**Andrzej POPENDA**

Czestochowa University of Technology

doi:10.15199/48.2019.12.44

# Mathematical modelling of transmission shafts based on electrical and mechanical similarities

***Abstract**. In the paper the kinematic structure of transmission shaft between driving motor and working mechanism is analysed. The analysis is based on electrical and mechanical similarities. The equivalent circuits, typical for electrical systems, are defined for the transmission shaft concerned.*

***Streszczenie.** W pracy zaprezentowano analizę struktury kinematycznej długiego wału napędowego łączącego silnik napędowy z mechanizmem roboczym. Analizę oparto na analogiach elektrycznych i mechanicznych. Zdefiniowano schematy zastępcze długiego wału napędowego, typowe dla układów elektrycznych. (**Modelowanie matematyczne długich wałów napędowych oparte na analogiach elektrycznych i mechanicznych**).*

**Keywords:** transmission shaft, lumped and distributed parameters, electrical and mechanical similarities.
**Słowa kluczowe:** długi wał napędowy, parametry skupione i rozłożone, analogie elektryczne i mechaniczne .

## Introduction

Electric motors are connected with working mechanisms via transmission shafts that are elements of mechanical power transmissions. Depending on length and cross-section, transmission shafts can demonstrate different susceptibilities to the impact of moment of torsion, as measured by a value of angle of twist. In the case of short mechanical connections, values of angles of twist are insignificant and they may be omitted by assumption of rigid mechanical connections. In the case of longer mechanical connections the values of angles of twist cannot be ignored and such connections should be considered as the elastic ones [1,2,3,4,5,6,7,8,9,10,11]. In the paper the kinematic structure of transmission shaft between driving motor and working mechanism is analysed. The analysis is based on electrical and mechanical similarities. The equivalent circuits, typical for electrical systems, are defined for the transmission shaft concerned.

The issues based on electrical and mechanical similarities were already considered in the previous papers of the author [1,2,12,13]. Identifying these similarities is very helpful for electricians in finding a relevant interpretation of mechanical systems, which is particularly important in the case of professionals dealing with electromechanical energy converters or drive systems. The comprehensive studies regarding mechanical connections used in drive systems can also be found in other publications, e.g. [6,7,8,9,10].

## Lumped parameters model of transmission shaft

The mass of real transmission shaft (or any element) is distributed continuously. Representing such transmission shaft by a model based on lumped parameters causes discrepancies in results of analysis in relation to accurate models [6]. These discrepancies decrease with the number of points of concentration in the model. Dividing a transmission shaft with distributed mass into several shorter elements, described by lumped parameters such as mass, elasticity and damping of the *i*-th element, it is possible to obtain the results of computer simulation not essentially different from the results obtained if the shaft is divided into infinite number of elements, that corresponds with the wave model [11]. The process of the abovementioned division is referred to as discretization of kinematic structure.

In the paper [1] the kinematic structure of transmission shaft (Fig. 1) transferred into $m$ discrete elements as well as the corresponding equivalent circuit (Fig. 2) were depicted, where $J_1,\ldots, J_m$, $C_{s,12},\ldots, C_{s,m\text{-}1,m}$, $S_{c,12},\ldots, S_{c,m\text{-}1,m}$, $D_{12},\ldots, D_{m\text{-}1,m}$ are moments of inertia, coefficients of torsional stiffness, coefficients of torsional susceptibility ($S_c = 1/C_s$) and coefficients of interior friction of the respective elements of divided transmission shaft; $D_1$, $D_m$ are coefficients of friction defined for bearings.

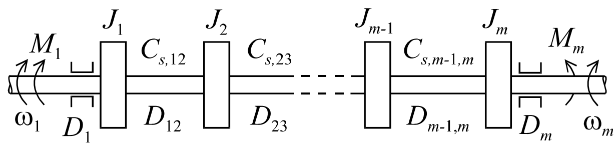


Fig. 1. Kinematic structure of transmission shaft transferred into $m$ discrete elements

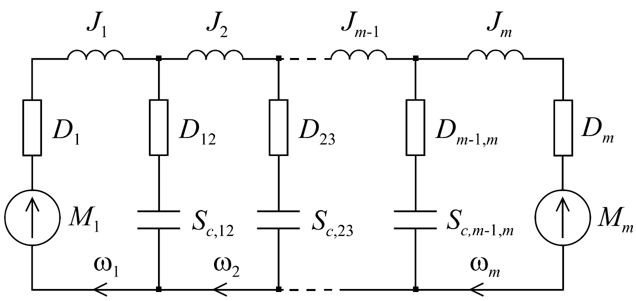


Fig. 2. Equivalent circuit corresponding with the structure depicted in Fig. 1

The $m$ equations of moments and torques as well as $m-1$ equations of angular velocities, corresponding with the Kirchhoff's circuit laws typically applied in order to analyse electrical circuits, may be written for the mechanical connection concerned [1]:

$$M_1 - D_1\omega_1 - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_1\omega_1\right) - D_{12}\left(\omega_1 - \omega_2\right) - M_{c,12} = 0 \tag{1}$$

$$\left.\begin{aligned} &M_{c,k-1,k} + D_{k-1,k}\left(\omega_{k-1} - \omega_k\right) - \\ &\quad - D_{k,k+1}\left(\omega_k - \omega_{k+1}\right) - \\ &\quad - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_k\omega_k\right) - M_{c,k,k+1} = 0 \end{aligned}\right\} \quad \text{for} \quad k = 2,\ldots,m-1 \tag{2}$$

$$\begin{aligned} &M_{c,m-1,m} + D_{m-1,m}\left(\omega_{m-1} - \omega_m\right) - \\ &\quad - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J_m\omega_m\right) - D_m\omega_m - M_m = 0 \end{aligned} \tag{3}$$

$$\omega_k - \omega_{k+1} = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(S_{c,k,k+1}M_{c,k,k+1}\right) \quad \text{for} \quad k = 1,\ldots,m-1 \tag{4}$$

On the basis of the equivalent circuit (Fig. 2) it can be concluded that the transmission shaft may be analysed in a similar way as the electric transmission line.

**Distributed parameter model of transmission shaft**

In order to formulate the distributed parameter model of transmission shaft, the following equivalent circuit of a shaft section, represented by two-port system, can be used (Fig. 3). The adopted length of the shaft section is $\Delta x$. It is assumed that the section $\Delta x$ is short enough to use, in respect of this section, the dependencies adequate for mathematical description of lumped parameter model.

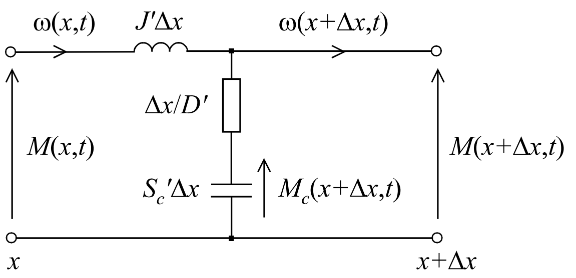


Fig. 3. Equivalent circuit of the transmission shaft section

Balancing moments of torsion around the closed loop and angular velocities at the point leads to the following dependencies:

(5) $$M(x,t) = J'\Delta x \frac{\partial \omega(x,t)}{\partial t} + M(x+\Delta x,t)$$

(6) $$\omega(x,t) = S_c'\Delta x \frac{\partial M_c(x+\Delta x,t)}{\partial t} + \omega(x+\Delta x,t)$$

(7) $$M(x+\Delta x,t) = D'S_c' \frac{\partial M_c(x+\Delta x,t)}{\partial t} + M_c(x+\Delta x,t)$$

where: $J'$, $D'$, $S_c'$ are specific (per unit of length) moment of inertia, coefficient of friction and coefficient of torsional susceptibility, respectively, $S_c' = \rho/GJ'$, $\rho$ is mass density in kg/m$^3$, $G$ is shear module in N/m$^2$. Transforming equations (5), (6) and (7) as well as assuming $\Delta x \to 0$ the following equations can be obtained:

(8) $$-\frac{\partial M(x,t)}{\partial x} = J' \frac{\partial \omega(x,t)}{\partial t}$$

(9) $$-\frac{\partial \omega(x,t)}{\partial x} = S_c' \frac{\partial M_c(x,t)}{\partial t}$$

(10) $$M_c(x,t) = M(x,t) + D' \frac{\partial \omega(x,t)}{\partial x}$$

The further equations result from the equations (8) as well as (9), (10) and (11):

(11) $$-\frac{1}{J'} \frac{\partial^2 M(x,t)}{\partial x^2} = \frac{\partial^2 \omega(x,t)}{\partial x \partial t}$$

(12) $$-\frac{\partial \omega(x,t)}{\partial x} = S_c' \frac{\partial M(x,t)}{\partial t} - \frac{S_c' D'}{J'} \frac{\partial^2 M(x,t)}{\partial x^2}$$

A lossless transmission shaft may also be considered, if $D' = 0$:

(13) $$-\frac{\partial M(x,t)}{\partial x} = J' \frac{\partial \omega(x,t)}{\partial t}, \qquad -\frac{\partial \omega(x,t)}{\partial x} = S_c' \frac{\partial M(x,t)}{\partial t}$$

If the shaft parameters do not change along the axis, the equations (13) may be decoupled in respect of moment of torsion and angular velocity:

(14) $$-\frac{\partial^2 M}{\partial x^2} = J'S_c' \frac{\partial^2 M}{\partial t^2}, \qquad -\frac{\partial^2 \omega}{\partial x^2} = J'S_c' \frac{\partial^2 \omega}{\partial t^2}$$

(15) $$-\frac{\partial^2 M}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 M}{\partial t^2}, \qquad -\frac{\partial^2 \omega}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 \omega}{\partial t^2}$$

$$v = \frac{1}{\sqrt{J'S_c'}} = \sqrt{\frac{G}{\rho}}$$

where: $v$ is phase velocity in m/s. In the equations (8) and (12) partial derivatives with respect to the variable $x$ may be approximated by finite differences [17]. To this end, the transmission shaft was divided into $m$ elements, defining division nodes as $x_k = k\Delta x$, $k = 1,2,\ldots,m$:

(16) $$\left.\frac{\partial M(x,t)}{\partial x}\right|_{x = x_k} \approx \frac{M(x_{k+1},t) - M(x_k,t)}{\Delta x} \approx \approx \frac{M(x_k,t) - M(x_{k-1},t)}{\Delta x} \Rightarrow$$

(17) $$\left.\frac{\partial M(x,t)}{\partial x}\right|_{x = x_k} \approx \frac{M(x_{k+1},t) - M(x_{k-1},t)}{2\Delta x}$$

The approximations of the derivative of angular velocity and the second derivative of moment of torsion are analogical:

(18) $$\left.\frac{\partial \omega(x,t)}{\partial x}\right|_{x = x_k} \approx \frac{\omega(x_{k+1},t) - \omega(x_{k-1},t)}{2\Delta x}$$

(19) $$\left.\frac{\partial^2 M(x,t)}{\partial x^2}\right|_{x = x_k} \approx \frac{M(x_{k+1},t) - 2M(x_k,t) + M(x_{k-1},t)}{2(\Delta x)^2}$$

where: $M(x_k,t) = M_k(t)$, $\omega(x_k,t) = \omega_k(t)$. Taking into account the dependencies (17), (18) and (19) in terms of equations (8) and (12), it is possible to derive the following equations for $k = 1,2,\ldots,m-1$:

(20) $$\frac{\mathrm{d}M_k}{\mathrm{d}t} = -\frac{\omega_{k+1}(t) - \omega_{k-1}(t)}{2S_c'\Delta x} + + \frac{D'}{J'} \frac{M_{k+1}(t) - 2M_k(t) + M_{k-1}(t)}{2(\Delta x)^2}$$

(21) $$\frac{\mathrm{d}\omega_k}{\mathrm{d}t} = -\frac{M_{k+1}(t) - M_{k-1}(t)}{2J'\Delta x}$$

with the initial conditions $M_k(0)$ i $\omega_k(0)$. The angular velocities at the beginning and the end of the transmission shaft may be calculated on the basis of the equations of motion for the rotors of both electric motor and working machine:

(22) $$J_e \frac{\mathrm{d}\omega_1(t)}{\mathrm{d}t} = M_e(t) - D_1\omega_1 - M_1(t)$$

(23) $$J_m \frac{\mathrm{d}\omega_m(t)}{\mathrm{d}t} = M_{m-1}(t) - D_m\omega_m - M_m(t)$$

where: $M_e(t)$ and $M_m(t)$ are motor output torque and working machine load torque.

## Results of computer simulations and discussion

Computer simulations on the behaviour of electromechanical assembly, containing a brushless DC (BLDC) motor connected with a rotating mass via the steel transmission shaft of parameters: length 1 m, diameter 0.02 m, mass density 7900 kg/m$^3$, shear module 7750 MN/m$^2$, were carried out. In the simulation investigations the model of BLDC motor of parameters: 4 kW, 400 V, 3000 rpm, 11.5 A, $J_e$ = 0.025 kgm$^2$, $R_s$ = 0.5 Ω, $L_\mu$ = 7.4 mH, $L_\sigma$ = 1.6 mH, back EMF = 212 V, presented in the paper [14], was adopted. The end of transmission shaft was loaded by a rotating mass $J_m$ = 0.11 kgm$^2$ or $J_m$ = $J_e$ = 0.025 kgm$^2$ as well as a mechanical torque of the rated value. In the simulations of transmission shaft both the model described by the equations (1) to (4) and the model described by the equations (20) to (23) were used for comparison. Time functions of electromagnetic torque, rotational speed and phase current of BLDC motor as well as moment of torsion applied to the transmission shaft are shown in Figs. 4 to 13, where Figs. 4 to 8 correspond to $J_m$ = 0.11 kgm$^2$, while Figs. 9 to 13 correspond to $J_m$ = 0.025 kgm$^2$.

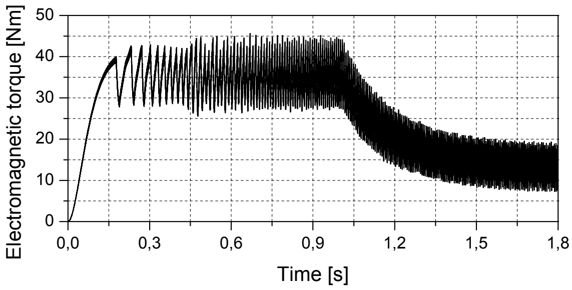


Fig. 4. Electromagnetic torque of BLDC motor

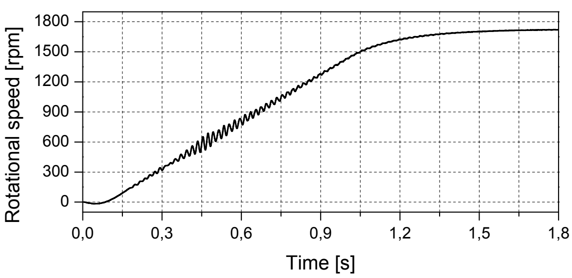


Fig. 5. Rotational speed of BLDC motor

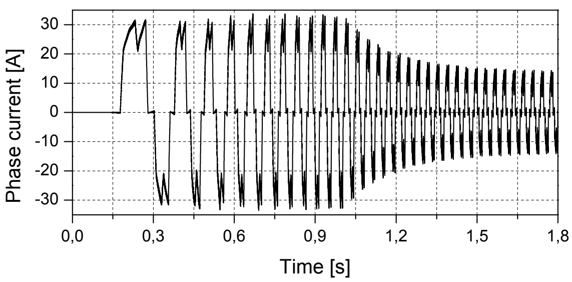


Fig. 6. Phase current of BLDC motor

In the case of transmission shaft described by the lumped parameters model the time functions of moment of torsion (Figs. 7 and 12) are similar to those of transmission shaft described by the distributed parameter model (Figs. 8 and 13). The structures of both models are similar, however due to the approximation of partial derivative by a central difference quotient in the equations (20) to (23), in contrast to the one-sided approximation used in the equations (1) to (4), it should be expected that simulation results are more accurate in the case of distributed parameter model.

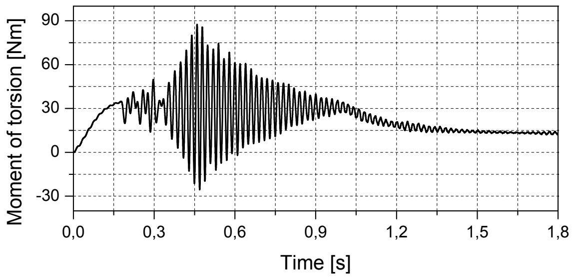


Fig. 7. Moment of torsion applied to the transmission shaft described by the equations (1) to (4)

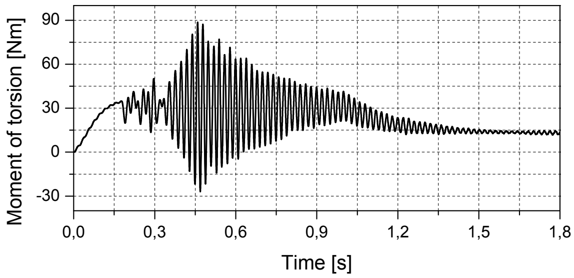


Fig. 8. Moment of torsion applied to the transmission shaft described by the equations (20) to (23)

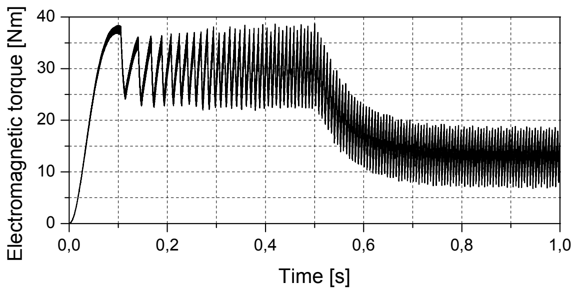


Fig. 9. Electromagnetic torque of BLDC motor

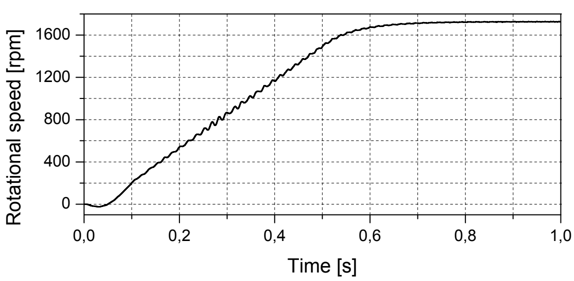


Fig. 10. Rotational speed of BLDC motor

A torque ripple (Figs. 4 and 9) and current ripple (Figs. 6 and 11) can be observed. These ripples are typical for a BLDC motor working without current feedback, i.e. a motor controlled only by armature voltage [14]. The torque ripple can be hazardous if its frequency and shaft resonant frequency are comparable, as the amplitudes of moment of torsion can exceed by a factor of several times the rated torque of motor or load torque for which the shafts are designed (see Figs. 7, 8 and 12, 13).

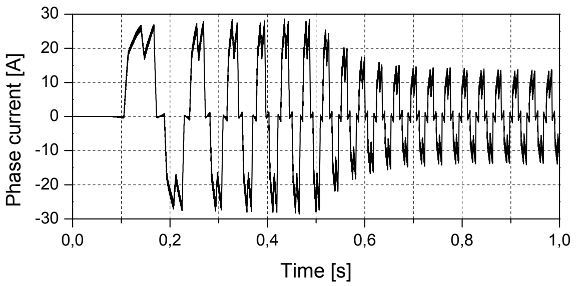


Fig. 11. Phase current of BLDC motor

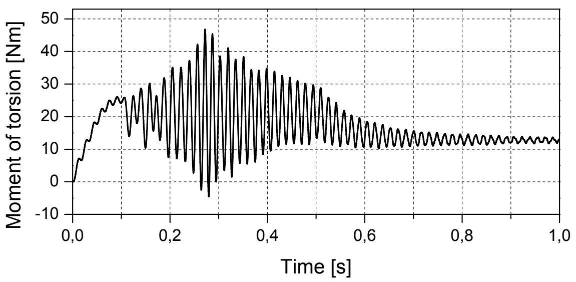


Fig. 12. Moment of torsion applied to the transmission shaft described by the equations (1) to (4)

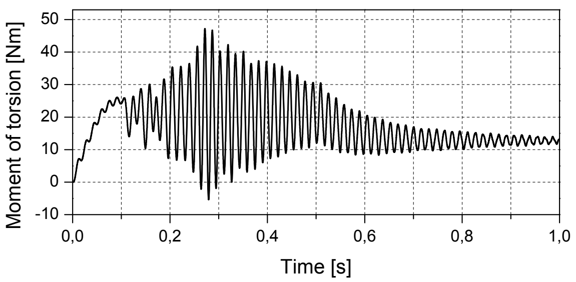


Fig. 13. Moment of torsion applied to the transmission shaft described by the equations (17) and (18)

## Conclusions

In the paper the kinematic structure of transmission shaft between driving motor and working mechanism is analyzed. The equations of torques and equations of angular velocities, analogous to the Kirchhoff's lows based equations typically applied in mathematical analysis of branched electrical circuits, are defined. Modelling of transmission shaft based on formal analogy between transmission shaft and electric transmission line is also proposed.

***Author***: *dr hab. inż. Andrzej Popenda prof. nadzw., Politechnika Częstochowska, Wydział Elektryczny, Al. Armii Krajowej 17, 42-200 Częstochowa, E-mail: popenda@el.pcz.czest.pl.*

## REFERENCES

[1] Popenda A.,, Mathematical modelling of real transmission shafts and mechanical connections with clearances, *Przegląd Elektrotechniczny* 93 (2017), nr 1, 189-192
[2] Popenda A., Simple mathematical models of transmission shafts and gear trains. Electrical and mechanical circuits, *Przegląd Elektrotechniczny* 92 (2016), nr 12, 137-140
[3] Popenda A., Modelowanie i symulacja dynamicznych stanów pracy układów napędowych do reaktorów polimeryzacji z silnikami indukcyjnymi specjalnego wykonania, Wydawnictwo Politechniki Częstochowskiej, Częstochowa 2011
[4] Popenda A., Rusek A., Model matematyczny i wybrane stany nieustalone głównego napędu reaktora polimeryzacji przy uwzględnieniu parametrów pracy komory mieszalnika, *Przegląd Elektrotechniczny* 84 (2008), nr 5, 84-87
[5] Popenda A., Modelling the load torques of electric drive in polymerization process, *Advances in Electrical and Electronic Engineering*, Vol. 6/2007, s. 102-107
[6] Lis M., Modelowanie matematyczne procesów nieustalonych w elektrycznych układach napędowych o złożonej transmisji ruchu, Wydawnictwo Politechniki Częstochowskiej, Częstochowa 2013
[7] Rusek. A., Stany dynamiczne układów napędowych z silnikami indukcyjnymi specjalnego wykonania, Wydawnictwo Politechniki Częstochowskiej, Częstochowa 2012
[8] Czaban A., Lis M.: Mathematical Modelling of Transient States in a Drive System with a Long Elastic Element, *Przegląd Elektrotechniczny*, 88 (2012), nr 12b, 167-170
[9] Rusek A., Czaban A., Lis M., Klatow K., Model matematyczny układu elektromechanicznego z długim sprężystym wałem napędowym, *Przegląd Elektrotechniczny*, 91 (2015), nr 12, 69-72
[10] Czaban A., Lis M., Model matematyczny i analiza układu napędowego silnika indukcyjnego z długim elementem sprężystym dla parametrów rozłożonych, *Prace Naukowe Instytutu Maszyn, Napędów i Pomiarów Elektrycznych Politechniki Wrocławskiej*, 32 (2012), nr 66, 224-230
[11] Szklarski L., Zarudzki J., Elektryczne maszyny wyciągowe, Wydawnictwo Naukowe PWN, Warszawa – Kraków 1998
[12] Popenda A., Mathematical modelling of induction motor with a saturated magnetic circuit during changes in moment of inertia, *Przegląd Elektrotechniczny*, 91 (2015), nr 12, 201-204
[13] Popenda A., Analiza i synteza wybranego układu transportu wody w aspekcie ekonomicznym, *Przegląd Elektrotechniczny*, 88 (2012), nr 8, 108-111
[14] Popenda A., Modelling of BLDC motor energized by different converter systems, *Przegląd Elektrotechniczny*, 94 (2018), nr 1, 81-84
[15] Andrzejewski A., Sposoby zmniejszania błędu estymacji momentu bezwładności, *Przegląd Elektrotechniczny*, 94 (2018), nr 12, 100-107
[16] Lis M., Szafraniec A., Model matematyczny synchronicznego układu pompowego o podatnej transmisji ruchu, *Maszyny Elektryczne – Zeszyty Problemowe*, 118 (2018), nr 2, 165-170
[17] Baron B., Kolańska-Płuska J., Zastosowanie metody ESDIRK (Kennedy-Carpenter) do badania stanów nieustalonych w linii długiej, *Poznan University of Technology Academic Journals. Electrical Engineering*, 81 (2015), 39-47
[18] Olesiak K., Analysis of the energy consumption by an industrial robot for the angular movement of individual axes, *Przegląd Elektrotechniczny*, 94 (2018), nr 12, 218-221
[19] Jakubiec B., Napęd bezszczotkowego silnika prądu stałego z rozmytym regulatorem prędkości, *Przegląd Elektrotechniczny*, 90 (2014), nr 12, 211-213