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Reduced Boundary Integral Equation for the Laplace, Poisson and Helmholtz Equations in Annular Region

***Abstract**. Boundary integral model corresponding to the Laplace, Poisson and Helmholtz equations for circular and annular regions was considered in the paper. By expanding the excitation and the solution into the Fourier series, the problem is reduced to purely algebraic for each angular harmonic. An example of use of the reduced boundary integral equation is given. As a kind of by-product, some definite integrals were found. (**Zredukowane równanie całkowo-brzegowe dla równania Laplace'a/Poissona oraz Helmholtza w obszarze pierścieniowym**).*

***Streszczenie.** Przedstawiono model całkowo-brzegowy dla równania Laplace'a, Poissona i Helmholtza w obszarach kołowych i pierścieniowych. Wykorzystując rozwinięcie wymuszenia i rozwiązania w szereg Fouriera, zagadnienie zredukowano do czysto algebraicznego dla każdej z harmonicznych kątowych. Zaprezentowano przykład zastosowania. Jako produkt uboczny obliczono pewne całki oznaczone.*

Keywords: boundary integral formulation, reduced BIE, mesh reduction, cylindrical symmetry.
Słowa kluczowe: model całkowo-brzegowy, zredukowane równanie całkowo-brzegowe, redukcja siatki, symetria cylindryczna.

Introduction

Mesh reduction is an important trend in computational methods. One of the methods belonging to that line is the boundary element method (BEM), e.g. [1-3]. It originates from the boundary integral equation (BIE), which describes the problem via the fundamental solution and the boundary values of field. The main reason of using it seems the fact of lowering the dimension of a problem. This results in smaller number of equations. For example, BIE formulation moves the calculations from a 3D region to its 2D boundary.

In case of certain symmetry a further reduction of the dimension is possible. Consider, for example, a 3D region of axial symmetry. Technically, BIE itself makes the problem 2D, but the axial symmetry makes it possible to consider the boundary in an axial cross-section, which is a line (1D). The derivation for several types of equations can be found for example in [2, 4]. A similar idea can be applied for circular or annular regions. In such a case the reduced boundary becomes a point or a set of isolated points. In fact, the reduced BIE (RBIE) becomes an algebraic equation then, and requires no numerical implementation in the form of BEM. This procedure for selected types of equations will be shown in this paper. From a general point of view, such an approach belongs to the trend of using the full information on the boundary shape, represented among others by so called parametric integral equation system [5]. It is worth mentioning that there are other trials for incorporating the cylindrical symmetry into BEM, e.g. using circular elements [6].

RBIE in polar coordinates

Consider annular domain Ω, whose internal and external radius is a and b, respectively (Fig. 1). Suppose function u satisfies the following equation:

$$\nabla^2 u - \kappa^2 u = -f, \tag{1}$$

where f – known function, κ – know constant. The corresponding boundary integral equation written for point X in domain Ω or on its boundary S is as follows [1, 2]:

$$\begin{aligned} &c(X)u(X) + \int_{S_1 \cup S_2} \frac{\partial G(X,Y)}{\partial n} u(Y)\,\mathrm{d}S_Y = \\ &= \int_{S_1 \cup S_2} G(X,Y)\frac{\partial u(Y)}{\partial n}\,\mathrm{d}S_Y + \int_{\Omega} f(Y) G(X,Y)\,\mathrm{d}\Omega_Y, \end{aligned} \tag{2}$$

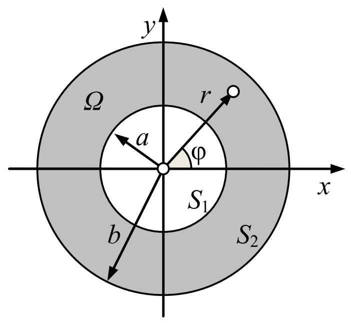

Fig.1. Annular domain

where $c(X)$ is the geometric coefficient, and $G(X, Y)$ is the fundamental solution for Eq. (1). In this case, boundaries S_1 and S_2 are circles, hence Eq. (2) can be rewritten in polar coordinates. Function $u(X)$ can be expanded into complex Fourier series with respect to angular coordinate as follows:

$$u(X) = u(r,\varphi) = \sum_{k=-\infty}^{\infty} u_k(r)\exp(\mathrm{j}k\varphi), \tag{3}$$

where j is the imaginary unit, and

$$u_k(r) = \frac{1}{2\pi}\int_0^{2\pi} u(r,\varphi)\exp(-\mathrm{j}k\varphi). \tag{4}$$

Similar formulas are for $f(X)$. Using the expansions in Eq. (2) and performing the integration with respect to angular coordinate, one obtains:

$$\begin{aligned} &c(r)u_k(r) - h_k(r,a)u_k(a) + h_k(r,b)u_k(b) = \\ &= -g_k(r,a)q_k(a) + g_k(r,b)q_k(b) + F_k(r), \end{aligned} \tag{5}$$

where

$$q_k(r) = \frac{\partial u_k(r)}{\partial r}, \tag{6}$$

$$g_k(r,\rho) = \rho\int_0^{2\pi} G(r,0;\rho,\theta)\exp(\mathrm{j}k\theta)\,\mathrm{d}\theta, \tag{7}$$

$$h_k(r,\rho) = \rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}g_k(r,\rho)\right), \tag{8}$$

(9) $$F_k(r)=\int_a^b f_k(\rho)g_k(r,\rho)\,\mathrm{d}\rho.$$

Eq. (5) is an algebraic equation. It does not require the discretization of the boundary into elements. The boundaries are now points corresponding to the radii of the boundaries. If a = 0, the ring degenerates to a punctured disk, and the terms involving a are absent in Eq. (5). Similarly, if $b = \infty$, the terms with b can be removed from Eq. (5).

Using RBIE

The hardest part in the above procedure is determining g_k and h_k. This will be described in the next sections. Here, it is just assumed that g_k and h_k are known. Eq. (5) can be now used to solve a boundary integral problem in annular region. For simplicity, it is assumed here that this is a Dirichlet problem, i.e. $u(a, \varphi)$ and $u(b, \varphi)$ are known. By Fourier series, also $u_k(a)$ and $u_k(b)$ are known. As in BEM, the first step is to determine the lacking boundary values, in this case $q_k(a)$ and $q_k(b)$. To achieve this, Eq. (5) is used twice: first with $r = a$, and then with $r = b$. Hence, the following system of equations is obtained

(10) $$\begin{cases} g_k(a,a)q_k(a)-g_k(a,b)q_k(b)= \\ \quad =F_k(a)+[h_k(a,a)-\tfrac{1}{2}]u_k(a)-h_k(a,b)u_k(b), \\ g_k(b,a)q_k(a)-g_k(b,b)q_k(b)= \\ \quad =F_k(b)+h_k(b,a)u_k(a)-[h_k(b,b)+\tfrac{1}{2}]u_k(b), \end{cases}$$

where $c(a) = c(b) = ½$. The system of equations is then solved with respect to $q_k(a)$ and $q_k(b)$. Having found these values, one can put any r ($a < r < b$) into Eq. (5) to obtain

(11) $$\begin{aligned} u_k(r)&=h_k(r,a)u_k(a)-h_k(r,b)u_k(b) \\ &\quad -g_k(r,a)q_k(a)+g_k(r,b)q_k(b)+F_k(r). \end{aligned}$$

Then Eq. (3) can be used to find $u(r, \varphi)$.

The Laplace/Poisson equation case

If κ = 0, Eq. (1) becomes the Laplace or Poisson equation, and the 2D fundamental solution is as follows:

(12) $$G=\frac{1}{2\pi}\ln\frac{1}{|X-Y|}=-\frac{1}{4\pi}\ln[r^2+\rho^2-2r\rho\cos(\theta-\varphi)],$$

where (r, φ) and (ρ, θ) are polar coordinates of points X and Y, respectively. Hence, after some transformations

(13) $$g_k(r,\rho)=-\frac{\rho}{4\pi}\int_0^{2\pi}\ln(r^2+\rho^2-2r\rho\cos\psi)\cos k\psi\,\mathrm{d}\psi.$$

It is difficult to evaluate the integral directly. Case of k = 0 can be found in [11] as formula (4.224-14), which gives

(14) $$g_0(r,\rho)=-\rho\ln\frac{r+\rho+|r-\rho|}{2}=-\rho\ln\max(r,\rho).$$

Then, by Eq. (8)

(15) $$h_0(r,\rho)=-\rho\frac{1-\operatorname{sgn}(r-\rho)}{r+\rho+|r-\rho|}=\begin{cases}-1 & \text{for } r<\rho, \\ -\tfrac{1}{2} & \text{for } r=\rho, \\ 0 & \text{for } r>\rho.\end{cases}$$

If $k \neq 0$, then g_k and h_k are hard to evaluate directly (e.g. Mathematica 7.0 fails). Tables of integrals [7] (formulas 4.397-6) can be helpful, but the integrals appear in [8]. The result is

(16) $$g_k(r,\rho)=\frac{\rho}{2|k|}\left(\frac{r+\rho-|r-\rho|}{r+\rho+|r-\rho|}\right)^{|k|}=\frac{\rho}{2|k|}\left(\frac{\min(r,\rho)}{\max(r,\rho)}\right)^{|k|},$$

(17) $$\begin{aligned} h_k(r,\rho)&=\frac{\operatorname{sgn}(r-\rho)}{2}\left(\frac{r+\rho-|r-\rho|}{r+\rho+|r-\rho|}\right)^{|k|}= \\ &=\frac{\operatorname{sgn}(r-\rho)}{2}\left(\frac{\min(r,\rho)}{\max(r,\rho)}\right)^{|k|}. \end{aligned}$$

If necessary, $F_k(r)$ given by Eq. (9) can be evaluated by splitting the integration interval as follows:

(18) $$F_k(r)=\int_a^r f_k(\rho)g_k(r,\rho)\,\mathrm{d}\rho+\int_r^b f_k(\rho)g_k(r,\rho)\,\mathrm{d}\rho.$$

Then the integrals can be found using appropriate forms for $g_k(r, \rho)$ in intervals $a \le \rho \le r$ and $r \le \rho \le b$.

The Helmholtz equation case

If $\kappa \neq 0$, Eq. (1) is the Helmholtz equation, for which the 2D fundamental solution is as follows:

(19) $$G=\frac{1}{2\pi}K_0\left[\kappa\sqrt{r^2+\rho^2-2r\rho\cos(\theta-\varphi)}\right],$$

where $K_0(z)$ is modified Bessel function of the second kind of order 0. Simple transformations give

(20) $$g_k(r,\rho)=\frac{\rho}{2\pi}\int_0^{2\pi}K_0\left[\kappa\sqrt{r^2+\rho^2-2r\rho\cos\psi}\right]\cos k\psi\,\mathrm{d}\psi.$$

This integral can be found in [7] as formula 6.681.13, but only for k = 0 and $r = \rho$. Integral h_k can be only found by Mathematica 7.0 for k = 0 and $r = \rho$, but it involves the special Meijer G function. The general case requires special methods. It can be shown (see Appendix) that

(21) $$g_k(r,\rho)=\rho I_k[\kappa\min(r,\rho)]K_k[\kappa\max(r,\rho)],$$

(22) $$\frac{h_k(r,\rho)}{\kappa\rho}=\begin{cases} I_k(\kappa r)K_k'(\kappa\rho) & \text{for } r<\rho, \\ \dfrac{I_k'(\kappa r)K_k(\kappa r)+I_k(\kappa r)K_k'(\kappa r)}{2} & \text{for } r=\rho, \\ I_k'(\kappa\rho)K_k(\kappa r) & \text{for } r>\rho, \end{cases}$$

where $I_k(z)$ and $K_k(z)$ are modified Bessel functions of the first and second kind, respectively, of order k, and the prime denotes their derivatives. Due to the fundamental property of the modified Bessel functions, it follows that

(23) $$I_k'(z)K_k(z)-I_k(z)K_k'(z)=\frac{1}{z},$$

and the case for $r = \rho$ in Eq. (22) can be simplified in the following way:

(24) $$\begin{aligned} h_k(r,r)&=\frac{1}{2}+\kappa r I_k(\kappa r)K_k'(\kappa r) \\ &=-\frac{1}{2}+\kappa r I_k'(\kappa r)K_k(\kappa r). \end{aligned}$$

Functions $g_k(r, \rho)$ and $h_k(r, \rho)$ for selected values of parameters are depicted in Fig. 2.

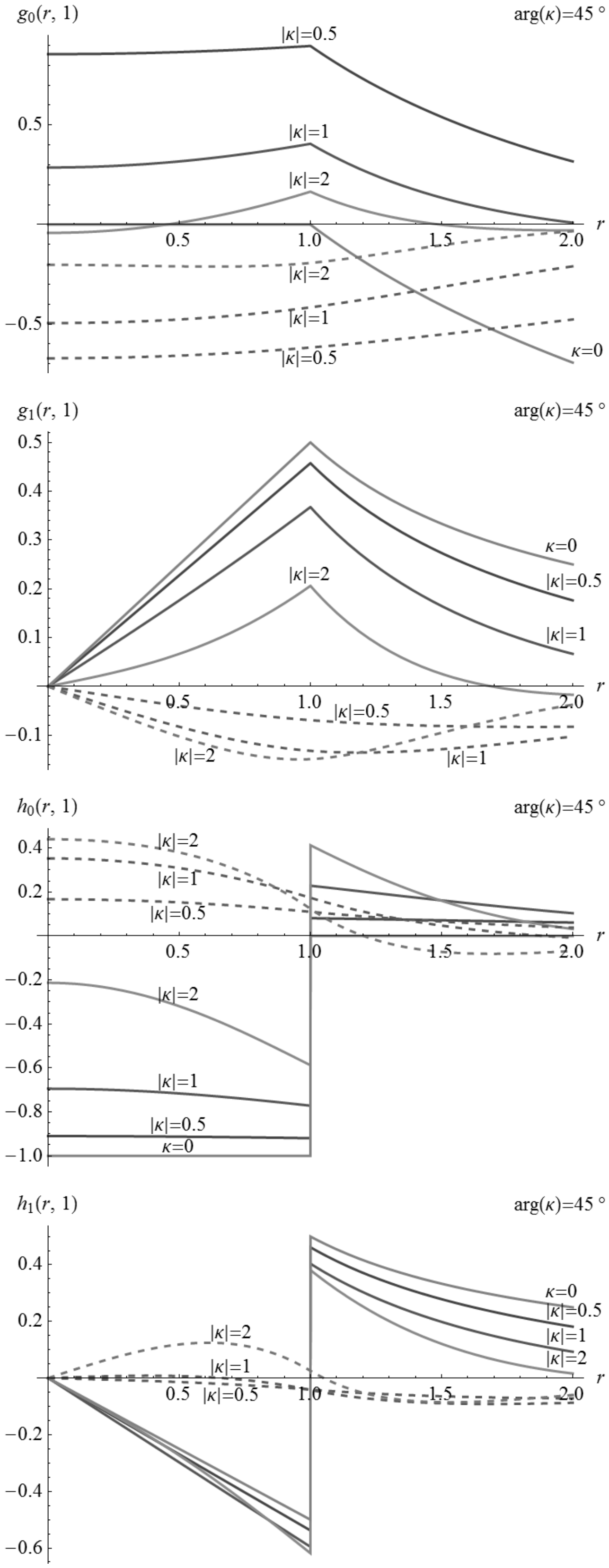

Fig.2. Functions $g_k(r, \rho)$ and $h_k(r, \rho)$ for selected k and κ (solid lines – real part, dashed lines – imaginary part)

Example

Consider a long, homogeneous cylinder of radius R and electrical conductivity $p\gamma_0$ placed in an open conductive medium of conductivity γ_0 in which an externally applied potential, $V_s(r, \varphi)$, exists. This potential can be expanded into Fourier series so that its k-th term equals

$$V_{sk}(r) = \frac{1}{2\pi} \int_0^{2\pi} V_s(r,\varphi) \exp(-\mathrm{j}k\varphi)\,\mathrm{d}\varphi. \tag{25}$$

Eq. (5) can be used for the cylinder itself (with a = 0 and b = R) as well as for the external region (with a = R and b = ∞). Taking into account the remarks below Eq. (9), the following system of equations is obtained:

$$\begin{cases} \frac{1}{2}V_k(R) + h_k(R,R)V_k(R) = g_k(R,R)q_k^{\text{int}}(R), \\ \frac{1}{2}V_k(R) - h_k(R,R)u_k(R) = \\ \qquad\qquad = -g_k(R,R)q_k^{\text{ext}}(R) + V_{sk}(R), \end{cases} \tag{26}$$

where functions g_k and h_k are given by Eqs. (16) and (17), and "int" and "ext" refer to the internal and external domain, respectively. The continuity of current across the boundary leads to relationship $pq_k^{\text{int}}(R) = q_k^{\text{ext}}(R)$. Thus,

$$V_k(R) = \begin{cases} V_{s0}(R) & \text{for } k = 0, \\ \frac{2}{p+1}V_{sk}(R) & \text{for } k \neq 0, \end{cases} \tag{27}$$

$$q_k^{\text{int}}(R) = \frac{1}{p}q_k^{\text{ext}}(R) = \begin{cases} 0 & \text{for } k = 0, \\ \frac{2}{p+1}\frac{|k|}{R}V_{sk}(R) & \text{for } k \neq 0. \end{cases} \tag{28}$$

In the next step, again Eq. (5) is used for the internal region (a = 0, b = R, r < R):

$$\begin{aligned} V_k^{\text{int}}(R) &= g_k(r,R)q_k^{\text{int}}(R) - h_k(r,R)V_k(R) = \\ &= \begin{cases} V_{s0}(R) & \text{for } k = 0, \\ \frac{2}{p+1}\left(\frac{r}{R}\right)^{|k|}V_{sk}(R) & \text{for } k \neq 0. \end{cases} \end{aligned} \tag{29}$$

Similarly, Eq. (5) is used for the external region (a = R, b = ∞, r > R) to obtain:

$$\begin{aligned} V_k^{\text{ext}}(R) &= -g_k(r,R)q_k^{\text{ext}}(R) + h_k(r,R)V_k(R) + V_{sk}(R) = \\ &= \begin{cases} V_{s0}(R) & \text{for } k = 0, \\ \left[\frac{1-p}{1+p}\left(\frac{R}{r}\right)^{|k|} + 1\right]V_{sk}(R) & \text{for } k \neq 0. \end{cases} \end{aligned} \tag{30}$$

The final expressions for potential inside and outside the cylinder are given by Eq. (3):

$$V^{\text{int}}(r,\varphi) = V_{s0}(R) + \frac{2}{1+p}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} V_{sk}(R)\left(\frac{r}{R}\right)^{|k|}\exp(\mathrm{j}k\varphi), \tag{31}$$

$$V^{\text{ext}}(r,\varphi) = V_s(r,\varphi) + \frac{1-p}{1+p}\sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} V_{sk}(R)\left(\frac{R}{r}\right)^{|k|}\exp(\mathrm{j}k\varphi). \tag{32}$$

The expressions are equivalent to those obtained via the method of separation of variables.

Extra results

By combining Eqs. (20) and (21) the following formula is obtained:

$$\begin{aligned} &\int_0^{2\pi} K_0\left[\kappa\sqrt{r^2+\rho^2-2r\rho\cos\psi}\right]\cos k\psi\,\mathrm{d}\psi = \\ &\qquad = 2\pi I_k\left[\kappa\min(r,\rho)\right]K_k\left[\kappa\max(r,\rho)\right] \end{aligned} \tag{33}$$

for integer k. Differentiating the formula with respect to r or ρ, one can obtain more integral formulas.

Conclusions

The RBIE reduces the dimension of the problem to 1D and makes the boundary integral equation an algebraic one. In such a case, no numerical implementation in the form of BEM is necessary. Its use in solving a boundary problem is equivalent to the method of separation of variables in polar coordinates.

Appendix – derivation of Eqs. (21) and (22)

To obtain g_k and h_k for the Helmholtz equation, let us consider the equation in polar coordinates

(a) $$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u}{\partial\varphi^2}-\kappa^2 u=0.$$

The method of separation of variables leads to the following general solution:

(b) $$u(r,\varphi)=\sum_{k=-\infty}^{\infty}\left[c_k I_k(\kappa r)+d_k K_k(\kappa r)\right]\exp(\mathrm{j}k\varphi).$$

To obtain a finite solution in circular region $0 \le r \le \rho$ it is necessary to put $d_k = 0$. Therefore the amplitude of k-th angular harmonic equals $c_k I_k(\kappa r)$. On the other hand, the corresponding RBIE is as follows:

(c) $$c(r)u_k(r)+h_k(r,\rho)u_k(r)=g_k(r,\rho)u_k'(r).$$

When $r < \rho$, $c(r) = 1$. Putting $u_k(r) = c_k I_k(\kappa r)$ and $c(r) = 1$ into this RBIE yields

(d) $$I_k(\kappa r)+h_k(r,\rho)I_k(\kappa\rho)=g_k(r,\rho)\kappa I_k'(\kappa\rho).$$

By using Eq. (8) this equation can be rewritten as follows:

(e) $$\rho\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}g_k(r,\rho)\right)I_k(\kappa\rho)-g_k(r,\rho)\kappa I_k'(\kappa\rho)=-I_k(\kappa r).$$

This is a differential equation with respect to $g_k(r, \rho)$ with constant r and variable ρ. It has the following solution:

(f) $$g_k(r,\rho)=\rho I_k(\kappa r)K_k(\kappa\rho)+A\rho I_k(\kappa\rho),\quad r<\rho,$$

where A is the integration constant, which can be found using the boundary condition visible from Eq. (20),

(g) $$g_k(0,\rho)=\delta_{k,0}\rho K_k(\kappa\rho),$$

where $\delta_{k,0}$ is the Kronecker delta. Hence $A = 0$.

The above procedure can be repeated for annular region $\rho < r < \infty$, but this time $c_k = 0$ is required to keep the solution meaningful at infinity. Hence, $K_k(\kappa r)$ should be used instead of $I_k(\kappa r)$. The relevant RBIE is now as follows:

(h) $$K_k(\kappa r)-h_k(r,\rho)K_k(\kappa\rho)=-g_k(r,\rho)\kappa K_k'(\kappa\rho)$$

(see remarks below Eq. (9)). This leads to the following result:

(i) $$g_k(r,\rho)=\rho I_k(\kappa\rho)K_k(\kappa r)+B\rho K_k(\kappa\rho),\quad r>\rho,$$

where B is the integration constant.

Hence, there are two forms of g_k, given by Eqs. (f) with A = 0 and (i) with certain B. To determine B it can be observed that g_k is a continuous function of r and ρ, as Eq. (20) shows. Therefore, forms (f) and (i) should be equal for $r = \rho$:

(j) $$rI_k(\kappa r)K_k(\kappa r)+BrK_k(\kappa r)=rI_k(\kappa r)K_k(\kappa r).$$

This is only possible when B = 0. Hence,

(k) $$g_k(r,\rho)=\begin{cases}\rho I_k(\kappa r)K_k(\kappa\rho) & r\le\rho,\\ \rho I_k(\kappa\rho)K_k(\kappa r) & r\ge\rho,\end{cases}$$

what can be rewritten as Eq. (21).

To determine h_k, Eq. (8) can be used.

(l) $$\frac{h_k(r,\rho)}{\rho}=\kappa I_k'\left[\min(r,\rho)\right]\frac{\partial\min(r,\rho)}{\partial\rho}K_k[\max(r,\rho)]+ + \kappa I_k\left[\min(r,\rho)\right]K_k'[\max(r,\rho)]\frac{\partial\max(r,\rho)}{\partial\rho}.$$

Observe that:

(m) $$\min(r,\rho)=\frac{r+\rho-|r-\rho|}{2},\quad \max(r,\rho)=\frac{r+\rho+|r-\rho|}{2}.$$

Hence,

(n) $$\frac{h_k(r,\rho)}{\rho}=\kappa\frac{1+\mathrm{sgn}(r-\rho)}{2}I_k'\left[\min(r,\rho)\right]K_k[\max(r,\rho)]+ +\kappa\frac{1-\mathrm{sgn}(r-\rho)}{2}I_k\left[\min(r,\rho)\right]K_k'[\max(r,\rho)],$$

what can be rewritten as Eq. (22).

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