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# Arbelos Theory in Electrical Engineering

***Streszczenie.*** *Arbelos jest częścią płaszczyzny ograniczoną trzema półokręgami, które są parami styczne i mają średnice leżące na tej samej linii. Jego nazwa nazwa wywodzi się z greckiego i oznacza nóż szewski. Kształt arbelos jest podobny do rozkładu linii pola magnetycznego - w idealnej formie - dla dwóch równoległych przewodów. Artykuł przedstawia dyskusję na temat analogii arbelos i zastosowania tej teorii w elektrotechnice.* ***Zastosowanie teorii Arbelos w elektrotechnice***

***Abstract***. *An arbelos is a plane figure bounded by three semicircles that are pairwise tangent and have diameters lying on the same line. Its name comes from Greek and means Shoemaker's knife. Arbelos shape have the same shape like lines of magnetic field distribution - in ideal form - for two parallel wires. The paper presents the discussion on the analogue of arbelos and their theoretical contributions in electrical engineering.*

**Słowa kluczowe**: arbelos, parbelos, podobieństwo, rozkład pola magnetycznego.
**Keywords**: arbelos, parbelos, similarity, magnetic field distribution.

## Introduction

Arbelos is a plane figure bounded by three pairwise tangent semicircles whose centers are lying on the same line (Fig.1).

Fig. 1 Arbelos.

It has been known since ancient times. According to *Book of Lemmas* [1], Archimedes of Syracuse gave the name to the figure and formulated a number of propositions concerning it. Nowadays we can only speculate whether ancient

Greek mathematicians investigated arbelos for practical reasons or just for its astounding geometric properties. Even in the twentieth century new results in the subject were obtained (see e.g. [2], [3], [4] and [6]). We briefly discuss basic properties of arbelos and its analogues, finally we consider an idea concerning application of arbelos and elliptic arbelos in indentifying assymetries in electrical circuits.

**"What Archimeds called an** αρβυλοζ"[4]

In this section we recall the well known properties of the classic arbelos. We refer to them in the following.

The common points of two semicircles of the arbelos are called the cusps and labeled $A, B, C,$ where $AC$ is the diameter of the largest semicircle.

**Property 1** *The length of the upper boundary of an arbelos is equal to the length of its lower boundary.*

**Property 2** *Under each lower arc of an arbelos construct a new arbelos similar to the original. Of the four new lower arcs, the middle two are congruent and their common length equals one half the harmonic mean of the lengths of the original lower arcs.*

**Property 3** *Let $D$ be the common point of the outer semicircle and the line perpendicular to $AB$ through $B$. Then the area of the arbelos is equal to the area of the circle whose diameter is $BD$.*

**Property 4** *The middle cusp and the midpoints of the semicircular arcs of an arbelos determine a rectangle whose area equals 2/π times the area of the arbelos.*

**Property 5** *Let the segment $BD$ be perpendicular to $AB$ and divide the arbelos into two regions, each of them bounded by this segment, a semicircle and an arc of the largest semicircle. Then the circles inscribed in these regions, known as the Twin Circles of Archimedes, are congruent.*

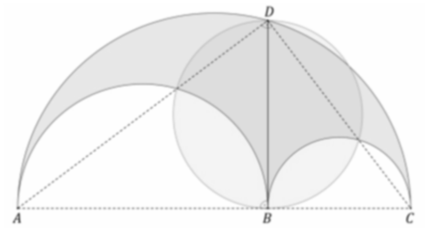


Fig.2. Illustration for Property 3.

## Generalizations

In 2013 J. Sondow published his article [5] on parbelos , a parabolic analogue of arbelos (see Figure 3). He replaced semicircles by the latus rectum arcs of parabolas and observed that the figures have properties corresponding to the ones of classic arbelos. It is remarkable that all the methods he used were available to the ancient Greeks.

Fig. 3. Parbelos.

It turns out that the techniques used in [5] can be under some additional conditions, semicircles can be replaced by three similar arcs: the graph of a continuous function $\mathrm{f}: [0, \mathrm{a}] \to [0, \infty)$ with $\mathrm{f}(0) = \mathrm{f}(\mathrm{a}) = 0$ and its images under appropriate homotheties (see [8]). He named this class of figures f-belos and showed that they have similar properties as arbelos and parbelos. In the following we use rather the name f-arbelos. In 2014 H. Khelif [9] described hyperbelos (hyperbolic arbelos). Interesting examples of f-arbelos are also ellarbelos (elliptic arbelos) with equal eccentricities [7].

In order to obtain a further class of figures resembling arbelos one can consider arcs which are not necessarily the graphs of continuous real-valued functions (see Figure 4).

Let $U$ be an arc in the plane $\mathbb{R}^2$, i.e. the image of the interval $[0,1]$ under a continuous injective function $\varphi: [0,1] \to \mathbb{R}^2$ with ends $\varphi(0) = A$, $\varphi(1) = C$, such that the line containing the segment $ineAC$ intersects $U$ only in $A$ and $C$.

Given a point $B \in \overline{AC}$, $B \notin \{A, C\}$, let $L$ be the arc obtained from $U$ by the homothety with center $A$ and ratio (scale factor) $\lambda = |AB| : |AC|$ and let $R$ be the image of $U$ under the homothety with center $C$ and ratio $1 - \lambda = |BC| : |AC|$. If $U \cap L = \{A\}$, $U \cap R = \{C\}$ and $L \cap R = \{B\}$ then the figure bounded by the curve $U \cup L \cup R$ is called the *generalized arbelos* generated by the arc $U$ or, in brief, the *$U$-arbelos*, with ratio $\lambda$. The points $A$, $B$, $C$ are its *cusps*.

Fig. 4. Examples of U-arbeloses.

Choosing an appropriate coordinate system, we may assume that $A(0,0)$, $B(0,b)$, $C(0,a)$, $a = |AC|$, $b = |AB|$, and $U$ is contained in the upper half-plane $\mathbb{R} \times [0, \infty)$. If $U$ is the graph of a continuous function $f : [0, a] \to [0, y)$ then the $U$-arbelos is clearly an $f$-arbelos. In this case the parametrizations $f_1$ and $f_2$ of $L$ and $R$ are given by the formulas

(1) $f_1 : [0, \lambda a] \to [0, \infty)$, $\quad f_1(x) = \lambda f\left(\frac{x}{\lambda}\right)$,

(2) $\quad f_2 : [\lambda a, a] \to [0, \infty)$, $\quad f_2(x) = (1 - \lambda) f\left(\frac{x - \lambda a}{1 - \lambda}\right)$.

To avoid intersecting $U$ by the arcs $L$ and $R$ (except the points $(0,0)$ and $(a, 0)$) it is necessary to assume that

(3) $\quad f_1(x) < f(x)$ *for each* $x \in (0, b]$, $f_2(x) < f(x)$ *for each* $x \in [b, a)$.

The condition (3) precludes the situations shown in Figure 5.

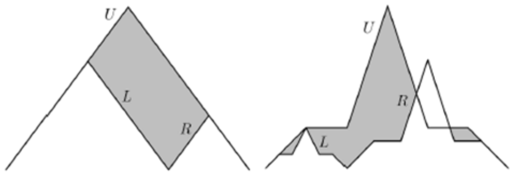

Fig. 5. Examples of figures which are not U-arbeloses.

The condition (3) is satisfied for every λ ∈ (0,1) whenever the function f is strictly convex (upward), i.e.

$$f(t \cdot x + (1 - t) \cdot y) > tf(x) + (1 - t)f(y)$$

for all $t \in (0,1)$ and $x, y \in [0, a]$, $x \neq y$.

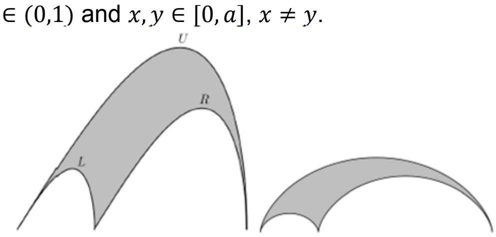

Fig. 6. U-arbeloses built of the graphs of convex functions: on the lemniscate of Gerono (on the left) and on the cycloid (on the right).

It turns out that generalized arbelos have several properties related to those of arbelos and f-arbelos. Moreover, the techniques used in [5] and [8] can be applied in more general setting and yield similar results.

**Property 6**. The length of the upper boundary of a generalized arbelos is equal to the length of its lower boundary.

**Property 7.** Under each lower arc of the $U$-arbelos construct a new $L$- and $R$-arbelos similar to the original. Of the four new lower arcs, the middle two are congruent and if their common length is finite then it equals one half the harmonic mean of the lengths of the original lower arcs.

**Property 8.** The area of the $U$-arbelos with parameter λ ∈ (0,1) is equal to

(4) $$S = 2\lambda(1 - \lambda)S_0$$

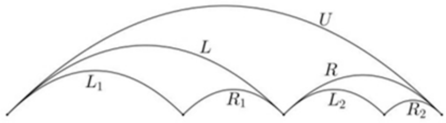

Fig.7. Illustration for Property 7.

where $S_0$ is the area of the region bounded by the upper arc $U$ and $\overline{AC}$.

In particular, if $U$ is a semicircle and $D$ is the common point of $U$ and the line perpendicular to $AC$ through $B$ (see Figure 1) then the area of the arbelos is equal to the area of the circle whose diameter is $BD$ ([1], Proposition 4).

**Corollary 9** Assume that the U-arbelos with parameter λ is well defined for every $\lambda \in (0,1)$. (This condition holds e.g. whenever *U* is the graph of a strictly convex (upward) function *f* : [0,*a*] → [0,∞) with *f*(0) = *f*(*a*) = 0.*)* Then the U-arbelos with the largest area is the one with parameter *λ* = 1/2.

**Property 10** Consider a point P on the upper arc of the U-arbelos with parameter *λ* and the images $P_1 \in L$, $P_2 \in R$ of *P* under the appropriate homotheties. The middle cusp *B* and the points *P*, $P_1$, $P_2$ determine a parallelogram whose area equals

(5) $$S_0 = \lambda(1 - \lambda) \cdot |AC| \cdot d(P, \overline{AC})$$

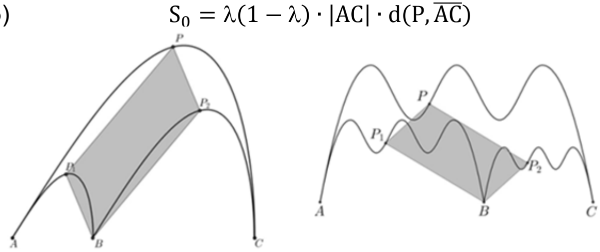

Fig. 8. Illustrations for Property 10.

**Corollary 11** The ratio of the areas of the U-arbelos and the parallelogram referred to in Property 10 does not depend on the parameter λ and equals

$$\frac{S_0}{S} = \frac{|AC| \cdot d(P, \overline{AC})}{2S_0}$$

where $S_0$ is the area enclosed by $U \cup \overline{AC}$.

If $P$ is the midpoint of the upper arc then the ratio equals *2/π* for classic arbelos and 3/4 for parbelos (compare [5], Property 3).

Using the First Mean Value Theorem for Integrals we obtain the following statement.

**Corollary 12** Let $U$ be the graph of a continuous function $f : [0,a] \to [0,\infty)$ with $f(0) = f(a) = 0$ and $\alpha \in (0,1/2)$. Then one can find such a point $P \in U$ that the area of the parallelogram referred to in Property 10 is equal to $\alpha$ times the area of the $U$-arbelos.

**Elliptic arbelos**

Now consider elliptic arbelos or ellarbelos, the class of figures whose boundaries consist of three semi ellipses instead of three semicircles.

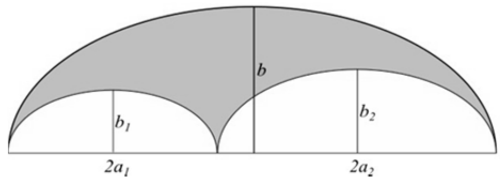

Fig. 9 Elliptic arbelos

We can assume that all considered ellipses are symmetric about the x-axis but their major axes are not necessarily parallel. The semi ellipses do not have to be similar, therefore the class of elliptic arbeloses goes beyond the class of $U$-arbelos. However we still require that both the $L$ and $R$ arcs meet the arc $U$ in exactly one point. If all semi ellipses are pairwise similar then the ellarbelos is a $U$-arbelos and hence it has all the properties listed in the previous section. In general case it is not so obvious whether every pair of semi ellipses intersect exactly in one point. The following lemma provides a criterion to check this condition.

**Lemma 13** *The ellipses*

$$E: \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$
$$E_1: \quad \frac{(x-x_1)^2}{a_1^2} + \frac{y^2}{b_1^2} = 1$$
$$E_2: \quad \frac{(x-x_2)^2}{a_2^2} + \frac{y^2}{b_2^2} = 1,$$

are pairwise tangent if and only if

$$b \geq \max\left\{ b_1\sqrt{1+\frac{a_2}{a_1}}, \quad b_2\sqrt{1+\frac{a_1}{a_2}} \right\}. \tag{6}$$

In the following we assume that the semi ellipses enclosing the ellarbelos satisfy the condition (6). The symbols $a$, $b$ denote the lengths of the semiaxes of the outer semi ellipse while $a1$, $b1$ and $a2$, $b2$ stand for the lengths of the semiaxes of the inner semi ellipses.

**Property 14** The area of ellarbelos equals

$$\frac{1}{2}(\pi(a_1+a_2)b - \pi a_1 b_1 - \pi a_2 b_2) =$$
$$= \frac{1}{2}\left(\pi(a_1+a_2)(b-b_1-b_2) + \pi a_1 b_2 + \pi a_2 b_1\right)$$

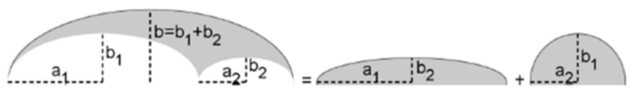

Fig. 10. Property 14 for $b = b1 + b2$.

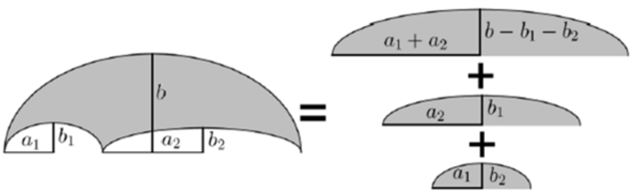

Fig. 11. Property 14 for $b > b1 + b2$.

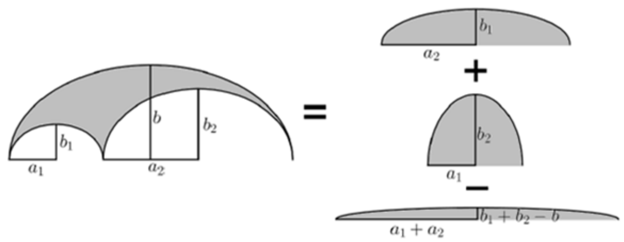

Fig. 12. Property 14 for $b < b1 + b2$.

In other words, the area of the ellarbelos is equal to

- the sum of the areas of the inner semi ellipses whenever b = b1 + b2 (Figure 10),
- the sum of the areas ot three semi ellipses: one with semiaxes a and b - b1 - b2, the second with semiaxes a1, b2 and the third with semiaxes a2, b1 – this equality holds when b > b1 + b2 (Figure 11),
- the sum of the areas of two semi ellipses with semiaxes a1, b2 and a2, b1, respectively, decreased by the area of the semi ellipse with semiaxes a and b1+b2 - b provided that b < b1 + b2 (Figure 12).

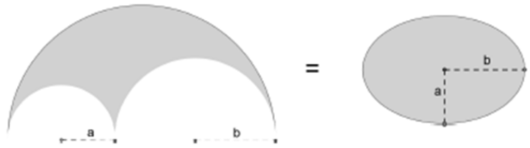

Fig. 13. Property 14 for classic arbelos.

In particular, the area of the classic arbelos is equal to the area of the ellipse whose semiaxes correspond to the radii of the smaller semicircles (Figure 13).

**Application of arbelos**

The arbelos theory can be used in identification of asymmetry of currents in electrical circuits. Let's take as an example cases: first where two parallel wires where the value of current is the same (Figure 14) and second where where the value of current is different (Figure 15).

Assuming the decoupling of fields (as shown in Fig. 14 and Fig. 15), we can use the arbelos theory to consider two cases where we can find functions that describe the magnetic field distribution and point to field asymmetry. Thus, we can identify the current asymmetry in a two-wire system or more.

For determination of magnetic field distribution using the analytical method of calculating the field distribution, let's use the Biot-Savart law.

$$d\vec{B} = \frac{\mu_0 I}{4\pi}\frac{d\vec{l}\times\vec{r}}{|\vec{r}|^3} \tag{7}$$

where: B – magnetic field, dl – length of wire, I – current, μ0 – magnetic permeability, r – distance from a wire

It's possible to find the magnetic field lines around the wires which shape will be similar to arbelos or parbelos. The proposed method is a new idea for identifying asymmetry in electrical systems (e.q. 3-phase system) when the function is specified. The method based on the comparison of magnetic field changes and, in simplified terms, on the basis of changes in surface area for arbelos. In this case we can use the field comparison algorithm, which can be easier with regard to complex measurement methods.

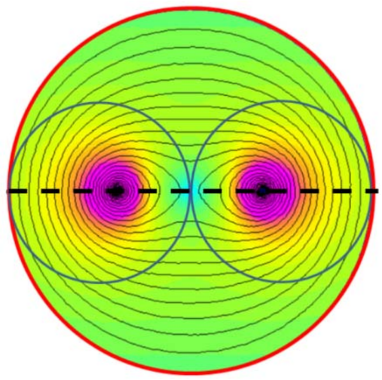
Fig. 14. Magnetic field distribution for two parallel wires with symmetrical current and with arbelos graph.

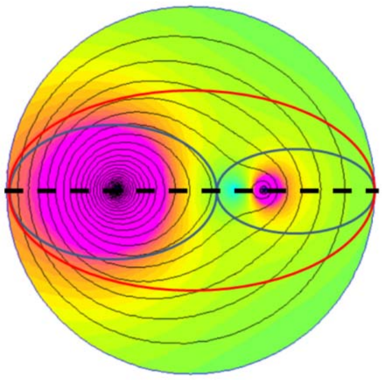

Fig.15. Magnetic field distribution for two parallel wires with asymmetrical current and ellarbelos graph.

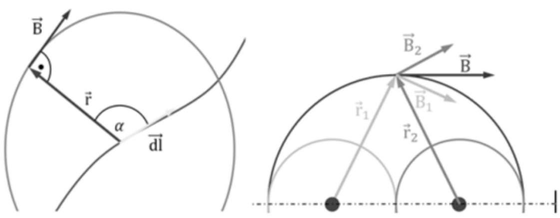


Fig.16. The graphical method of determining the magnetic field and their applications to arbelos.

**Conclusion**

Despite the ancient origin arbelos theory seems to be promising even nowadays. In the paper we present classic theorems and a number of recent results in the field. Moreover, we discuss the way to apply them in identification of asymmetry of currents in electrical circuits. As far as we know, the theory might be useful also in determining the static moments of arc rod constructions or in problems of structural stability and durability of constructions.

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