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Positive linear systems and electrical circuits with inverse state matrices

***Abstract**. The positive linear systems and electrical circuits with inverse state matrices are addressed. It is shown that: 1) If the eigenvalues of asymptotically stable matrix $A \in M_n$ are located outside the unit circle then the eigenvalues of asymptotically stable matrix $-A^{-1} \in \Re_+^{n\times n}$ are positive and located inside the unit circle; 2) If the eigenvalues of asymptotically stable matrix $A \in \Re_+^{n\times n}$ are positive and located inside the unit circle then the eigenvalues of the matrix $-A^{-1} \in M_n$ are located in the left half plane outside the unit circle. A new problem concerning the discretization of positive linear systems and electrical circuits is formulated and solved.*

***Streszczenie.** W pracy przedstawiono dodatnie układy liniowe i obwody elektryczne z odwrotnością macierzy stanu. Zostało pokazane że: 1) Jeżeli wartości własne asymptotycznie stabilnej macierzy $A \in M_n$ są ulokowane poza obszarem koła jednostkowego wtedy wartości własne asymptotycznie stabilnej macierzy $-A^{-1} \in \Re_+^{n\times n}$ są dodatnie i leżą wewnątrz koła jednostkowego; 2) Jeżeli wartości własne asymptotycznie stabilnej macierzy $A \in \Re_+^{n\times n}$ są dodatnie i leżą wewnątrz koła jednostkowego wtedy wartości własne macierzy $-A^{-1} \in M_n$ leżą w lewej półpłaszczyźnie poza obszarem koła jednostkowego. Sformułowany zostanie nowy problem dotyczący dyskretyzacji dodatnich układów liniowych oraz obwodów elektrycznych. (**Dodatnie układy liniowe i obwody elektryczne z odwrotnością macierzy stanu**).*

Keywords: positive, linear, system, electrical circuit, discretization, stability.
Słowa kluczowe: dodatni, liniowy, obwód elektryczny dyskretyzacja, stabilność.

Introduction

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive systems theory is given in monographs [2, 10]. Variety of models having positive behavior can be found in engineering, especially in electrical circuits [20], economics, social sciences, biology and medicine, etc. [2, 10].

The positive electrical circuits have been analyzed in [4-9, 11, 13]. The constructability and observability of standard and positive electrical circuits has been addressed in [5], the decoupling zeros in [6] and minimal-phase positive electrical circuits in [8]. A new class of normal positive linear electrical circuits has been introduced in [9]. Positive fractional linear electrical circuits have been investigated in [11], positive linear systems with different fractional orders in [14, 15] and positive unstable electrical circuits in [16]. Zeroing of state variables in descriptor electrical circuits has been addressed in [19] and the realization problem of positive linear systems in [1]. Determinants of the matrices of solutions to the standard and positive linear electrical circuits have been addressed in [7] and positive electrical circuits with zero transfer matrices in [12].

In this paper the positive linear systems and electrical circuits with inverse state matrices will be addressed.

The paper is organized as follows. In section 2 some definitions and theorems concerning linear positive systems and matrix functions are recalled. The relationship between stability of positive continuous-time and discrete-time linear systems is addressed in section 3. Discretization of continuous-time positive linear systems is analyzed and new problem and its solution are presented in section 4. The new problem is illustrated on positive electrical circuits in section 5. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, M_n - the set of $n\times n$ Metzler matrices (real matrices with nonnegative off-diagonal entries), I_n - the $n\times n$ identity matrix, A^T denotes the transpose of the matrix A.

Preliminaries

Consider the continuous-time linear system

(1) $$\dot{x} = Ax + Bu,$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$ are the state and input vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$.

Definition 1. [2, 10] The continuous-time linear system (1) is called (internally) positive if $x(t) \in \Re_+^n$, $t \ge 0$ for any initial conditions $x(0) \in \Re_+^n$ and all inputs $u(t) \in \Re_+^m$, $t \ge 0$.

Theorem 1. [2, 10] The continuous-time linear system (1) is positive if and only if

(2) $$A \in M_n,\ B \in \Re_+^{n\times m}.$$

Definition 2. [2, 10] The positive continuous-time system (1) for $u(t) = 0$ is called asymptotically stable if

(3) $$\lim_{t\to\infty} x(t) = 0 \text{ for any } x(0) \in \Re_+^n.$$

Theorem 2. [2, 10] The positive continuous-time linear system (1) for $u(t) = 0$ is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficient of the characteristic polynomial

(4) $$p_n(s) = \det[I_n s - A] = s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0$$

are positive, i.e. $a_i > 0$ for $i = 0,1,...,n-1$.

2) There exists strictly positive vector $\lambda^T = [\lambda_1 \ \cdots \ \lambda_n]^T$, $\lambda_k > 0$, $k = 1,...,n$ such that

(5) $$A\lambda < 0.$$

If the matrix A is nonsingular then we can choose $\lambda = A^{-1}c$, where $c \in \Re^n$ is strictly positive.

Consider the discrete-time linear system

(6) $$x_{i+1} = Ax_i + Bu_i,\ i \in Z_+ = \{0,1,...\},$$

where $x_i \in \Re^n$, $u_i \in \Re^m$ are the state and input vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$.

Definition 3. [2, 10] The discrete-time linear system (6) is called (internally) positive if $x_i \in \Re_+^n$, $i \in Z_+$ for any initial conditions $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 3. [2, 10] The discrete-time linear system (6) is positive if and only if

(7) $$A \in \Re_+^{n\times n},\ B \in \Re_+^{n\times m}.$$

Definition 4. [2, 10] The positive discrete-time system (6) for $u_i = 0$ is called asymptotically stable if

(8) $$\lim_{i\to\infty} x_i = 0 \text{ for any } x_0 \in \Re_+^n.$$

Theorem 4. [2, 10] The positive discrete-time linear system (6) for $u_i = 0$ is asymptotically stable if and only if one of the following equivalent conditions is satisfied:

1) All coefficient of the characteristic polynomial

(9) $$p_n(z) = \det[I_n(z+1) - A] = z^n + a_{n-1}z^{n-1} + ... + a_1 z + a_0$$

are positive, i.e. $a_i > 0$ for $i = 0,1,...,n-1$.

2) There exists strictly positive vector $\lambda^T = [\lambda_1 \ \cdots \ \lambda_n]^T$, $\lambda_k > 0$, $k = 1,...,n$ such that

(10) $$(A - I_n)\lambda < 0.$$

If the matrix $(A - I_n)$ is nonsingular then we can choose $\lambda = (A - I_n)^{-1}c$, where $c \in \Re^n$ is strictly positive.

Consider a matrix $A \in \Re^{n\times n}$ with the minimal characteristic polynomial

(11) $$\Psi(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2}...(\lambda - \lambda_r)^{m_r},$$

where λ_1, λ_2, … , λ_r are the eigenvalues of the matrix A and $\sum_{i=1}^{r} m_i = m \le n$. It is assumed that the function $f(\lambda)$ is well-defined on the spectrum $\sigma_A = \{\lambda_1, \lambda_2, ..., \lambda_r\}$ of the matrix A, i.e.

(12) $$f(\lambda_k),\ f^{(1)}(\lambda_k) = \left.\frac{df(\lambda)}{d\lambda}\right|_{\lambda=\lambda_k}, ...,\quad f^{(m_k-1)}(\lambda_k) = \left.\frac{d^{m_k-1}f(\lambda)}{d\lambda^{m_k-1}}\right|_{\lambda=\lambda_k},\ k = 1,...,r$$

are finite [3, 18].

In this case the matrix $f(A)$ is well-defined and it is given by the Lagrange-Sylvester formula [3, 18]

(13) $$f(A) = \sum_{i=1}^{r} Z_{i1}f(\lambda_i) + Z_{i2}f^{(1)}(\lambda_i) + ... + Z_{im_i}f^{(m_i-1)}(\lambda_i),$$

where

(14) $$Z_{ij} = \sum_{k=j-1}^{m_i-1} \frac{\Psi_i(A)(A - \lambda_i I_n)^k}{(k-j+1)!(j-1)!}\frac{d^{k-j+1}}{d\lambda^{k-j+1}}\left[\frac{1}{\Psi_i(\lambda)}\right]_{\lambda=\lambda_i}$$

and

(15) $$\Psi_i(\lambda) = \frac{\Psi(\lambda)}{(\lambda - \lambda_i)^{m_i}},\ i = 1,...,r.$$

In particular case where the eigenvalues λ_1, λ_2, … , λ_n of the matrix A are distinct ($\lambda_i \ne \lambda_j$, $i \ne j$) and

(16) $$\varphi(\lambda) = \Psi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n),$$

then the formula (13) has the form

(17) $$f(A) = \sum_{k=1}^{n} Z_k f(\lambda_k),$$

where

(18) $$Z_k = \prod_{\substack{i=1\\ i\ne k}}^{n} \frac{A - \lambda_i I_n}{\lambda_k - \lambda_i}.$$

Theorem 5. Let $f(\lambda)$ be well-defined on the spectrum σ_A of the matrix $A \in \Re^{n\times n}$. If λ_k, $k = 1,...,n$ is the eigenvalue of the matrix A then $f(\lambda_k)$, $k = 1,...,n$ are the eigenvalues of the matrix $f(A) \in \Re^{n\times n}$.

Proof. The proof is given in [3, 18].

In particular cases from Theorem 5 we have the following corollaries.

Corollary 1. If λ_k, $k = 1,...,n$ are the eigenvalues of the nonsingular matrix $A \in \Re^{n\times n}$ then λ_k^{-1}, $k = 1,...,n$ are the eigenvalues of the inverse matrix A^{-1}.

Corollary 2. If λ_k, $k = 1,...,n$ are the eigenvalues of the matrix $A \in \Re^{n\times n}$ then $-\lambda_k$, $k = 1,...,n$ are the eigenvalues of the matrix $-A$.

Relationship between stability of continuous-time and discrete-time positive linear systems

In this section the relationship between asymptotic stability of continuous-time and corresponding discrete-time positive linear systems will be investigated.

Theorem 6. If the eigenvalues of asymptotically stable matrix $A \in M_n$ are located outside the unit circle then the eigenvalues of asymptotically stable matrix $-A^{-1} \in \Re_+^{n\times n}$ are positive and located inside the unit circle.

Proof. First using the induction method we shall show that if $A \in M_n$ is asymptotically stable then $-A^{-1} \in \Re_+^{n\times n}$.

The hypothesis is true for $n = 1$ since $A = [a_{11}] \in M_1$ and $a_{11} < 0$. The hypothesis is also true for $n = 2$ since for

(19) $$A_2 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \in M_2$$

asymptotically stable we have

(20) $$-A_2^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}}\begin{bmatrix} -a_{22} & a_{12} \\ a_{21} & -a_{11} \end{bmatrix} \in \Re_+^{n\times n}$$

since $a_{11}a_{22} - a_{12}a_{21} > 0$.

Assuming that the hypothesis is true for $n-1$ we shall show that it is also valid for $n = 2,3,...$.

It is easy to check that if the matrix A_n has the form

(21) $$A_n = \begin{bmatrix} A_{n-1} & u_n \\ v_n & a_{nn} \end{bmatrix},$$

$$u_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{n-1,n} \end{bmatrix}, v_n = [a_{n1} \quad a_{n2} \quad \cdots \quad a_{n,n-1}]$$

then its inverse is given by

(22) $$A_n^{-1} = \begin{bmatrix} A_{n-1}^{-1} + \frac{A_{n-1}^{-1}u_n v_n A_{n-1}^{-1}}{a_n} & -\frac{A_{n-1}^{-1}u_n}{a_n} \\ -\frac{v_n A_{n-1}^{-1}}{a_n} & \frac{1}{a_n} \end{bmatrix},$$

where

(23) $$a_n = a_{nn} - v_n A_{n-1}^{-1} u_n .$$

From (22) and (23) it follows that if $-A_{n-1}^{-1} \in \Re_+^{(n-1)\times(n-1)}$ then $-A_n^{-1} \in \Re_+^{n\times n}$ since $a_n < 0$.

If the eigenvalues of asymptotically stable $A \in M_n$ are located outside the unit circle then by Corollary 1 the eigenvalues of A^{-1} are located inside the unit circle and by Corollary 2 the eigenvalues of $-A^{-1} \in \Re_+^{n\times n}$ are positive and located inside the unit circle. □

Example 1. Consider the system (1) for $u(t) = 0$ and

(24) $$A = \begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix} \in M_2 .$$

The characteristic polynomial of (24) has the form

(25) $$\det[I_2 s - A] = \begin{vmatrix} s+3 & -1 \\ -1 & s+3 \end{vmatrix} = s^2 + 6s + 8$$

and the eigenvalues of the matrix (24) are $s_1 = -2$, $s_2 = -4$.

The inverse matrix of (24) has the form

(26) $$A^{-1} = -\begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix}$$

and its eigenvalues are: $s_1^{-1} = -\frac{1}{2}$, $s_2^{-1} = -\frac{1}{4}$.

The eigenvalues of the matrix

(27) $$-A^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix} \in \Re_+^{2\times 2}$$

are $z_1 = -s_1^{-1} = \frac{1}{2}$, $z_2 = -s_2^{-1} = \frac{1}{4}$.

Theorem 7. If the eigenvalues of asymptotically stable matrix $A \in \Re_+^{n\times n}$ are positive and located inside the unit circle then the eigenvalues of the matrix $-A^{-1} \in M_n$ are located in the left half plane outside the unit circle.

Proof. By Corollary 1 if the eigenvalues of $A \in \Re_+^{n\times n}$ are positive and located inside the unit circle then the eigenvalues of the matrix A^{-1} are located outside the unit circle and by Corollary 2 the eigenvalues of $-A^{-1} \in M_n$ are located in the left half plane outside the unit circle. □

Example 2. Consider the discrete-time linear system

(28) $$x_{i+1} = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.4 \end{bmatrix} x_i ,\ i \in Z_+ = \{0,1,...\} .$$

The characteristic polynomial of the matrix

(29) $$A_d = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & 0.4 \end{bmatrix} \in \Re_+^{2\times 2}$$

has the form

(30) $$\det[I_2 z - A_d] = \begin{bmatrix} z-0.3 & -0.1 \\ -0.2 & z-0.4 \end{bmatrix} = z^2 - 0.7z + 0.1$$

and the eigenvalues of (29) are: $z_1 = 0.2$, $z_2 = 0.5$. Therefore, the system (28) is asymptotically stable.

The inverse matrix of (29) has the form

(31) $$A_d^{-1} = \begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$$

and its eigenvalues are $z_1^{-1} = 5$, $z_2^{-1} = 2$. Therefore, the matrix (31) is unstable.

The matrix

(32) $$-A_d^{-1} = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix} \in M_2$$

is the matrix of asymptotically stable continuous-time linear system with its eigenvalues $s_1 = z_1^{-1} = -5$, $s_2 = z_2^{-1} = -2$ located in the left half plane outside the unit circle.

Discretization of positive continuous-time linear systems, problem formulation and its solution

Consider the positive autonomous continuous-time linear system

(33) $$\dot{x}(t) = Ax(t) ,$$

where $x(t) \in \Re_+^n$, $t \ge 0$ and $A \in M_n$.

Applying the approximation

(34) $$\dot{x}(t) \approx \frac{x_{i+1} - x_i}{h} ,\ x_{i+1} = x(t+h) ,\ x_i = x(t) ,\ i \in Z_+$$

to (33) we obtain the discrete-time linear system

(35) $$x_{i+1} = A_d x_i ,\ i \in Z_+ ,$$

where

(36) $$A_d = I_n + hA .$$

The discrete-time system (35) is positive if and only if $A_d \in \Re_+^{n\times n}$. From (36) it follows that the system (35) is positive for $A \in M_n$ if and only if the discretization step h satisfies the condition

(37) $h \le \frac{1}{\max|a_{ii}|}$ for $i = 1,...,n$,

where a_{ii} is the diagonal entry of $A \in M_n$.

Theorem 8. The eigenvalues s_i , $i = 1,...,n$ of the matrix $A \in M_n$ with the eigenvalues z_i , $i = 1,...,n$ of the matrix $A_d \in \Re_+^{n\times n}$ are related by

(38) $z_i = 1 + hs_i$.

Proof. The proof follows immediately from Theorem 5 applied to (36). □

Theorem 9. The discrete-time system (35) is asymptotically stable for $h > 0$ if and only if the continuous-time system (33) is asymptotically stable and

(39) $h < -\frac{2\alpha_l}{\alpha_l^2 + \beta_l^2}$, $l = 1,...,n$.

Proof. Let $s_l = \alpha_l + j\beta_l$, $l = 1,...,n$, then the discrete-time system (35) is asymptotically stable if and only if

(40)
$$\left|z_l^2\right| = \left|1 + hs_l\right|^2 = (1 + h\alpha_l)^2 + (h\beta_l)^2$$
$$= 1 + 2h\alpha_l + (h\alpha_l)^2 + (h\beta_l)^2 < 1$$

or

(41) $2\alpha_l + h(\alpha_l^2 + \beta_l^2) < 0$.

Solving (41) with respect to *h* we obtain (4.7). □

From (39) it follows that $h > 0$ if and only if $\alpha_l < 0$, $l = 1,...,n$, i.e. the continuous-time system (33) is asymptotically stable. □

In section 3 it was shown that if the matrix $A \in M_n$ of the system (33) is asymptotically stable then the matrix $-A^{-1} \in \Re_+^{n\times n}$ is also asymptotically stable (Theorem 6) and it can be considered as a matrix of positive discrete-time linear system. It has been shown that by suitable choice of the discretization step *h* it is possible to obtain from positive continuous-time system (33) the positive discrete-time system (35). The following problem arises: under which conditions it is possible to find a suitable discretization step *h* such that

(42) $-A^{-1} = I_n + hA$

or equivalently

(43) $I_n + A + hA^2 = 0$.

Theorem 10. There exists a discretization step *h* such that (43) holds if and only if

(44) $1 + s_k + hs_k^2 = 0$ for $k = 1,...,n$,

where s_k is the eigenvalue of the matrix A.

Proof. It is well-known [3, 18] that the equality (43) holds if and only if the equation (44) is satisfied for all eigenvalues s_k of the matrix A. □

If the matrix A has only one n-multiple real eigenvalue $s_1 = -\alpha$ such that $1 - \alpha < 0$ then from (44) we have

(45) $h = \frac{\alpha - 1}{\alpha^2} > 0$.

In this case from Theorem 10 we have the following corollary.

Corollary 3. There exists a discretization step given by (45) if the matrix A has only one multiple real eigenvalue $s_1 = -\alpha$ such that $\alpha - 1 > 0$.

Example 3. Consider the positive continuous-time system (33) with

(46) $A = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} \in M_2$

with double real eigenvalue $s_1 = s_2 = -\alpha = -2$.

The inverse matrix of (46) has the form

(47) $A^{-1} = -\begin{bmatrix} 0.5 & 0.25 \\ 0 & 0.5 \end{bmatrix}$ and $-A^{-1} = \begin{bmatrix} 0.5 & 0.25 \\ 0 & 0.5 \end{bmatrix}$.

From (45) we have

(48) $h = \frac{\alpha - 1}{\alpha^2} = \frac{1}{4}$.

The matrix (46) satisfies the equality (43) for $h = \frac{1}{4}$, i.e.

(49) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 4 & -4 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Positive electrical circuits

Consider the electrical linear circuit shown in Figure 1 with given resistances R_1 , R_2 , R_3 , inductances L_1 , L_2 and source voltages e_1 , e_2 .

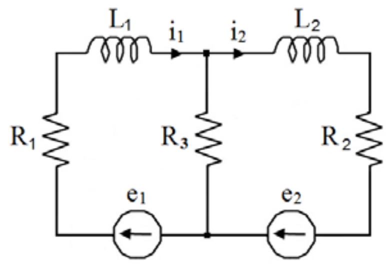

Fig. 1. Positive electrical circuit

Using the Kirchhoff's laws we can write the equations

(50)
$$e_1 = R_1 i_1 + L_1 \frac{di_1}{dt} + R_3(i_1 - i_2),$$
$$e_2 = R_2 i_2 + L_2 \frac{di_2}{dt} + R_3(i_2 - i_1),$$

which can be written in the form

(51a) $\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix} + B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$,

where

(51b) $$A=\begin{bmatrix} -\frac{R_1+R_3}{L_1} & \frac{R_3}{L_1} \\ \frac{R_3}{L_2} & -\frac{R_2+R_3}{L_2} \end{bmatrix},\ B=\begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix}.$$

The electrical circuit is positive since $A \in M_2$ and $B \in \Re_+^{2\times 2}$ for the positive resistances R_1, R_2, R_3 and inductances L_1, L_2.

In particular case for $R_1 = R_2 = 2$, $R_3 = 1$ and $L_1 = L_2 = 1$ we obtain

(52a) $$A=\begin{bmatrix} -3 & 1 \\ 1 & -3 \end{bmatrix},$$

(52b) $$B=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The characteristic polynomial of the matrix (52a) has the form

(53) $$\det[I_2 s - A]=\begin{bmatrix} s-3 & -1 \\ -1 & s+3 \end{bmatrix}=s^2+6s+8$$

and the eigenvalues of the matrix are: $s_1=-2$, $s_2=-4$.

The inverse matrix of (52a) has the form

(54) $$A^{-1}=-\begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix}$$

and its eigenvalues are: $s_1^{-1}=-\frac{1}{2}$, $s_2^{-1}=-\frac{1}{4}$.

By Corollary 2 the eigenvalues of the matrix

(55) $$-A^{-1}=\begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix} \in \Re_+^{2\times 2}$$

are $z_1=-s_1^{-1}=\frac{1}{2}$, $z_2=-s_2^{-1}=\frac{1}{4}$ and the matrix as the matrix of positive discrete-time linear system is asymptotically stable.

Note that for $R_3=1$ the condition (45) is not satisfied since for $s_1=-2$ we have $h=\frac{1}{4}$ and for $s_2=-4$, $h=\frac{3}{16}$.

For $R_3=0$, $R_1=R_2=2$ and $L_1=L_2=1$ we obtain $s_1=s_2=-2$ and $h=\frac{1}{4}$.

Consider the electrical circuit shown in Figure 2 with given resistances R_1, R_2, R_3, capacitances C_1, C_2 and source voltage e.

Using the Kirchhoff's laws we can write the equations

(56) $$e=(R_1+R_3)C_1\frac{du_1}{dt}+R_3C_2\frac{du_2}{dt}+u_1,$$
$$e=R_3C_1\frac{du_1}{dt}+(R_2+R_3)C_2\frac{du_2}{dt}+u_2,$$

which can be written in the form

(57) $$\frac{d}{dt}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}=A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}+Be,$$

where

(58)
$$A=\begin{bmatrix} -\frac{R_2+R_3}{C_1[R_1(R_2+R_3)+R_2R_3]} & \frac{R_3}{C_1[R_1(R_2+R_3)+R_2R_3]} \\ \frac{R_3}{C_2[R_1(R_2+R_3)+R_2R_3]} & -\frac{R_1+R_3}{C_2[R_1(R_2+R_3)+R_2R_3]} \end{bmatrix},$$
$$B=\begin{bmatrix} \frac{R_2}{C_1[R_1(R_2+R_3)+R_2R_3]} \\ \frac{R_1}{C_2[R_1(R_2+R_3)+R_2R_3]} \end{bmatrix}.$$

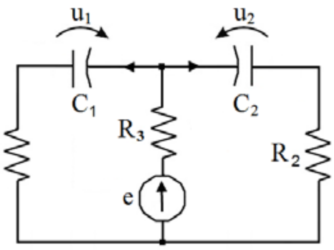

Fig. 2. Positive electrical circuit

The electrical circuit is positive since $A \in M_2$ and $B \in \Re_+^{2\times 1}$ for the positive resistances R_1, R_2, R_3 and capacitances C_1, C_2.

In particular case for $R_1=R_2=0.2$, $R_3=0.1$ and $C_1=C_2=1$ we obtain

(59a) $$A=\begin{bmatrix} -\frac{15}{4} & \frac{5}{4} \\ \frac{5}{4} & -\frac{15}{4} \end{bmatrix},$$

(59b) $$B=\begin{bmatrix} \frac{5}{2} \\ \frac{5}{2} \end{bmatrix}.$$

The characteristic polynomial of the matrix (59a) has the form

(60) $$\det[I_2 s - A]=\begin{bmatrix} s+\frac{15}{4} & -\frac{5}{4} \\ -\frac{5}{4} & s+\frac{15}{4} \end{bmatrix}=s^2+\frac{15}{2}s+\frac{25}{2}$$

and the eigenvalues of the matrix (59a) are $s_1=-\frac{5}{2}$, $s_2=-5$.

The inverse matrix of (59a) has the form

(61) $$A^{-1}=-\begin{bmatrix} 0.3 & 0.1 \\ 0.1 & 0.3 \end{bmatrix}$$

and its eigenvalues are: $s_1^{-1}=-\frac{2}{5}$, $s_2^{-1}=-\frac{1}{5}$.

By Corollary 2 the eigenvalues of the matrix

(62) $$-A^{-1}=\begin{bmatrix}0.3 & 0.1\\0.1 & 0.3\end{bmatrix}\in\Re_+^{2\times 2}$$

are $z_1=-s_1^{-1}=\frac{2}{5}$, $z_2=-s_2^{-1}=\frac{1}{5}$ and the matrix as the matrix of positive discrete-time linear system is asymptotically stable.

Note that for $R_1=R_2=0.2$, $R_3=0.1$ and $C_1=C_2=1$ the condition (44) is not satisfied since for $s_1=-\frac{5}{2}$ we have $h=\frac{6}{25}$ and for $s_2=-5$, $h=\frac{4}{25}$.

The considerations can be easily extended to positive electrical circuits composed of resistances, inductances, capacitances and source voltages (currents) [10, 20].

Concluding remarks

It has been shown that:

1) If the eigenvalues of asymptotically stable matrix $A\in M_n$ are located outside the unit circle then the eigenvalues of asymptotically stable matrix $-A^{-1}\in\Re_+^{n\times n}$ are positive and located inside the unit circle (Theorem 6).
2) If the eigenvalues of asymptotically stable matrix $A\in\Re_+^{n\times n}$ are positive and located inside the unit circle then the eigenvalues of the matrix $-A^{-1}\in M_n$ are located in the left half plane outside the unit circle (Theorem 7).
3) The eigenvalues s_i, $i=1,...,n$ of the matrix $A\in M_n$ with the eigenvalues z_i, $i=1,...,n$ of the matrix $I_n+hA\in\Re_+^{n\times n}$ are related by (38) (Theorem 8).
4) The discrete-time system (35) is asymptotically stable for $h>0$ if and only if the continuous-time system (33) is asymptotically stable and satisfies (39) (Theorem 9).

The problem under which condition it is possible to find a suitable discretization step h such that the equality (43) holds has been formulated and solved (Theorem 10). The problem has been applied to positive electrical circuits.

The considerations have been illustrated by numerical examples. An open problem is an extension of these considerations to fractional positive linear systems and electrical circuits [17, 20].

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