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Algorithm reducing influence of frequency fluctuations and a noise on undersampling of periodical signals

Streszczenie. *Zaprezentowano nowy algorytm pomiaru sygnałów okresowych, wykorzystujący podpróbkowanie Σ oraz metodę kompensacji błędów pomiarowych, spowodowanych przez wahania częstotliwości za pomocą podpróbkowania Σ - Δ. Wyznaczono charakterystyki filtrów cyfrowych, realizujących ten algorytm. Przeanalizowano zależność błędów pomiarowych od krotności podpróbkowania, liczby próbek, błędów częstotliwości oraz mocy zakłóceń na przykładzie typowych przebiegów okresowych.* ***Nowy algorytm pomiaru sygnałów okresowych, wykorzystujący podpróbkowanie Σ oraz metodę kompensacji błędów pomiarowych***

Abstract. *New algorithm of the measurement of periodical signals, utilizing the sigma undersampling and a method of a compensation of the errors arising from frequency fluctuations basing on Σ - Δ undersampling are presented. Characteristics of FIR filters realizing this algorithm are calculated. An analysis of the dependence of the errors on the undersampling factor, the number of the samples per period, a power of the noise and fluctuations of the frequency is performed for typical periodical signals.*

Keywords: periodical signal, sigma undersampling, frequency fluctuations, Gaussian noise;
Słowa kluczowe: sygnał okresowy, podpróbkowanie sigma, szumy Gaussowskie

Introduction

Present-day digital signal processors (DSP) enable their applications in wider range of measurements. However in very high frequency applications the undersampling, i.e. the sampling of the measured signal using the sampling frequency lower than Nyquist's frequency, can be more profitable than the standard sampling [1-5]. Especially, the sigma undersampling, when the measured signal is integrated before the sampling, can reduce an influence of a noise for measurements of small signals. It is obvious, that a degree of this reduction increases proportionally to the extension of the time of the integration on the condition, that expected value of the noise equals 0.

The method reducing the influence of the fluctuations of the frequency basing on sigma undersampling was worked out earlier [6-7]. The significant reduction of the errors was achieved however the digital data processing system was characterized by one inconvenience - a nonlinear element, calculating the geometrical mean value of DFTs of particular series of the results of the measurement. An elimination of this attribute and a creation of the less complicated algorithm of the signal processing is a purpose of this work.

Sigma undersampling

Every periodical signal $x_0(t)$ of a frequency f_0 can be expressed in the form of Fourier's series.

$$x_0(t) = X_0 + \sum_{m=1}^{\infty} X_m \cdot \cos(2\pi m f_0 t + \varphi_m), \tag{1}$$

where X_0 is the mean value of the signal and X_m and φ_m are the amplitude and the phase of its m^{th} harmonic, respectively.

To apply the sigma undersampling, the signal must be integrated during the sampling period T_{d0}, according to following condition [1-5]

$$T_{d0} = f_0^{-1} \cdot \left(M + N^1\right), \tag{2}$$

where M is the undersampling factor and N - the number of samples per period.

When the measured signal is integrated, the spectrum of the output signal of the integrator $y(t)$ is similar to the spectrum of the signal $x_0(t)$ [6-7].

$$\begin{aligned} y(t) &= \int_{t-T_{d0}/2}^{t+T_{d0}/2} x(t) \cdot dt = \frac{X_0(MN+1)}{N \cdot f_0} + \\ &+ \frac{f_0^{-1}}{\pi} \sum_{m=1}^{\infty} \frac{X_m}{m} \sin\left(\frac{m\pi}{N} + mM\pi\right) \cos(2\pi m f_0 t + \varphi_m) \end{aligned} \tag{3}$$

A comparison of (1) and (3) leads to the conclusion, that the samples of the original signal $x_0(k)$ are strictly connected with the values of the integrals $y(k)$. Therefore they can be obtained at the output of the NOI filter, which transfer function $H_\Sigma(f)$ satisfies the following condition [6-7]

$$X_0(mf_0) = H_\Sigma(mf_0) \cdot Y(mf_0), \tag{4}$$

where $X_0(f)$ and $Y(f)$ are the spectra of the signals $x_0(t)$ and $y(t)$, respectively.

$$H_\Sigma(z) = \sum_{n=0}^{N-1} h_\Sigma(n) \cdot z^{-n}, \tag{5}$$

The coefficients $h_\Sigma(k)$ can be calculated by means of Inverse Discrete Fourier's Transform of the transfer function of the filter.

$$\begin{aligned} h_\Sigma(n) &= \frac{Nf_0}{MN+1} + (-1)^k \frac{\pi N f_0}{2} + \\ &+ 2\pi f_0 \cdot \sum_{m=1}^{N/2-1} \frac{m \cos\dfrac{2 \cdot m \cdot n \cdot \pi}{N}}{\sin\left(mM\pi + \dfrac{m\pi}{N}\right)} \end{aligned} \tag{6}$$

When the frequencies of the signal and the sampling generator fluctuate, the signal $x_0(t)$ is sampled in different phases in the subsequent periods. Therefore the ideal values of the sampling period T_{d0} and the frequency f_0 are replaced by the real values T_d and f [6-8].

$$f = f_0 \cdot (1 + \delta f), \tag{7}$$

$$T_d = T_{d0} \cdot (1 + \delta T_d). \tag{8}$$

δf and δT_d are the relative variations of the sampling period and the frequency of the signal, respectively.

After the undersampling and the filtering the value of k^{th} sample of the measured signal can be expressed as

$$x(k) = X_0 + \sum_{m=1}^{\infty} X_m \cos\left[mk\left(\frac{2\pi}{N} + (MN+1)\Delta\varphi_1\right) + \varphi_m\right], \tag{9}$$

where

$$\Delta\varphi_1 = \frac{2\pi}{N}\left(\delta f + \delta T_d + \delta f \cdot \delta T_d\right). \tag{10}$$

$\Delta\varphi_1$ is the difference between the phases of 1[st] harmonics of the ideal signal $x_0(t)$ and the real signal $x(t)$.

The inaccuracy of the sigma undersampling arises from these phase errors. It is obvious that the errors of the measurement of the signal increase for higher numbers of samples. This effect can be limited, when the undersampling is repeated multiply and when it starts from different phases of the measured signal for every set of N samples. One should also notice, that the phase errors for the undersampling are ($N{\cdot}M$+1) times greater than the errors during the usual sampling. The formula (10) makes possible to estimate the ranges of N and M, when the undersampling gives acceptable results. For typical frequency stability in range of 10^{-6} the phase error for N-th sample of the 1st harmonic should be less than 1º. This condition is fulfilled when $N{\cdot}M$< 1400

Using the trigonometric formula (9) can be expressed as

(11)
$$x(k)=X_0+\sum_{m=1}^{\infty}X_m\left[\cos\left(\frac{2\pi mk}{N}+\varphi_m\right)\cos mk(MN+1)\Delta\varphi_1\right]+$$
$$-\sum_{m=1}^{\infty}X_m\left[\sin\left(\frac{2\pi mk}{N}+\varphi_m\right)\sin mk(MN+1)\Delta\varphi_1\right]$$

The second component in (9) is the corrective one. Its value is proportional to a derivative of the measured signal.

Sigma-delta undersampling

The derivative of the signal $x_0(t)$ is expressed by following formula

(12) $$\frac{dx_0(t)}{dt}=-\sum_{m=1}^{\infty}2m\pi f_0X_m\cdot\sin(2\pi mf_0t+\varphi_m).$$

A signal proportional to the derivative of the signal $x_0(t)$ can be obtained by means of the sigma-delta undersampling [8-14], basing on the subtraction of 2 succeeding integrals, according to

(13)
$$y_0(t)=\int_{t}^{t+T_{d0}/2}x_0(t)\cdot dt-\int_{t-T_{d0}/2}^{t}x_0(t)\cdot dt=$$
$$=-\frac{2}{\pi f_0}\sum_{m=1}^{\infty}\frac{X_m}{m}\cdot\sin^2\left(\frac{mM\pi}{2}+\frac{m\pi}{2N}\right)\cdot\sin(2m\pi f_0t+\varphi_m).$$

Analogically as in the case of the sigma undersampling, the samples of the derivative of the measured signal can be obtained as the output signal of the FIR digital filter, which transfer function $H_\Delta(f)$ satisfies the following condition

(14) $$X_1(mf_0)=H_\Delta(mf_0)\cdot Y_0(mf_0),$$

where $X_1(f)$ and $Y_0(f)$ are the spectra of the derivative of the signal $x_0(t)$ and $y_0(t)$, respectively. This transfer function can be calculated using IDFT of $H_\Delta(m)$

(15) $$H_\Delta(z)=\sum_{n=1}^{N-1}h_\Delta(n)\cdot z^{-n}$$

(16)
$$h_\Delta(n)=(-1)^n\frac{\pi^2N^2f_0^2}{\sin\left(\frac{NM\pi}{4}+\frac{\pi}{4}\right)}+$$
$$+2\pi^2f_0^2\cdot\sum_{m=1}^{N/2-1}\frac{m^2\cos\frac{2mn\pi}{N}}{\sin^2\left(\frac{mM\pi}{2}+\frac{m\pi}{2N}\right)}.$$

Analogically as in the case of the sigma undersampling the fluctuations of the frequencies should be taken into considerations. Basing on (7) and (8) the values of the samples of the derivative of the signal can be expressed as

(17)
$$x'(k)=-2\pi f_0\sum_{m=1}^{\infty}mX_m\left[\sin\left(\frac{2\pi mk}{N}+\varphi_m\right)\cos mk(MN+1)\Delta\varphi_1\right]+$$
$$-2\pi f_0\sum_{m=1}^{\infty}mX_m\left[\cos\left(\frac{2\pi mk}{N}+\varphi_m\right)\sin mk(MN+1)\Delta\varphi_1\right]$$

where $x'(k)$ denotes a derivative of the signal $x(t)$ for $t=k{\cdot}T_d$. Formulas (11) and (17) enable a creation of the signal processing algorithm.

Algorithm of signal processing

According to (3)-(6) and (11)-(17) the algorithm of the signal processing can be formulated. The purpose of this algorithm is the maximum reduction of the influence of the frequency fluctuations on the results of the measurements and the accurate reconstruction of the signal $x_0(t)$ basing on the values of the integrals. The integral (3) can be divided into 2 integrals: from t-T_{d0}/2 to t and from t to t+T_{d0}/2. The sum of these integrals gives the value proportional to the sample of the signal $x(k)$, whereas the difference - the value of the sample of the derivative of this signal. The values of the signal and its derivative are obtained on the outputs of the filters H_Σ and H_Δ.

Since the fluctuations of both frequencies are in range of 10^{-6}, the values of phase errors are close to 0. Therefore formulas (11) and (17) can be expressed approximately:

(18) $$x(k)=x_0(k)-mk(MN+1)\cdot\Delta\varphi_1\cdot x_0'(k)$$

(19) $$x'(k)=x_0'(k)-mk(MN+1)\cdot\Delta\varphi_1\cdot x_0(k),$$

where $x_0'(k)$ denotes a derivative of the signal $x_0(t)$ at the moment k. A solution of these equations enables the calculations of the values $x_0(k)$ and $x_0'(k)$ on the condition that $\Delta\varphi_1$ is known.

The value of the error $\Delta\varphi_1$ can be obtained basing on 2 first values of the 1st harmonic of the measured signal. They can be separated using FFT and IFFT. The values of the 1st harmonic $x_1(k)$ and its derivative $x_1'(k)$ for k=0 are equal:

(20) $$x_1(0)=X_1\cos\varphi_1,$$

(21) $$x_1'(0)=X_1\sin\varphi_1,$$

whereas for k=1:

(22)
$$x_1(1)=X_1\cos\left(\frac{2\pi}{N}+\varphi_1\right)\cos(MN+1)\Delta\varphi_1+$$
$$-X_1\sin\left(\frac{2\pi}{N}+\varphi_1\right)\sin(MN+1)\Delta\varphi_1,$$

(23)
$$x_1'(1)=X_1\sin\left(\frac{2\pi}{N}+\varphi_1\right)\cos(MN+1)\Delta\varphi_1+$$
$$+X_1\cos\left(\frac{2\pi}{N}+\varphi_1\right)\sin(MN+1)\Delta\varphi_1.$$

The solution of the set of the equations (21)-(24) enables calculations of all parameters necessary to the creation of the signal processing algorithm.

Results of simulations

The estimation of the accuracy of the method described above was performed using the simulation in MATLAB. Calculations were made separately for every harmonic and they were repeated P=10^6 times. The fluctuations of the frequency of the signal and the sampling period were random values described by Gaussian distribution with variance δf (the frequency of the input signal) or δT_d (the sampling period) and mean value 0. Gaussian noise of the mean value 0 and the variance σ was added to the output signal as the simulation of the errors of the deceiver. The undersampling factor M, the number of the samples per period N and the signal to noise ratio, defined as

$$SNR = \frac{2\cdot\sigma^2}{A_m^2} \tag{25}$$

are the parameters of the simulations.

The error δ, defined as

$$\delta = \frac{\sum_{p=1}^{P}\sum_{k=0}^{N-1}\left|\left[x(k) - A_m\cdot\cos\left(\frac{2\cdot\pi\cdot m\cdot k}{N} + \varphi_m\right)\right]\right|}{P\cdot N\cdot A_m} \tag{26}$$

was chosen as a criterion of the preciseness of the method. The results of the simulations are presented in Figs. 1-10.

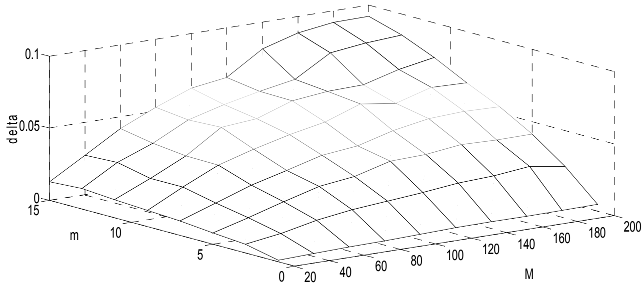

Fig. 1. Dependence of error δ on m and M, N = 256, 1/SNR = 0,1, δf = δT_d= 10^{-6}

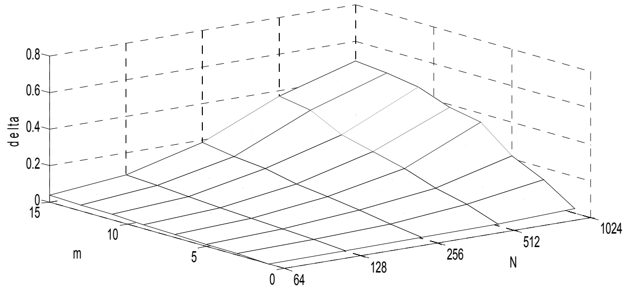

Fig. 2. Dependence of error δ on m and N, M = 100 , 1/SNR = 0,1, δf = δT_d= 10^{-6}

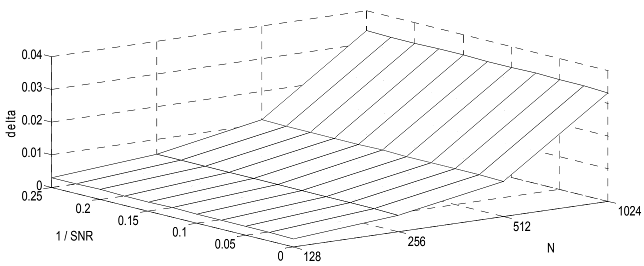

Fig.3. Dependence of error δ for 1st harmonic on N and 1/SNR, M=100, δf = δT_d= 10^{-6}

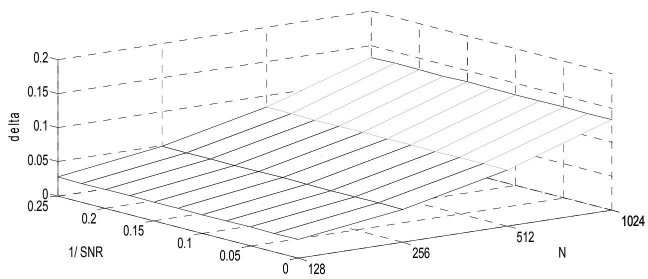

Fig. 4 Dependence of error δ for 5th harmonic on N and 1/SNR, M=100, δf = δT_d= 10^{-6}

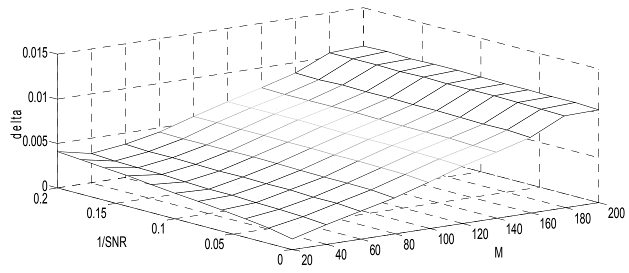

Fig.5. Dependence of error δ for 1st harmonic on M and 1/SNR, N=256, δf = δT_d= 10^{-6}

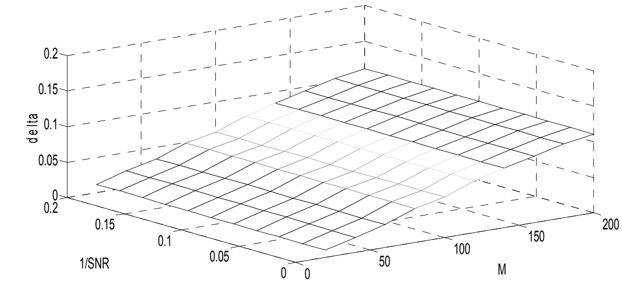

Fig.6. Dependence of error δ for 5th harmonic on M and 1/SNR, N=256, δf = δT_d= 10^{-6}

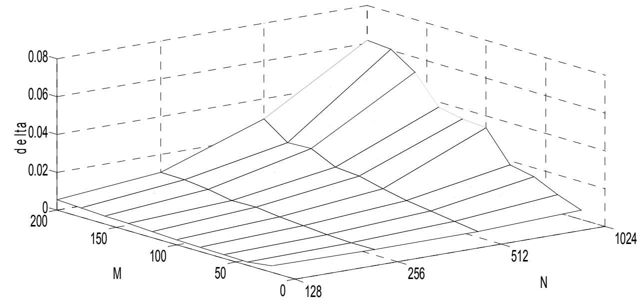

Fig. 7. Dependence of error δ for 1st harmonic on N and M, 1/SNR=0,1, δf = δT_d= 10^{-6}

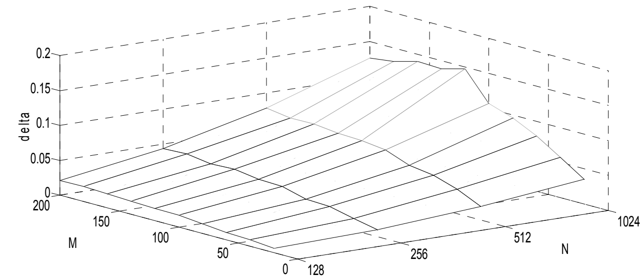

Fig. 8. Dependence of error δ for 5th harmonic on N and M, 1/SNR=0.1, δf = δT_d= 10^{-6}

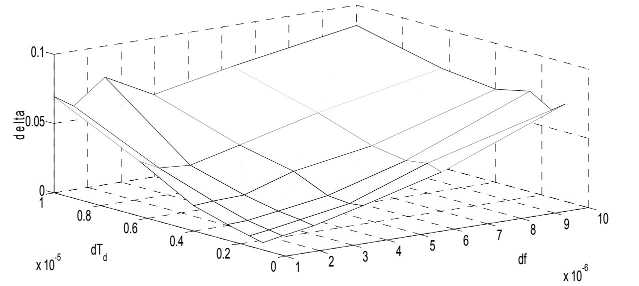

Fig. 9. Dependence of error δ for 1st harmonic on δf and δT_d , N =256, M =100, 1/SNR=0,1

The simulation of the method was also performed for an exemplary periodical signal. The triangular signal $x_0(t)$ with the mean value 0 was chosen. In this case the conditions of the calculations were identical as previously with 2 exceptions. For the triangular signal SNR should be defined as

(27) $$SNR = \frac{3 \cdot \sigma^2}{x_{0\max}^2},$$

where x_{0max} is the amplitude of the triangle, whereas the accuracy of the method is estimated by means of the error δ_1.

(28) $$\delta_1 = \frac{\sum_{p=1}^{P} \sum_{k=0}^{N-1} \left| \left[x(k) - x_0(k) \right] \right|}{P \cdot N \cdot x_{0\max}}.$$

The results of the simulation for the triangular signal are presented in Figs. 11-16.

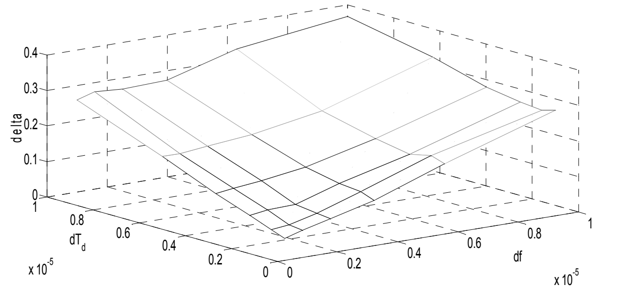

Fig. 10. Dependence of error δ for 5th harmonic on δf and δT_d, N =256, M =100, 1/SNR=0,1

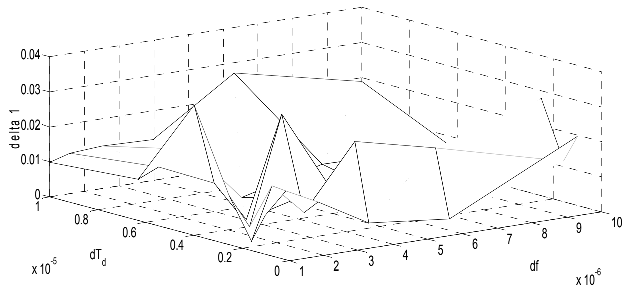

Fig. 11. Dependence of error δ_1 on δf and δT_d for triangular signal, N =256, M = 100, 1/SNR=0

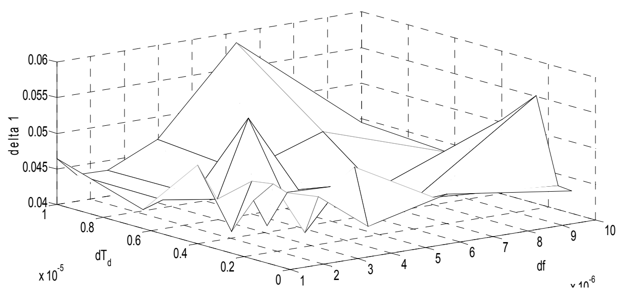

Fig. 12. Dependence of error δ_1 on δf and δT_d for triangular signal, N =256, M = 100, 1/SNR=0,1

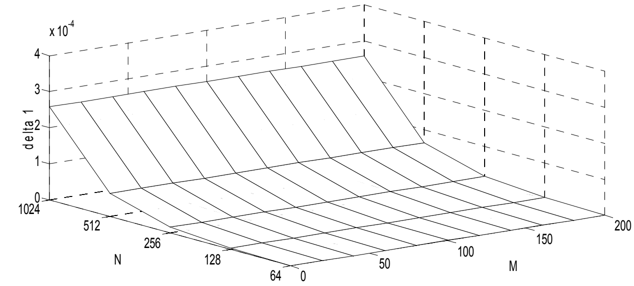

Fig. 13. Dependence of error δ_1 on N and M for triangular signal, δf = δT_d= 10^{-6}, 1/SNR=0

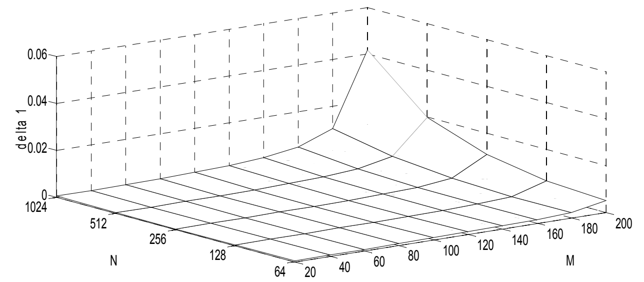

Fig. 14. Dependence of error δ_1 on N and M for triangular signal, δf = δT_d= 10^{-6}, 1/SNR=0,1

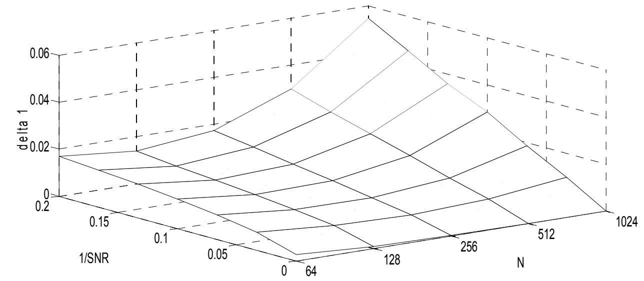

Fig. 15. Dependence of error δ_1 on N and 1/SNR, for triangular signal, M = 100, δf = δT_d= 10^{-6}

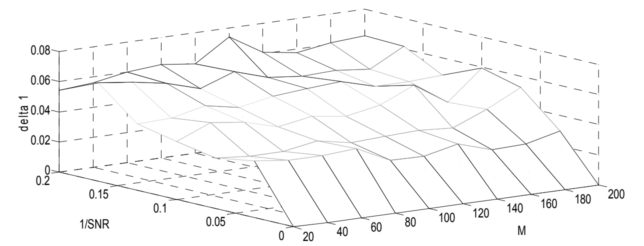

Fig. 16. Dependence of error δ_1 on M and 1/SNR, for triangular signal, N = 256, δf = δT_d= 10^{-6}

Conclusions

Presented results of the simulations both for particular harmonics of the measured signal, as well for the triangular signal prove, that the sigma undersampling connected to the sigma-delta undersampling can be the effective method of the measurement of periodical signals. Properly selected NOI filters in cooperation with sampling system enable calculations of the corrective components reducing the errors, arising from the variations of the frequency well as the noise of the detectors of the signal.

The most important advantage of presented algorithm is the possibility of its application to the measurements of low level signals.

A comparison of this algorithm to the algorithm worked out earlier [6-8] reveals an advantage of the new solution, consisting in its simplicity, the easier application, the absence of nonlinear elements and less values of the errors.

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REFERENCES

[1] Sun Z., Laneman J. N.: Performance Metrics, Sampling Schemes, and Detection Algorithms for Wideband Spectrum Sensing. *IEEE Trans. on Signal Processing*, vol. 62, no. 19, Oct. 2014, 5107-5118

[2] Race P.E., Leino R.E., Styer D.: Use of the Symmetrical Number System in Resolving Single-Frequency Undersampling Aliases. *IEEE Trans. on Signal Proc.*, vol. 45, no. 5, May 1997, 1153-1160
[3] Bhatta D., Banerjee A., Deyati S., Tzou N., Chatterjee A.: Low Cost Signal Reconstruction Based Testing of RF Components Using Incoherent Undersampling. *14th Latin American Test Workshop,* Cordoba, Spain, China, 3-5 Apr. 2013, 1-5
[4] Tzou N., Bhatta D., Hsiao S.-W., Choi H. W., Chatterjee A.: Low-Cost Wideband Periodic Signal Reconstruction Using Incoherent Undersampling and Back-end Cost Optimization, *IEEE Int. Test Conf.* Anaheim, 5-8 Nov. 2012, 1-10
[5] Duc H. L., Jabbour C., Desgreys P., Jamin O., Nguyen V. T.: A Fully Digital Background Calibration of Timing Skew in Undersampling TI-ADC. *IEEE Transactions on Circuits and Systems,* vol. 63, no .1, Jan 2016, 99-103
[6] Nikołajew A.: Analiza możliwości zastosowania procesorów sygnałowych w pomiarach światłowodów. *Przegląd Elektrotechniczny,* R. 91, 11(2015), 229-234
[7] Nikołajew A.: Effect of Disturbances on Sigma Undersampling of Periodical Signals, *Przegląd Elektrotechniczny*, R. 92, 2(2016), 78-81
[8] Nikołajew A.: An Application of ΣΔ Undersampling to an Estimation of Linear System's Impulse Response", *Przegląd Elektrotechniczny*, R. 92, 2(2016), 73-77
[9] Donoho D. L., Tanner J.: Precise Undersampling Theorems. *Proc. of IEEE* 2010, vol. 98, no. 6, 913-924
[10] Tzou N., Bhatta D., Hsiao S.-W., Chatterjee A.: Periodic Jitter and Bounded Uncorrelated Jitter Decomposition Using Incoherent Undersampling. *Design, Automation & Test in Europe Conference and Exhibition,* 2013, 1667-1672
[11] Reddy K., Pavan S.: Fundamental Limitations of Continuous-Time Delta-Sigma Modulators Due To Clock Jitter. *IEEE Trans. on Circuits and Systems*, vol. 54, no. 10, Oct. 2007, 2184-2194.
[12] Naderi A., Sawan M., Savaria Y.: On the Design of Undersampling Continuous-Time Bandpass Delta-Sigma Modulators for Gigahertz Frequency A/D Conversion. *IEEE Trans. on Circuits and Systems*, vol. 55, no. 11, Dec. 2008, 3488-3499.
[13] Liang H., Li X., Xia X.-G.: Adaptive Frequency Estimation with Low Sampling Rates Based on Robust Chinese Remainder Theorem and IIR Notch Filter. *4th IEEE Conf. on Industrial Electronics and Applications*, Xi'an, China, 2009, 2999-3004.
[14] Kammoun A., Beilleau N., Aboushady H.: Undersampled LC Bandpass ΣΔ Modulators with Feedback FIRDACs. *IEEE International Symp. on Circuits and Systems,* 2006, 4427-4430