Determining the dwell time constraint for switched $\mathcal{H}_\infty$ filters

Abstract. This paper presents an algorithm for determining the minimum dwell time constraint for switched linear $\mathcal{H}_\infty$ fault detection filters. When applying switched systems, ensuring the stability is a crucial part, which can be guaranteed, when we switch slowly enough between the subsystems, more precisely the intervals between two consecutive switching, called dwell time, are large enough. The problem formulation is based on multiple Lyapunov functions and is expressed through a special form of linear matrix inequalities (LMIs), which include a nonlinear term with the dwell time. This represents a multivariable time dependent optimization problem. To solve this special formulated LMIs, we propose an algorithm, called $T_\infty$-iteration, which is a combination of the procedure of interval halving with an LMI solver. The results of the illustrative example suggest further benefits.

Streszczenie. W artykule zaprezentowano algorytm do określenia minimum czasowego między kolejnymi przełączeniami liniowego filtru do detekcji błędu $\mathcal{H}_\infty$. Sformułowanie problemu bazuje na wielokrotnym funkcji Lapunova. Do rozwiązania tego problemu zaproponowano algorytm $T_\infty$-iteration który jest kombinacją procedury połowiania interwału z wykorzystaniem solvera LMI. Agorytm do określania minimum czasowego między kolejnymi przełączeniami liniowego filtru do detekcji błędu $\mathcal{H}_\infty$:

Keywords: Switched linear system, Dwell time constraint, Switched linear fault detection filter, MFARE

Introduction

Switched systems for purpose of nonlinear control have been studied extensively in the past two decades and useful results are now available, see e.g. [1], [2], [3], [4] and [5]. As it was stated by several authors e.g. (Liberzon and Morse in 1999, Hespana in 2004, Chen and Saif in 2004, Colaneri in 2008,) the asymptotic stability can be ensured when we switch slow enough between the subsystems, more precisely the intervals between two consecutive switchings -called dwell time-, are large enough. This problem has been specially addressed in the synthesis of switched state estimator of Luenberger type, e.g. (Prandini in 2003, Chen and Saif in 2004) and it is also a crucial part in our objective of the designing a switched linear $\mathcal{H}_\infty$ fault detection filter. In earlier researches different methods have been proposed for determining the minimum dwell time, see [4], [6], [7], [8], [9] and [10]. The most commonly used and powerful algorithms, like e.g. the representation based on Kronecker products (Geromel and Colaneri, 2006) or Logic-Based Switching Algorithms (Hespana, 1998), are based on multiple Lyapunov functions and expressed in form of linear matrix inequalities (LMIs), see in [6], [7], [9], [10] and [11].

Since we deal with $\mathcal{H}_\infty$ filtering, the basic Lyapunov theorem needs to be extended to cope with performance requirements such as the root mean square (RMS) property of a switched system, which corresponds normally to determining an upper bound of the minimum dwell time. To this aim, in our research we consider a method used by (Geromel and Colaneri, 2008) for $\mathcal{H}_\infty$ nonlinear control and we have adopted it to the classical $\mathcal{H}_\infty$ detection filtering problem, see in [12], [13], [14], [15] and [16]. More exactly, the concept of the switched $\mathcal{H}_\infty$ control in [7] can be associated to the switched $\mathcal{H}_\infty$ filtering problem by duality and sufficient stability conditions can be derived.

LMIs are nowadays widely used powerful tools for solving complex optimization problems in field of control engineering, see e.g. [17], [18], [19] and [20]. The common used advanced methods, however, refer to a LMI solver only accepts formulation where the decision variables are included in linear terms. On the contrary, our problem is formulated as a LMIs which include the term of matrix-exponential function with the dwell time, is consequently nonlinear. As a result, the task cannot be treated as a simple feasibility problem, see e.g. [17], [21] and [22]. Despite the widespread referring to this special LMI formulation, however, there cannot be found any solution algorithm about it in the control literature. To this aim, in this paper we present an algorithm to calculate the common minimum dwell time, within a pre-specified accuracy, assuring each specified $\mathcal{H}_\infty$ level calculated separately for each single filter.

The contents of this paper are as follows. After the introduction, in Section II the dwell time condition for assuring stability of the switched linear $\mathcal{H}_\infty$ filter is presented. The main outcome is a special form of LMIs including the nonlinear term with the dwell time, which represents a multivariable time dependent optimization problem. Section III presents the proposed numerical algorithm for the calculation of the common minimum dwell time assuring each specified $\mathcal{H}_\infty$ level. In Section IV the algorithm $T_\infty$-iteration is applied for an illustrative example in MATLAB. In Section V the main results are summarized and concluded the paper is concluded with some final remarks.

Stability of the State Estimation Error involving the dwell time constraint

The synthesis technique proposed below is originated from results (Geromel and Colaneri, 2008) with focus on the application to robust nonlinear control, see in [7] and [6]. We have adopted this concept to a $\mathcal{H}_\infty$ detection filtering problem and it will be introduced in this chapter.

However, in order to improve the detection’s performance we form our concept slightly different from theirs. That means, instead of calculation of the minimum dwell time assuring a common specified $\mathcal{H}_\infty$ level for each controller, we determine the common minimum dwell time to each specified $\mathcal{H}_\infty$ level calculated separately for each single filter.

In the following we are referring to the concept in [12], which’s system-description has been extended to a switched linear system.

Extending the switched linear system representation in [6] to the concept of perturbed system, see in [12], the extended switched linear system subjected to disturbance and faults, can be represented in state space form as follows:

\[
\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + B_{\sigma(t)}v(t) + \sum_{i=1}^{k} L_{\sigma(t)}v_i(t),
\]

\[
y(t) = C_{\sigma(t)}x(t),
\]

(1)
where for all \( t \geq 0 \), \( x(t) \in \mathbb{R}^n \) is the state vector, \( \xi \in \mathbb{R}^n \) is the arbitraril fixed initial condition, \( u(t) \in \mathbb{R}^m \) is the input vector, \( y(t) \in \mathbb{R}^p \) is the output vector, \( \sigma(t) : [0, \infty) \to \Theta \) is the piecewise constant switching function. \( A_{\sigma(t)} \in \mathbb{R}^{nxn} \), \( B_{\sigma(t)} \in \mathbb{R}^{nxm} \), \( C_{\sigma(t)} \in \mathbb{R}^{pxn} \) are an appropriate matrices. Assume that the pairs \((A_{\sigma(t)}, C_{\sigma(t)})\) are observable for all \( t \geq 0 \). For further consideration denote \( n_\text{S} \) the number of subsystems, \( \Theta = \{1, \ldots, n_\text{S}\} \) an index set and \( q = 1, \ldots, n_\text{S} \) the sequence number of the switchings. \( B_{\sigma(t)} = [B_{\sigma_1}, L_\sigma] \) denotes the worst-case input direction and \( \chi(t) \in \ell_2 \) is the input function for all \( t \in \mathbb{R}^+ \) representing the worst-case effects of modelling uncertainties and external disturbances. The cumulative effect of a number of \( k \) faults appearing in the system can be expressed as

\[ \nu_{\sigma(t)} = 0 \]

\[ \nu_{\sigma(t)} = 0 \]

The state estimator for the system description (1) can be formulated as

\[
\hat{x}(t) = (A_{\sigma(t)} - Y_{\sigma(t)} C_{\sigma(t)}^T C_{\sigma(t)}) \hat{x}(t) + B_{\sigma(t)} \hat{w}(t) + \sum_{i=1}^k L_i \sigma(t) v_i(t),
\]

(2)

where \( \hat{x}(t) \) and \( \tilde{e}(t) \) are defined as

\[
\tilde{e}(t) = x(t) - \hat{x}(t),
\]

(4)

As the switching occurs within the finite set of \( q \in \Theta = \{1, \ldots, n_\text{S}\} \) subsystems, the system description in (1) and consequently in (2) and (3) can be simply represented by the matrices \((A_q, B_q, B_{q\sigma}, C_q, C_{q\sigma}, L_{q\sigma}, Y_q)\), for each \( q \in \Theta \). Assume that all matrices \( A_q, q \in \Theta \) are Hurwitz.

By duality we can associate the \( K \) control problem of switched linear system described in [12] to our switched \( K \) filtering task, which’s synthesis is based on the Modified Riccati Equation (MFARE), that can be formulated for switched linear system as

\[
A_q Y_q + Y_q A_q^T + Y_q C_q^T C_q - Y_q + B_{q\sigma} B_{q\sigma}^T = 0,
\]

for all \( q \in \Theta \). In (5) the \( Y_q > 0 \) are positive rational constants and \( Y_q \in \mathbb{R}^{pxp} \) denote the decision variables which are positive definite matrices.

Following the steps of the synthesis procedure in [7], the MFARE can be factorized in form of Riccati Equation as

\[
W_q Y_q + Y_q H_q^T + Q_q = 0, \quad \forall q \in \Theta,
\]

where the associated matrices are

\[
W_q = Y_q C_q^T C_q \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},
\]

(7)

\[
H_q = \begin{bmatrix} A_q - W_q C_q \\ C_q \end{bmatrix},
\]

(8)

\[
Q_q = \begin{bmatrix} W_q & 0 \\ 0 & I \end{bmatrix} W_q + B_{q\sigma} B_{q\sigma}^T.
\]

(9)

We have note that the optimal gain \( W_q \) is determined from the unique stabilizing solution to MFARE and the matrix \( H_q \) is Hurwitz for each \( q \in \Theta \). Since \( Q_q \) depends on the \( Y_{\text{min}} \) value, \( Q_q \geq 0 \) is not guaranteed for any \( q \in \Theta \). However, (6) admit a positive definite solution, since that was created by factorizing the MFARE. It is to note that for solving the LMI in (11) is the condition \( Q_q \geq 0 \) necessary, hence, if \( Q_q \geq 0 \) does not hold, \( Y_q > Y_{\text{min}} \) can always be chosen such that \( Q_q \geq 0 \) holds.

For any \( \sigma(t) : [0, \infty) \to \Theta \) and for all \( t \in (t_p, t_{p+1}) \), where \( t_{p+1} = t_p + T_p \) with \( T_p \geq T_q > 0 \) and at \( t = t_{p+1} \) the switching jumps to \( \sigma(t) = j \in \Theta \), where the corresponding solution of the Lyapunov function along a trajectory of the switched filter state estimation error (2) is expressed by

\[
V(\tilde{x}(t)) = \tilde{x}(t_{p+1})^T Z_{\text{i}} \tilde{x}(t_{p+1}) = \tilde{x}(t)^T H^T Z_e H \tilde{x}(t),
\]

(10)

where \( Z_i \in \mathbb{R}^{nxn} \) is positive definite matrix.
The $H_{\infty}$ control problem described in [7] can be associated to the $H_{\infty}$ filtering problem by duality. By using the time varying solution Riccati equation in finite time interval such that any $\alpha(t) : [0, \infty) \rightarrow \theta$ further satisfies the additional constraint $T_\theta \geq t_{j+1} - t_j$ for all $q > 0$, for some $T_\theta \geq T_d > 0$ is given [7]. Based on (6) and the Lyapunov function formulated along a trajectory of the filter error system in (10) one can derive a time varying LMIs, which can be used to obtain the common minimum dwell time for each specified $H_{\infty}$ level of each particular filter.

Assume that for a given $T_d$ there exists a collection of positive definite matrices $\{Z_u, \ldots, Z_n\}$ of compatible dimensions such that the LMI

$$
H_i Z_i + Z_i H_i^T + Q_i < 0,
\forall q \in \Theta,
$$

(11)

$$
e^{\mu_i t_k} Z_i e^{\mu_i t_k} - Z_i + Y_q < 0,
\forall q \neq j \in \Theta,
$$

hold under the worst-case input assumption in (1) for any switching signal $\alpha(t) : [0, \infty) \rightarrow \Theta$ satisfying the condition $T_\theta = t_{j+1} - t_j \geq T_{\text{dmin}}$. Then, the equilibrium solution of the state estimation error (3) is globally asymptotically stable.

Numerical method for finding the common minimal dwell time by means of $T_d$–iteration

As we have shown it in the previous chapter, the problem of determining the common minimum dwell time can be obtained by solving the set of LMIs (11). According to this idea, by means of combining of an algorithm interval halving for fixed scalar $T_d$ the LMIs can be treated as an a feasibility problem and the common minimum dwell time calculated.

After solving the MFARE in (5) and doing the factorization, the $Y_q$, $H_q$ and $Q_q$ matrices previously obtained for each $q \in \theta$. In [16] it is explained how the MFARE as a LMI can be formulated and solved. Then (11) can be represented via the following optimization problem:

$$
\begin{array}{l}
\min T_d \\
\text{s.t. } Z_q > 0 \\
\quad H_i Z_i + Z_i H_i^T + Q_i < 0 \\
\quad e^{\mu_i t_k} Z_i e^{\mu_i t_k} - Z_i + Y_q < 0 \\
\quad \forall q \neq j \in \Theta
\end{array}
$$

(12)

The main benefit of the LMI formulation is, that it defines a convex constraint with respect to the variable vector. For that reason, it has a convex feasible set which can be found guaranteed by means of convex optimization procedure. When using an LMI solver, however, it usually only accepts formulation where the decision variables are included in linear terms. Unfortunately the LMIs in (12) include the term of matrix-exponential with the design scalar variable $T_d$ are nonlinear, consequently the task cannot be treated as a feasibility problem, see in [18], [19], [20] and [22], so the $T_d$ obtained. To overcome this difficulty we implemented an algorithm called $T_d$-iteration, in which an interval halving method is used iteratively. The algorithm reduces gradually the value of the $T_d$ scalar variable until the constraints of the LMIs in (12) are no longer feasible, consequently any of $Z_q$ matrices, has no longer a positive definite solutions. The $T_{\text{dmin}}$ which is so reached, is within the limits given by an arbitrarily small tolerance $\varepsilon > 0$ and is the common minimum dwell time, such holds that $T_{\text{dmin}} \leq \tau_D$.

The algorithm for the feasibility problem of determining the minimum dwell time can be formulated as follows (remark: the algorithm has been also implemented in MATLAB so some commands are referred to them):

1. Calculate $T_d$, the midpoint of the interval, which is assigned by $a$ and $b$. That is $T_d = a + (b - a)/2$;
2. Calculate the matrix exponential function $e^{\mu_i t_k}$ for the fixed $T_d$ value and substituting its values in (11);
3. Solve the LMIs in (11) as a feasibility problem by the MATLAB function $\text{feasp}$ [22], which returns both the scalar value of $t_{\text{min}}$ as a measure of the feasibility and the feasibility decision vector $x$;
4. Call the MATLAB function $\text{dec2mat}$ which returns the solutions for $Z_q$;
5. If the feasibility criteria with fixed $T_d$ are not satisfied, that is $t_{\text{min}} > 0$, then the upper and lower bounds of interval are changed; Otherwise the value of $T_d$ is saved, that is $T_{\text{dmin}} = T_d$ and the iteration is continued;
6. Examine whether the new interval assigned by $b-a$ reached the relative accuracy of the solution - called epsilon:
   - If not, the iteration is repeated;
   - If yes, the iteration is finished and the $Z_q$ matrices are calculated based on the previous value of $T_d$.

Additionally $T_{\text{dmin}} = T_{\text{dmax}}$.

An illustrative example

$T_d$ – iteration algorithm

In this section the presented $T_d$–iteration algorithm will be implemented for synthesis of switched linear $H_{\infty}$ filter based on 3 subsystems and in case study.

Consider the matrices $Y_q$, $H_q$ and $Q_q$ were previously calculated from (5), (8) and (9). The conditions of the definiteness for each matrix are satisfied according to conditions in (6). The input matrices of the 3 subsystems are:

for the subsystem 1:

$$
H_i = \begin{bmatrix}
-107.4991 & 15.4019 & 27.8936 \\
80.9958 & -564.3523 & 1.9649 \\
0.1119 & 0.3748 & -8.6469
\end{bmatrix},
$$

$Q_1 = 10^s \begin{bmatrix}
0.1810 & -0.2182 & -0.0002 \\
-0.2182 & 2.7837 & 0.0006 \\
-0.0002 & 0.0006 & 0.0000
\end{bmatrix}
$$

$Y_1 = \begin{bmatrix}
81.6978 & -17.0191 & -0.1127 \\
-17.0191 & 244.1844 & 0.2370 \\
-0.1127 & 0.2370 & 0.0103
\end{bmatrix}$

for the subsystem 2:
for the subsystem 3:
\[ H_3 = \begin{bmatrix} -78.3760 & 25.7621 & 27.8720 \\ 132.6617 & -626.8514 & 1.9901 \\ 0.1106 & 0.4382 & -8.5647 \end{bmatrix}, \]
\[ Q_3 = \begin{bmatrix} 0.0854 & -0.2658 & -0.0001 \\ -0.0001 & 0.0008 & 0.0000 \end{bmatrix}, \]
\[ Y_3 = \begin{bmatrix} 48.0245 & -19.4905 & -0.1127 \\ -0.1127 & 0.2733 & 0.0142 \end{bmatrix}. \]

The \( T_d \)-iteration including the LMI computation with the feasibility problem was implemented in MATLAB, hereby only its symbolic algorithm is shown, see below! (Note that some commands are from LMI-Toolbox).

**Algorithm** for finding the common minimum dwell time by means of \( T_d \)-iteration

1. while \( i < i_{\max} \) and \((b - a) > \epsilon_s\)
2. \( T_d = a + (b - a)/2 \)
3. \[ H_kZ_k + Z_kH_k^T + Q_k < 0 \]
4. \( t_{\text{min},x_{\text{feas}}} = \text{fseasp}(\text{lmis}) \)
5. \( Z_q = \text{dec2mat}(t_{\text{min},x_{\text{feas}}}, Z_k) \)
6. if \( t_{\text{min}} < 0 \)
7. \( a = T_d \)
8. else
9. \( b = T_d \)
10. \( T_{dm} = T_d \)
11. end if
12. end while

\[ T_{\text{min}} = T_{dm} \]

The obtained values feasible solutions for \( Z_q \) are denoted by \( Z_s1, Z_s2, Z_s3 \) and shown below. The corresponding eigenvalues \( \text{eig}(Z_q) \) are denoted by \( \text{eig}(Z_s1), \text{eig}(Z_s2), \text{eig}(Z_s3) \) and shown in the row 10 of the Table 1. The positive eigenvalues prove the positive definiteness of \( Z_q \) and the feasibility as well.

<table>
<thead>
<tr>
<th>( Z_{s1} )</th>
<th>( Z_{s2} )</th>
<th>( Z_{s3} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0.4073 )</td>
<td>( 0.0542 )</td>
<td>( 0.5203 )</td>
</tr>
<tr>
<td>( 0.0542 )</td>
<td>( 0.0518 )</td>
<td>( 0.0806 )</td>
</tr>
<tr>
<td>( 0.5203 )</td>
<td>( 0.0806 )</td>
<td>( 1.8642 )</td>
</tr>
<tr>
<td>( 0.5122 )</td>
<td>( 0.0832 )</td>
<td>( 0.6322 )</td>
</tr>
<tr>
<td>( 0.0832 )</td>
<td>( 0.0573 )</td>
<td>( 0.1137 )</td>
</tr>
<tr>
<td>( 0.6322 )</td>
<td>( 0.1137 )</td>
<td>( 1.8605 )</td>
</tr>
</tbody>
</table>

It is to note that in the rows 6, 8 and 11 we did not get a feasible solution, because the scalar \( t_{\text{min}} \) returned with a positive value, which means that the associated \( Z_q \) pencil contains eigenvalues on or very near the imaginary axis. Of course, this resulted in infeasibility. In such cases according to the algorithm interval halving, in these steps the upper- and lower bounds of an interval changed to ensure a proper distance between the \( t_m \) eigenvalues and the imaginary axis.

The iteration ran until the new interval assigned by \( b-a \) is obtained with a \( T_d \)-iteration and repeated it 10-times for the \( T_{\text{min}} = 0.0410s \) is obtained by \( T_{\text{min}} = -9.7372 \). The computational cost associated to solving the \( q \) independent LMIs plus the iteration. Despite the multivariable time dependent optimization problem, by means of combination an algorithm interval halving with an LMI solver, the determining the common minimum dwell time could be applied. An evolution of measures for the feasibility \( t_{\text{min}} \) step by step during the iteration is also plotted and shown in Fig. 1.
The nature of switched systems. Conditions can be easily examined, step by step during the iteration. One can easily perform iteration steps or the impact of changing the relative analyse the impact of the algorithm, it gives more flexibility to examine the solution LMIs, due to the combination with the interval halving specifications and constraints can be expressed through

Despite the complexity of the multivariable time dependent optimization problem, by means of the $T_2$-iteration, the common minimum dwell time assuring each specified $H_\infty$ level of each single filter could be calculated. The in the case study implemented algorithm resulted positive definite solutions for $T_2$ and also the corresponding common minimum dwell time $T_{\text{dmin}} = 0.0410$s. On the other hand the $T_2$-iteration has to face of successive numerical computation of the quadratic matrix inequalities resulted in a proportional computation cost. We have found the solution after running the code in MATLAB after 0.5 second CPU time on a PC with Intel® Celeron® CPU B815 (1.60 GHz).

Apart from the advantage that a variety of design specifications and constraints can be expressed through LMIs, due to the combination with the interval halving algorithm, it gives more flexibility to examine the solution during the entire design process. For example, it is easy to analyse the impact of the $T_2$ value on the number of iteration steps or the impact of changing the relative accuracy on the solution. One can easily perform experiments and get answers e.g. to the following questions: How does the iteration converge? How do the eigenvalues of the decision variable change? How close are they to the imaginary axis? Issues with such explicit conditions can be easily examined, step by step during the iterations, which can also be useful for better understanding the nature of switched systems.

Conflicts of Interest
The authors declare that there are no conflicts of interest regarding the publication of this paper.

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Fig.1. Changing the $T_{\text{dmin}}$ step by step during the iteration

Conclusion
This paper was concerned with a numerical algorithm for determining the minimum dwell time constraint for ensuring stability of switched linear $H_\infty$ fault detection filter. Despite the complexity of the multivariable time dependent optimization problem, by means of the $T_2$-iteration, the common minimum dwell time assuring each specified $H_\infty$ level of each single filter could be calculated. The in the case study implemented algorithm resulted positive definite solutions for $T_2$ and also the corresponding common minimum dwell time $T_{\text{dmin}} = 0.0410$s. On the other hand the $T_2$-iteration has to face of successive numerical computation of the quadratic matrix inequalities resulted in a proportional computation cost. We have found the solution after running the code in MATLAB after 0.5 second CPU time on a PC with Intel® Celeron® CPU B815 (1.60 GHz).

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Fig.2. Schematic diagram of the control strategy