\overline{A}_{k2})}{\left|\overline{A}_{k1}\right|\left|\overline{A}_{k2}\right|}=-\dfrac{2}{\dfrac{L}{R^2C}+\dfrac{C}{L}+\dfrac{L}{C}}$ for $k=2,3,4$.

Proof. Using (35b), (16b) and (16d) we obtain

(54a) $$\overline{A}_{21}=\begin{bmatrix}-\frac{1}{RC}\\ -\frac{1}{C}\\ \frac{1}{L}\\ 0\end{bmatrix},\quad \overline{A}_{22}=\begin{bmatrix}0\\ \frac{1}{L}\\ -\frac{1}{C}\\ -\frac{1}{RC}\end{bmatrix}$$

and

(54b) $$\cos\varphi_2=\frac{(\overline{A}_{21},\overline{A}_{22})}{|\overline{A}_{21}||\overline{A}_{22}|}=-\frac{2}{\frac{L}{R^2C}+\frac{C}{L}+\frac{L}{C}}$$

since

(54c) $$|\overline{A}_{21}|=|\overline{A}_{22}|=\sqrt{\left(\frac{1}{RC}\right)^2+\left(\frac{1}{L}\right)^2+\left(\frac{1}{C}\right)^2}\ .$$

Proof for $k=3.4$ is similar. □

Remark 1. For the electrical circuit shown in Fig. 1 the angle between two matrices A_{11} and A_{12} is given by

(55) $$\cos\varphi_1=\frac{(\overline{A}_{11},\overline{A}_{12})}{|\overline{A}_{11}||\overline{A}_{12}|}=-\frac{2}{\frac{R^2C}{L}+\frac{L}{C}+\frac{C}{L}}<0$$

and is different from the ones of Fig. 2.

Now let us consider the angles between the state matrices of the electrical circuits A_{k1} and their inverses A_{k1}^{-1} for $k=1,...,4$.

Taking into account that for the electrical circuit shown in Fig. 1

(56) $$A_{11}=\begin{bmatrix}0 & \frac{1}{C}\\ -\frac{1}{L} & -\frac{R}{L}\end{bmatrix} \text{ and } A_{11}^{-1}=\begin{bmatrix}-RC & -L\\ C & 0\end{bmatrix}$$

we obtain

(57a) $$\overline{A}_{11}=\begin{bmatrix}0\\ \frac{1}{C}\\ -\frac{1}{L}\\ -\frac{R}{L}\end{bmatrix},\quad \overline{A}_{11}^{-1}=\begin{bmatrix}-RC\\ -L\\ C\\ 0\end{bmatrix}$$

and

$$\cos\overline{\varphi}_1=\frac{(\overline{A}_{11},\overline{A}_{11}^{-1})}{|\overline{A}_{11}||\overline{A}_{11}^{-1}|}$$

(57b) $$=-\frac{\frac{C}{L}+\frac{L}{C}}{\sqrt{\left(\frac{1}{C}\right)^2+\left(\frac{1}{L}\right)^2+\left(\frac{R}{L}\right)^2}\sqrt{(RC)^2+L^2+C^2}}<0\ .$$

For the electrical circuit shown in Fig. 2a we have

(58a) $$\overline{A}_{21}=\begin{bmatrix}-\frac{1}{RC}\\ -\frac{1}{C}\\ \frac{1}{L}\\ 0\end{bmatrix},\quad \overline{A}_{21}^{-1}=\begin{bmatrix}0\\ L\\ -C\\ -\frac{L}{R}\end{bmatrix}$$

and

$$\cos\overline{\varphi}_2=\frac{(\overline{A}_{21},\overline{A}_{21}^{-1})}{|\overline{A}_{21}||\overline{A}_{21}^{-1}|}$$

(58b) $$=-\frac{\frac{C}{L}+\frac{L}{C}}{\sqrt{\left(\frac{1}{RC}\right)^2+\left(\frac{1}{L}\right)^2+\left(\frac{R}{L}\right)^2}\sqrt{L^2+C^2+\left(\frac{L}{R}\right)^2}}<0\ .$$

In a similar way for the electrical circuit shown in Fig. 2b we obtain

(59a) $$\overline{A}_{31}=\begin{bmatrix}-\frac{1}{RC}\\ \frac{1}{C}\\ -\frac{1}{L}\\ 0\end{bmatrix},\quad \overline{A}_{31}^{-1}=\begin{bmatrix}0\\ -L\\ C\\ -\frac{L}{R}\end{bmatrix}$$

and

$$\cos\overline{\varphi}_3=\frac{(\overline{A}_{31},\overline{A}_{31}^{-1})}{|\overline{A}_{31}||\overline{A}_{31}^{-1}|}$$

(59b) $$=-\frac{\frac{C}{L}+\frac{L}{C}}{\sqrt{\left(\frac{1}{RC}\right)^2+\left(\frac{1}{L}\right)^2+\left(\frac{1}{C}\right)^2}\sqrt{L^2+C^2+\left(\frac{L}{R}\right)^2}}<0$$

For the electrical circuit shown in Fig. 2c we obtain

(60a) $$\overline{A}_{41}=\begin{bmatrix}-\frac{1}{RC}\\ \frac{1}{C}\\ -\frac{1}{L}\\ 0\end{bmatrix},\quad \overline{A}_{41}^{-1}=\begin{bmatrix}0\\ -L\\ C\\ -\frac{L}{R}\end{bmatrix}$$

and

$$\cos\overline{\varphi}_4=\frac{(\overline{A}_{41},\overline{A}_{41}^{-1})}{|\overline{A}_{41}||\overline{A}_{41}^{-1}|}$$

(60b) $$=-\frac{\frac{C}{L}+\frac{L}{C}}{\sqrt{\left(\frac{1}{RC}\right)^2+\left(\frac{1}{C}\right)^2+\left(\frac{1}{L}\right)^2}\sqrt{L^2+C^2+\left(\frac{L}{R}\right)^2}}<0$$

Corollary 2. From comparison of (58b) and (60b) we have $\cos\overline{\varphi}_2=\cos\overline{\varphi}_4$ and all $\cos\overline{\varphi}_k$, $k=1,2,3,4$ are negative.

The above considerations can be extended to linear electrical circuits for $n>2$.

Concluding remarks

Generalized Frobenius matrices and their inverses have been applied in analysis of the linear electrical circuits. The properties of generalized Frobenius matrices have been analyzed. It has been shown that if the state matrix of electrical circuit has generalized Frobenius form then its inverse system matrix has also Frobenius form. The notion of an angle between state matrices of linear electrical circuits has been proposed and its basic properties have been investigated. The considerations have been illustrated by examples of linear electrical circuits with state matrices in generalized Frobenius forms. The considerations can be easily extended to fractional linear electrical circuits.

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***Authors**: prof. dr hab. inż. Tadeusz Kaczorek, Politechnika Białostocka, Wydział Elektryczny, ul. Wiejska 45d, 15-351 Białystok, E-mail: kaczorek@ee.pw.edu.pl.*

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