Tadeusz KACZOREK
Bialystok University of Technology

doi:10.15199/48.2017.02.61

Determinants of the matrices of solutions to the standard and positive linear electrical circuits

***Abstract**. Determinants of the matrices of solutions to the standard and positive time-invariant and time-varying linear electrical circuits are addressed. Necessary and sufficient conditions for the positivity and asymptotic stability of the linear time-varying electrical circuits are established. It is shown that the determinants of the matrices of solutions to the standard and positive linear electrical circuits are nonzero and they decrease to zero for time tending to infinity if the electrical circuit contains at least one resistance and tends to 1 if the electrical circuit does not contain resistances. Determinants of the matrices of solutions of asymptotically stable electrical circuits tend to zero for time tending to infinity.*

***Streszczenie.** Praca jest poświęcona analizie wyznaczników macierzy rozwiązań standardowych i dodatnich obwodów elektrycznych o stałych i zmiennych w czasie parametrach. Podano warunki konieczne i wystarczające dodatniości i stabilności asymptotycznej obwodów liniowych o zmiennych w czasie parametrach. Wykazano, że wyznaczniki macierzy rozwiązań standardowych i dodatnich obwodów o stałych i zmiennych w czasie parametrach są niezerowe i dążą do zera dla czasu dążącego do nieskończoności, gdy obwód elektryczny zawiera przynajmniej jedną rezystancję i dąży do 1, gdy obwód nie zawiera rezystancji. Wyznaczniki macierzy rozwiązań stabilnych asymptotycznie obwodów elektrycznych dążą do zera dla czasu dążącego do nieksończoności. (**Wyznaczniki macierzy rozwiązań standardowych i dodatnich liniowych obwodów elektrycznych**)*

Keywords: determinant of solution matrix, standard and positive, time-invariant and time-varying, electrical circuit, stability.
Słowa kluczowe: wyznacznik macierzy rozwiązań, standardowy i dodatni, o parametrach stałych i zmiennych w czasie, obwód elektryczny, stabilność.

Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of the state of the art in positive systems theory is given in the monographs [7, 15]. Variety of models having positive behaviour can be found in engineering, economics, social sciences, biology, medicine, etc.

The Lyapunov, Bohl and Perron exponents and stability of time-varying discrete-time linear systems have been investigated in [1-6]. The positivity and stability of time-varying linear systems have been addressed in [12, 16, 18, 20, 22, 23, 28] and the stability of continuous-time linear systems with delays in [26]. The fractional positive linear systems have been analyzed in [10, 11, 13, 19, 21, 24, 25, 29]. The positive electrical circuits and their reachability have been considered in [17] and the controllability and observability in [9]. The stability and stabilization of positive fractional linear systems by state-feedbacks have been analyzed in [25]. The normal positive electrical circuits has been introduced in [14].

In this paper the determinants of the matrices of solutions to the standard and positive electrical circuits will be addressed.

The paper is organized as follows. In section 2 some preliminaries on linear time-varying systems and the Jacobi equality are recalled. The determinants of the matrix solution of standard time-invariant linear electrical circuits are investigated in section 3. Similarly, problems for positive electrical circuits are analyzed in section 4. The standard and positive time-varying linear electrical circuits are addressed in section 5. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix.

Preliminaries

Consider the linear time-varying autonomous system

(1) $$\dot{x} = A(t)x\ ,\ \dot{x} = \frac{dx}{dt},\ x_0 = x(t_0)\ ,$$

where $x = x(t) \in \Re^n$, the entries $a_{ij}(t)$, $i, j = 1,...,n$ of the matrix $A(t) = [a_{ij}(t)] \in \Re^{n\times n}$ are continuous functions of time t.

Let $x_k(t)$, $k = 1,...,n$ be the solution of the equation (1). Then the matrix $X(t) = [x_1(t) \ \cdots \ x_n(t)] \in \Re^{n\times n}$ is also the solution of the equation (1) and

(2) $$\dot{X}(t) = A(t)X(t)\ .$$

The general solution of the equation (2) for any given initial condition matrix $C = X(t_0)$ has the form $X(t)C$. It is well-known [8] that the determinant $|X(t)| = \det X(t)$ of the solution matrix $X(t)$ satisfies the Jacobi equality

(3) $$\det X(t) = ce^{\int_0^t \operatorname{tr} A(\tau)d\tau}\ ,$$

where $\operatorname{tr} A(t) = \sum_{i=1}^n a_{ii}(t)$ is the trace of the matrix $A(t)$ and c is a constant.

From (3) we have the following conclusion.

Conclusion 1. The matrix $X(t)$ for any time t is the nonsingular matrix, i.e.

(4) $$\det X(t) \neq 0 \text{ for any } t \in [t_0, +\infty)\ .$$

In particular case if $A \in \Re^{n\times n}$ is a constant matrix (independent of time t), then the general solution of the equation

(5) $$\dot{X}(t) = AX(t)\ ,\ A = [a_{ij}] \in \Re^{n\times n}\ ,\ i, j = 1,...,n$$

has the form

(6) $X(t) = e^{At}C \in \Re^{n\times n}$,

where $C = X(0)$ is a matrix of initial conditions.
In this case the Jacobi equality has the form

(7) $\det e^{At} = ce^{(\mathrm{tr}\, A)t}$, $\mathrm{tr}\, A = \sum_{i=1}^{n} a_{ii}$,

where c is a constant.

Theorem 1. The determinant of the matrix $X(t)$ of the system (2) satisfies the condition

(8) $\lim_{t\to\infty} \det X(t) = 0$,

if

(9) $\int_0^t \mathrm{tr}\, A(\tau)d\tau < 0$ for $t \in [0,+\infty)$.

Proof. The proof follows immediately from (3) since

(10) $\lim_{t\to\infty} e^{\int_0^t \mathrm{tr}\, A(\tau)d\tau} = 0$

if (9) is satisfied. □

In particular case for the system (5) we have the following theorem.

Theorem 2. If the system (5) is asymptotically stable then

(11) $\lim_{t\to\infty} \det X(t) = 0$.

Proof. If the system (5) is asymptotically stable then the eigenvalues s_k of the matrix A satisfies $\mathrm{Re}\, s_k < 0$, $k = 1,...,n$ and

(12) $\mathrm{tr}\, A = \sum_{k=1}^{n} s_k < 0$

and this implies (11). □

Standard time-invariant linear electrical circuits

We start the analysis of standard time-invariant linear electrical circuits with simple examples.

Example 1. Consider the electrical circuit shown in Fig. 1 with given resistance R, inductance L, capacitance C and source voltage e.

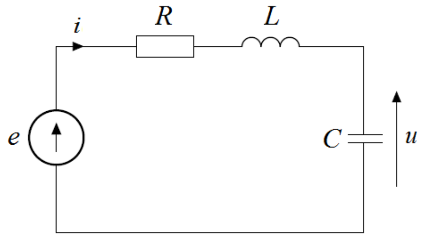

Fig.1. Electrical circuit of Example 1.

The electrical circuit is described by the equation

(13a) $$\frac{d}{dt}\begin{bmatrix} u \\ i \end{bmatrix} = A\begin{bmatrix} u \\ i \end{bmatrix} + Be,$$

where

(13b) $$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix}.$$

If $R > 0$ and $L > 0$, $C > 0$ then from (7) we have

(14) $$\lim_{t\to\infty} \det e^{At} = \lim_{t\to\infty} e^{-\frac{R}{L}t} = 0$$

since $\mathrm{tr}\, A = -\frac{R}{L}$ and

(15) $$\lim_{t\to\infty} \det e^{At} = 1$$

for $R = 0$ and $L > 0$, $C > 0$.

Therefore, the $\det e^{At}$ of the electrical circuit decreases to zero if $R > 0$ and is equal to 1 if $R = 0$.

Example 2. Consider the linear electrical circuit shown in Fig. 2 with given resistances R_1, R_2, R_3, inductances L_1, L_2, capacitances C_1, C_2 and source voltages e_1, e_2.

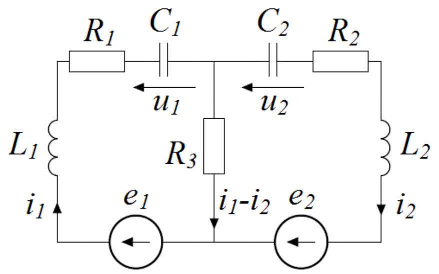

Fig.2. Electrical circuit of Example 2.

Using Kirchhoff's laws we may write for the electrical circuit the equations

(16) $$\begin{aligned} e_1 &= (R_1 + R_3)i_1 + L_1\frac{di_1}{dt} + u_1 - R_3 i_2, \\ e_2 &= (R_2 + R_3)i_2 + L_2\frac{di_2}{dt} + u_2 - R_3 i_1, \\ i_1 &= C_1\frac{du_1}{dt}, \; i_2 = C_2\frac{du_2}{dt}. \end{aligned}$$

The equations (16) can be written in the form

(17a) $$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} u_1 \\ u_2 \\ i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where

(17b) $$A = \begin{bmatrix} 0 & 0 & \frac{1}{C_1} & 0 \\ 0 & 0 & 0 & \frac{1}{C_2} \\ -\frac{1}{L_1} & 0 & -\frac{R_1+R_3}{L_1} & \frac{R_3}{L_1} \\ 0 & -\frac{1}{L_2} & \frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix},$$
$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}.$$

From (17b) we have

(18) $\operatorname{tr} A = -\frac{R_1 + R_3}{L_1} - \frac{R_2 + R_3}{L_2}$

and from (7)

(19) $\lim_{t \to \infty} \det e^{At} = 0$

if at least one of the resistances R_1, R_2, R_3 is nonzero. Note that

(20) $\lim_{t \to \infty} \det e^{At} = 1$

if and only if all resistances R_1, R_2, R_3 are zero.

In general case we have the following theorem.

Theorem 3. If the electrical circuit contains at least one resistance then

(21) $\lim_{t \to \infty} \det e^{At} = 0$

and

(22) $\lim_{t \to \infty} \det e^{At} = 1$

if the electrical circuit contains only inductances and capacitances.

Proof. From equations written by the one of Kirchhoff's laws it follows that the system matrix A of the electrical circuit has at least one negative entry on the main diagonal and $\operatorname{tr} A < 0$ and this implies (21). If the electrical circuit contains only inductances and capacitances then $\operatorname{tr} A = 0$ and by (3) we obtain (22). □

Theorem 4. Asymptotically stable electrical circuit consisting of resistances, inductances and capacitances satisfies the condition (21).

Proof. The proof follows immediately from Theorem 2.

Example 3. Consider the electrical circuit shown in Fig. 3 with given resistances R_1, R_2, inductances L_1, L_2 and source voltages e_1, e_2.

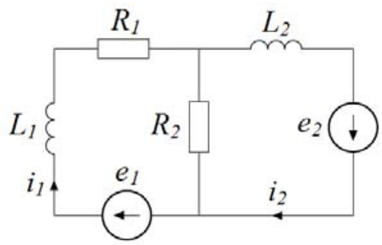

Fig.3. Electrical circuit of Example 3.

Using Kirchhoff's laws we can write the equations

(23)
$$e_1 = (R_1 + R_2)i_1 - R_2 i_2 + L_1 \frac{di_1}{dt},$$
$$e_2 = -R_2 i_1 + R_2 i_2 + L_2 \frac{di_2}{dt}.$$

The equations (23) can be written in the form

(24a) $$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where

(24b) $$A = \begin{bmatrix} -\frac{R_1 + R_2}{L_1} & \frac{R_2}{L_1} \\ \frac{R_2}{L_2} & -\frac{R_2}{L_2} \end{bmatrix}, \; B = \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}.$$

Note that

(25) $$\det A = \begin{vmatrix} -\frac{R_1 + R_2}{L_1} & \frac{R_2}{L_1} \\ \frac{R_2}{L_2} & -\frac{R_2}{L_2} \end{vmatrix} = \frac{R_1 R_2}{L_1 L_2} = 0$$

if at least one of the resistances R_1, R_2 is zero.

Therefore, the electrical circuit is unstable if $R_1 = 0$ or $R_2 = 0$. Let assume that $R_1 \neq 0$ and $R_2 = 0$. Then

(26) $$\det[I_2 s - A] = \begin{vmatrix} s + \frac{R_1}{L_1} & 0 \\ 0 & s \end{vmatrix} = s\left(s + \frac{R_1}{L_1}\right) = 0$$

gives $s_1 = -\frac{R_1}{L_1}$, $s_2 = 0$ and

(27) $$\lim_{t \to \infty} \det e^{At} = \lim_{t \to \infty} \det \begin{bmatrix} e^{-\frac{R_1}{L_1}t} & 0 \\ 0 & 1 \end{bmatrix} = \lim_{t \to \infty} e^{-\frac{R_1}{L_1}t} = 0.$$

Therefore, the electrical circuit for $R_1 \neq 0$ and $R_2 = 0$ is unstable but the condition (21) is satisfied.

Positive time-invariant linear electrical circuits

Consider electrical circuits composed of resistances, inductances, capacitances and voltage (current) sources. As the state variables the components of the state vector $x(t)$ we choose the voltages on the capacitors and the currents in the coils. Using Kirchhoff's laws we may describe the linear electrical circuits by the equations [29]

(28a) $\dot{x}(t) = Ax(t) + Bu(t)$,

(28b) $y(t) = Cx(t)$,

where $x(t) \in \Re^n$, $u(t) \in \Re^m$ and $y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n \times n}$, $B \in \Re^{n \times m}$, $C \in \Re^{p \times n}$.

Definition 1. [29] The electrical circuit (28) is called (internally) positive if $x(t) \in \Re_+^n$, $y(t) \in \Re_+^p$, $t \in [0, +\infty)$ for all $x_0 = x(0) \in \Re_+^n$ and $u(t) \in \Re_+^m$, $t \in [0, +\infty)$.

Theorem 5. [29] The electrical circuit (28) is positive if and only if

(29) $A \in M_n$, $B \in \Re_+^{n \times m}$, $C \in \Re_+^{p \times n}$.

Definition 2. [29] The positive electrical circuit (28) is called asymptotically stable if

(30) $\lim_{t \to \infty} x(t) = 0$ for $x_0 \in \Re_+^n$.

Theorem 6. [29] The positive electrical circuit is asymptotically stable if and only if

(31) $\operatorname{Re} \lambda_k < 0$ for $k = 1, ..., n$,

where λ_k is the eigenvalue of the matrix $A \in M_n$.

Note that the electrical circuits shown in Fig. 1 and Fig. 2 are not positive since the matrices A defined by (13b) and (17b) have negative off-diagonal entries for all nonzero values of the parameters R, L, C.

The electrical circuit shown in Fig. 3 is positive since the matrices defined by (24b) satisfy the condition (29) for all values of the resistances R_1, R_2 and nonzero values of the inductances L_1, L_2.

Example 4. [29] Consider the electrical circuit shown in Fig. 4 with given resistances R_k, $k=1,...,n$, inductances L_2, L_4, …, L_{n_2}, capacitances C_1, C_3, …, C_{n_1} and source voltages e_0, e_2, e_4, …, e_{n_2}, $n=n_1+n_2$.

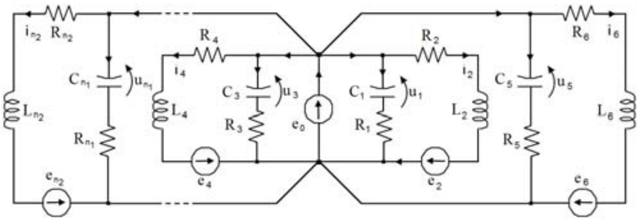

Fig.4. Electrical circuit of Example 4.

Using Kirchhoff's laws we may write the equations

$$\begin{aligned} & e_0 = R_k C_k \frac{du_k}{dt} + u_k,\ k=1,3,...,n_1, \\ & e_0 + e_j = L_j \frac{di_j}{dt} + R_j i_j,\ j=2,4,...,n_2. \end{aligned} \tag{32}$$

The equations (32) can be written in the form

$$\frac{d}{dt}\begin{bmatrix} u \\ i \end{bmatrix} = A\begin{bmatrix} u \\ i \end{bmatrix} + Be, \tag{33a}$$

where

$$u = \begin{bmatrix} u_1 \\ u_3 \\ \vdots \\ u_{n_1} \end{bmatrix},\ i = \begin{bmatrix} i_2 \\ i_4 \\ \vdots \\ i_{n_2} \end{bmatrix},\ e = \begin{bmatrix} e_0 \\ e_2 \\ e_4 \\ \vdots \\ e_{n_2} \end{bmatrix}, \tag{33b}$$

and

$$A = \mathrm{diag}\left[-\frac{1}{R_1C_1}\quad -\frac{1}{R_3C_3}\quad \cdots \quad -\frac{1}{R_{n_1}C_{n_1}}\quad -\frac{R_2}{L_2}\quad -\frac{R_4}{L_4}\quad \cdots \quad -\frac{R_{n_2}}{L_{n_2}}\right] \in M_{n_2},$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 & \cdots & 0 \\ \frac{1}{R_3C_3} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ \frac{1}{R_{n_1}C_{n_1}} & 0 & 0 & \cdots & 0 \end{bmatrix} \in \Re_+^{n_1\times m},$$

$$B_2 = \begin{bmatrix} \frac{1}{L_2} & \frac{1}{L_2} & 0 & \cdots & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{L_{n_2}} & 0 & 0 & \cdots & \frac{1}{L_{n_2}} \end{bmatrix} \in \Re_+^{n_2\times m}. \tag{33c}$$

From (33) it follows that the electrical circuit is positive for all nonzero values of R_k, $k=1,2,...,n$, inductances L_j, $j=2,4,...,n_2$ and capacitances C_l, $l=1,3,...,n_1$. Note that the positive electrical circuit is unstable if at least one of the resistances R_2, R_4, …, R_{n_2} is zero.

Theorem 7. If at least one of the entry $\frac{1}{R_jC_j}$, $j=1,3,...,n_1$ or $\frac{R_k}{L_k}$, $k=2,4,...,n_2$ of the matrix A of the positive electrical circuit shown in Fig. 4 is nonzero then

$$\lim_{t\to\infty} \det e^{At} = 0. \tag{34}$$

Proof. If at least one diagonal entry of the matrix A is nonzero then the $\mathrm{tr}\,A < 0$ and the condition (34) is satisfied. □

Remark 1. Theorem 7 is valid for the asymptotically stable and unstable positive electrical circuit.

Standard and positive time-varying linear electrical circuits

Consider time-varying linear electrical circuits composed of resistances, inductances, capacitances depending on time t and voltage (current) sources. Similarly, as for time-invariant linear electrical circuits we choose as the state variables the voltages on the capacitors and the currents in the coils. Using Kirchhoff's laws we may describe the electrical circuits by the equations

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{35a}$$

$$y(t) = C(t)x(t), \tag{35b}$$

where

where $x(t)\in\Re^n$, $u(t)\in\Re^m$ and $y(t)\in\Re^p$ are the state, input and output vectors and $A(t)\in\Re^{n\times n}$, $B(t)\in\Re^{n\times m}$, $C(t)\in\Re^{p\times n}$ are continuous-time matrices.

Theorem 8. [20] The solution of the equation (35a) for given initial condition $x(t_0)\in\Re^n$ and input $u(t)\in\Re^m$ has the form

$$x(t) = \Omega_{t_0}^t(A)x_0 + \int_{t_0}^t K(t,\tau)B(\tau)u(\tau)d\tau, \tag{36a}$$

where

$$\begin{aligned} & K(t,\tau) = \Omega_{t_0}^t(A)[\Omega_{t_0}^\tau(A)]^{-1}, \\ & \Omega_{t_0}^t(A) = I_n + \int_{t_0}^t A(\tau)d\tau + \int_{t_0}^t A(\tau)\int_{t_0}^\tau A(\tau_1)d\tau_1 d\tau + ... \end{aligned} \tag{36b}$$

Definition 3. The electrical circuit described by (35) is called (internally) positive if $x(t)\in\Re_+^n$, $y(t)\in\Re_+^p$, $t\in[t_0,+\infty)$ for any initial conditions $x_0\in\Re_+^n$ and all inputs $u(t)\in\Re_+^m$, $t\in[t_0,+\infty)$.

Theorem 9. Let $A(t)\in\Re^{n\times n}$, $t\in[t_0,+\infty)$. Then $\Omega_{t_0}^t(A)\in\Re_+^{n\times n}$, $t\ge t_0$ if and only if $A(t)\in M_n$, $t\in[t_0,+\infty)$.

Proof. The proof is given in [20].

Theorem 10. The electrical circuit described by (35) is positive if and only if

(37) $A(t) \in M_n$, $B(t) \in \Re_+^{n \times m}$, $C(t) \in \Re_+^{p \times n}$, $t \in [t_0, +\infty)$.

Proof. The proof is given in [20].

Definition 4. The positive electrical circuit described by the equation

(38) $\dot{x}(t) = A(t)x(t)$, $A(t) \in M_n$

is called asymptotically stable if

(39) $\lim_{t \to \infty} x(t) = 0$ for all $x_0 \in \Re_+^n$.

Theorem 11. The positive electrical circuit described by (37) is asymptotically stable if and only if one of the conditions is satisfied:

1) All coefficients of the polynomial

(40) $$\det[I_n s - A] = s^n + a_{n-1}(t)s^{n-1} + ... + a_1(t)s + a_0(t)$$

are positive, i.e. $a_k(t) > 0$ for $k = 0,1,...,n-1$ and $t \in [t_0, +\infty)$.

2) All leading principal minors $M_k(t)$, $k = 1,...,n$ of the matrix $-A(t)$ are positive, i.e.

(41) $$M_1(t) = -a_{11}(t) > 0,\ M_2(t) = \begin{vmatrix} -a_{11}(t) & -a_{12}(t) \\ -a_{21}(t) & -a_{22}(t) \end{vmatrix} > 0,\ ...,\ M_n(t) = \det[-A(t)] > 0.$$

Example 5. Consider the time-varying linear electrical circuit shown in Fig. 5 with given nonzero resistances $R_1(t)$, $R_2(t)$, $R_3(t)$, inductances $L_1(t)$, $L_2(t)$ and source voltages $e_1(t)$, $e_2(t)$.

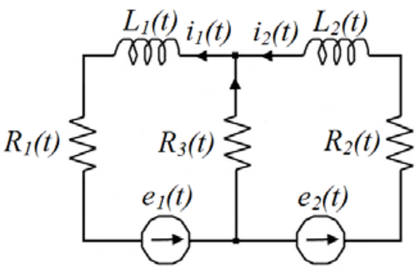

Fig.5. Electrical circuit of Example 5.

Using Kirchhoff's laws we may write the equations

(42)
$$e_1(t) = \left[R_1(t) + \frac{dL_1(t)}{dt}\right]i_1(t) + L_1\frac{di_1(t)}{dt} + R_3(t)[i_1(t) - i_2(t)],$$
$$e_2(t) = \left[R_2(t) + \frac{dL_2(t)}{dt}\right]i_2(t) + L_2\frac{di_2(t)}{dt} + R_3(t)[i_2(t) - i_1(t)].$$

The equations (42) can be written in the form

(43a) $$\frac{d}{dt}\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} = A(t)\begin{bmatrix} i_1(t) \\ i_2(t) \end{bmatrix} + B(t)\begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix},$$

where

(43b)
$$A(t) = \begin{bmatrix} -\dfrac{R_1(t) + R_3(t) + \dfrac{dL_1(t)}{dt}}{L_1(t)} & \dfrac{R_3(t)}{L_1(t)} \\ \dfrac{R_3(t)}{L_2(t)} & -\dfrac{R_2(t) + R_3(t) + \dfrac{dL_2(t)}{dt}}{L_2(t)} \end{bmatrix}$$
$$B(t) = \begin{bmatrix} \dfrac{1}{L_1(t)} & 0 \\ 0 & \dfrac{1}{L_2(t)} \end{bmatrix}.$$

From (43b) it follows that $A(t) \in M_2$ and $B(t) \in \Re_+^{2 \times 2}$, $t \in [t_0, +\infty)$ if $R_k(t) > 0$, $k = 1,2,3$, $L_1(t) > 0$, $L_2(t) > 0$, $t \in [t_0, +\infty)$ and the electrical circuit is a positive time-varying one.

Using (43b) we obtain

(44)
$$\det[I_2 s - A(t)]$$
$$= \begin{bmatrix} s + \dfrac{R_1(t) + R_3(t) + \dfrac{dL_1(t)}{dt}}{L_1(t)} & -\dfrac{R_3(t)}{L_1(t)} \\ -\dfrac{R_3(t)}{L_2(t)} & s + \dfrac{R_2(t) + R_3(t) + \dfrac{dL_2(t)}{dt}}{L_2(t)} \end{bmatrix}$$
$$= s^2 + \left[\frac{R_1(t) + R_3(t) + \dfrac{dL_1(t)}{dt}}{L_1(t)} + \frac{R_2(t) + R_3(t) + \dfrac{dL_2(t)}{dt}}{L_2(t)}\right]s$$
$$+ \frac{\left[R_1(t) + R_3(t) + \dfrac{dL_1(t)}{dt}\right]\left[R_2(t) + R_3(t) + \dfrac{dL_2(t)}{dt}\right] - R_3^2(t)}{L_1(t)L_2(t)}.$$

From (44) by Theorem 11 it follows that the positive time-varying electrical circuit is asymptotically stable if

(45) $$R_k(t) + \frac{dL_k(t)}{dt} > 0 \text{ for } k = 1,2.$$

Note that the trace of the matrix $A(t)$ defined by (43b) is

(46)
$$\text{tr}\, A(t) = -\left[\frac{R_1(t) + R_3(t) + \dfrac{dL_1(t)}{dt}}{L_1(t)} + \frac{R_2(t) + R_3(t) + \dfrac{dL_2(t)}{dt}}{L_2(t)}\right] < 0$$

for $t \in [t_0, +\infty)$

if the condition (45) is satisfied.

From the Jacobi equality (3) we have

(47) $$\lim_{t \to \infty} \det \Omega_{t_0}^t(A) = 0$$

if (46) holds.

Note that the asymptotic stability condition (45) of the positive electrical circuit is stronger than the condition (46) for (47).

In general case we have the following theorem.

Theorem 12. If the positive time-varying linear electrical circuit is asymptotically stable then (47) holds.

Proof. Note that by Theorem 11 if the positive electrical circuit is asymptotically stable then

$$a_{n-1}(t) = -\sum_{k=1}^{n} s_k(t) = \operatorname{tr} A(t) < 0 \tag{48}$$

and by Jacobi equality (3) the condition (47) is satisfied. □

Remark 2. Note that the condition (47) can be also satisfied if the positive electrical circuit is unstable, for example if $a_0(t) = \det A(t) = 0$.

Concluding remarks

The determinants of the matrices of solutions to the standard and positive time-invariant and time-varying linear electrical circuits have been addressed. Necessary and sufficient conditions for the positivity (Theorem 10) and stability (Theorem 11) of time-varying linear electrical circuits have been presented. It has been shown that the determinants of the matrices of solutions to the standard time-invariant (Theorem 3) and to the positive (Theorem 12) time-varying linear electrical circuits are nonzero and they decrease to zero for time tending to infinity for asymptotically stable electrical circuits (Theorems 4 and 12). The considerations have been illustrated by examples of standard and positive time-invariant and time-varying linear systems. The considerations can be extended to fractional linear electrical circuits.

***Authors**: prof. dr hab. inż. Tadeusz Kaczorek, Bialystok University of Technology, Faculty of Electrical Engineering, Wiejska 45D, 15-351 Bialystok, e-mail: kaczorek@ee.pw.edu.pl.*

REFERENCES

[1] Czornik A., Newrat A., Niezabitowski M., Szyda A., On the Lyapunov and Bohl exponent of time-varying discrete linear systems, 20th Mediterranean Conf. on Control and Automation (MED), Barcelona, (2012), 194-197
[2] Czornik A., Klamka J., Niezabitowski M., On the set of Perron exponents of discrete linear systems, Proc. of World Congress of the 19th International Federation of Automatic Control, Kapsztad, (2014), 11740-11742
[3] Czornik A., Newrat A., Niezabitowski M., On the Lyapunov exponents of a class of the second order discrete time linear systems with bounded perturbations, Dynamical Systems: an International Journal, 28 (2013), No. 4, 473-483
[4] Czornik A., Niezabitowski M., Lyapunov Exponents for Systems with Unbounded Coefficients, Dynamical Systems: an International Journal, 28 (2013), No. 2, 140-153
[5] Czornik A., Niezabitowski M., On the stability of discrete time-varying linear systems, Nonlinear Analysis: Hybrid Systems, (2013), No. 9, 27-41
[6] Czornik A., Niezabitowski M., On the stability of Lyapunov exponents of discrete linear system, Proc. of European Control Conf., Zurich, (2013), 2210-2213
[7] Farina L., Rinaldi S., Positive Linear Systems; Theory and Applications, J. Wiley, New York, 2000
[8] Gantmacher F.R., The Theory of Matrices, Chelsea Pub. Comp., London, 1959.
[9] Kaczorek T., Controllability and observability of linear electrical circuits, *Electrical Review*, 87 (2011), No. 9a, 248-254
[10] Kaczorek T., Minimum energy control of fractional positive electrical circuits, *Archives of Electrical Engineering*, 65 (2016), No. 2, 191-201
[11] Kaczorek T., Fractional positive continuous-time linear systems and their reachability, *Int. J. Appl. Math. Comput. Sci.*, 18 (2008), No. 2, 223-228
[12] Kaczorek T., Fractional standard and positive descriptor time-varying discrete-time linear systems, *Progress in Automation, Robotics and Measuring Techniques*, Advances in Intelligent Systems and Computing, 350 (2015), 101-112
[13] Kaczorek T., New stability test of positive standard and fractional linear systems, *Circuits and Systems*, (2011), No. 2, 261-268
[14] Kaczorek T., Normal positive electrical circuits, *IET Control Theory and Applications*, doi: 10.1049/iet-cta.2014.0183
[15] Kaczorek T., Positive 1D and 2D Systems, Springer-Verlag, London, 2002
[16] Kaczorek T., Positive and stable time-varying continuous-time linear systems and electrical circuits*, Poznan University of Technology Academic Journals: Electrical Engineering*, (2015), No. 81, 11-19
[17] Kaczorek T., Positive electrical circuits and their reachability, *Archives of Electrical Engineering*, 60 (2011), No. 3, 283-301 and also Selected classes of positive electrical circuits and their reachability, Monograph Computer Application in Electrical Engineering, Poznan University of Technology, Poznan, 2012.
[18] Kaczorek T., Positive descriptor time-varying discrete-time linear systems and their asymptotic stability, *TransNav*, 9 (2015), No. 1, 83-89
[19] Kaczorek T., Positive linear systems consisting of n subsystems with different fractional orders*, IEEE Trans. Circuits and Systems*, 58 (2011), No. 6, 1203-1210
[20] Kaczorek T., Positive time-varying continuous-time linear systems and electrical circuits, *Bull. Pol. Acad. Sci. Techn.*, 63 (2015), No. 4, 837-842
[21] Kaczorek T., Positivity and reachability of fractional electrical circuits, *Acta Mechanica et Automatica*, 5 (2011), No. 2, 42-51.
[22] Kaczorek T., Positivity and stability of fractional descriptor time-varying discrete-time linear systems, *Int. J. Appl. Math. Comput. Sci.*, 26 (2016), No. 1, 6-13
[23] Kaczorek T., Positivity and stability of time-varying discrete-time linear systems, *Intelligent Information and Database Systems*, Lecture Notes in Computer Science, 9011 (2015), 295-303
[24] Kaczorek T., Selected Problems of Fractional Systems Theory, Springer-Verlag, 2011.
[25] Kaczorek T., Stability and stabilization of positive fractional linear systems by state-feedbacks, *Bull. Pol. Acad. Sci. Techn.*, 58 (2010), No. 4, 517-554
[26] Kaczorek T., Stability of positive continuous-time linear systems with delays, *Bull. Pol. Acad. Sci. Techn.*, 57 (2009), No. 4, 395-398
[27] Kaczorek T., Vectors and Matrices in Automation and Electrotechnics, WNT, Warsaw, 1998 (in Polish)
[28] Kaczorek T., Borawski K., Positivity and stability of time-varying fractional discrete-time linear systems, *Measurement Automation Monitoring*, 61 (2015), No. 3, 84-87
[29] Kaczorek T., Rogowski K., Fractional Linear Systems and Electrical Circuits, *Studies in Systems, Decision and Control*, 13 (2015), Springer