Waldemar WÓJCIK[1], Nurlan TEMIRBEKOV[2], Dossan BAIGEREYEV[3]

Lublin University of Technology, Lublin, Poland (1), D.Serikbayev East Kazakhstan State Technical University, Ust-Kamenogorsk, Kazakhstan (2), D.Serikbayev East Kazakhstan State Technical University, Ust-Kamenogorsk, Kazakhstan (3)

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Fractional flow formulation for three-phase non-isothermal flow in porous media

Abstract. *The present paper focuses on the simulation of three-phase non-isothermal compressible flow in porous media taking into account capillary effects. We propose a new formulation of the considered non-isothermal problem in which the gradients of capillary pressure functions are eliminated from the pressure and temperature equations by the introduction of a change of variables for the pressure. The mentioned change of variables is referred to as the global pressure. A computational algorithm for the numerical implementation of the problem using the finite difference method is proposed. A priori estimate for the solution of the difference problem is obtained. The results of numerical experiments on the example of a one-dimensional problem are presented.*

Streszczenie. *W pracy autorzy koncentrują się na symulacji trójfazowego nie izotermicznego ściśliwego przepływu w środowiskach porowatych przy uwzględnieniu efektów kapilarnych. Zaproponowano nowe sformułowanie zagadnienia w którym gradienty funkcji ciśnienia kapilarnego są wyeliminowane z równań ciśnienia i temperatury poprzez zamiane zmiennych dla ciśnienia globalnego. Zaproponowano implementację algorytmu wykorzystując schemat różnicowy. Wyniki obliczeń dla jednowymiarowego zagadnienia przedstawiono w zakończeniu pracy.* ***(Sformułowanie przepływów wielofazowych na przykładzie trójfazowych przepływów nieizotermicznych w porowatych mediach)***

Keywords: three-phase non-isothermal flow, fractional flow, capillary pressure, finite difference method, a priori estimate.
Słowa kluczowe: trójfazowe przepływy nieizotermiczne, przepływy wielofazowe, ciśnienie kapilarne, metoda różnic skończonych, estymacja a priori.

Introduction

Study of three-phase non-isothermal multiphase flow in porous media has received considerable attention in recent years because of its importance in variety of industrial processes, particularly in the extraction of heavy oil. Thermal recovery methods provide one of the highest recovery rates, and they are the most widely used methods in the heavy oil industry.

Extensive studies have been performed to simulate non-isothermal flows in porous media recently. Different approaches to solve non-isothermal flow problems with different assumptions about the physical data were studied in [1, 2, 3, 4, 5, 6, 7, 8]. The most common numerical approach to solve the three-phase non-isothermal flow problem is based on the choice of the pressure of one of the phases, two saturations and temperature as the primary variables, and this approach is usually referred to as the *phase formulation*. However, the choice of the phase pressure as the main unknown involves certain difficulties encountered in obtaining the numerical solution of the three-phase flow problem as well as during its mathematical analysis. Some of them, in relation to the isothermal case, are described in [9, 10, 11]. These difficulties are mainly related to the unbounded increase of the derivatives of capillary pressures when saturations approach corresponding residual values [9].

To overcome some of these shortcomings in obtaining numerical solutions to the three-phase isothermal flow problems, so-called *fractional flow (global pressure) formulation* is widely used. This approach was initially proposed in [12, 13] for the simulation of two-phase flow, and then generalized to the three-phase case. The idea of the global pressure approach is to replace the three-phase flow with the flow of some fluid which is described by Darcy's law.

Using the concept of the global pressure, in [13] the existence of the solution to the incompressible fluid flow problem for the degenerate and nondegenerate cases is proved. The uniqueness of the problem is proved in a particular case. In recent papers, new theoretical results of solvability of the two-phase flow problem using this approach with different assumptions about the physical data were obtained (cf., [14, 15, 16, 17, 18] and references therein).

A number of papers are devoted to the numerical simulation of isothermal flow by introducing the global pressure. Studies conducted in recent years show that this approach can be successfully applied for the simulation of two-phase compressible [11, 19], three-phase compressible [20], multicomponent compressible [21], non-isothermal two-phase incompressible [22] flow, as well as for the case of porous media with a discontinuity [10]. In [20], it is shown that from a computational point of view, solving the three-phase flow problem in the global pressure formulation is more efficient than in the phase formulation. To our knowledge, using the global pressure approach for the simulation of three-phase compressible non-isothermal flow has not been studied.

In this paper, the idea of introducing the global pressure is generalized for obtaining the numerical solution to the three-phase non-isothermal flow problem. The purpose of this work is to eliminate the gradients of the capillary pressure functions, leading to the unlimited growth of the solution at residual saturations, from the pressure and temperature equations by introducing a change of variables. Following the original papers [12, 13], the sought change of variables is referred to as *the global pressure* in the present work. A new formulation of the non-isothermal problem, which consists of four partial differential equations with respect to the global pressure, temperature and two saturations is derived. A computational algorithm for the considered problem with the use of the finite difference method is suggested. An a priori estimate is derived for the solution of the grid problem. In conclusion, the results of the numerical simulation are presented for a one-dimensional model problem.

Governing equations

This section describes the mathematical model used in the present paper. The following assumptions are made about the physical data. The flow is assumed to obey the generalized Darcy's law. For simplicity, we assume that the porous media is homogeneous and isotropic, gravitational forces are neglected. The phases are in local thermal equilibrium which means that in any elementary volume, the fluids saturating the porous media and rock have the same temperature. In addition, oil is assumed to be homogeneous non-evaporable fluid and phase transitions may occur in the water-steam system.

In this case, the three-phase non-isothermal compressible flow in a bounded domain $\Omega \subset R^d$ $(d = 1, 2, 3)$ is usually described by the equations:

$$\frac{\partial}{\partial t}\left(\phi\rho_\alpha s_\alpha\right)+\nabla\cdot\left(\rho_\alpha\vec{u}_\alpha\right)+I_\alpha=q_\alpha,\quad \alpha=w,\,o,\,g, \tag{1}$$

$$\vec{u}_\alpha=-\frac{kk_\alpha}{\mu_\alpha}\nabla p_\alpha,\quad \alpha=w,\,o,\,g, \tag{2}$$

$$\frac{\partial}{\partial t}\left(\phi\sum_\alpha\rho_\alpha s_\alpha i_\alpha+(1-\phi)\,\rho_r i_r\right)+\nabla\cdot\sum_\alpha\rho_\alpha\vec{u}_\alpha i_\alpha-$$

$$-\nabla\cdot\left(k_T\nabla T\right)=q_T, \tag{3}$$

$$\rho_\alpha=\rho_\alpha\left(p_\alpha,T\right),\quad \alpha=w,\,o,\,g \tag{4}$$

where subscripts w, o, g, r denote the phases of water, oil, heat transfer agent, and rock, respectively; ϕ and k are the porosity and conductivity of the porous media; p_α, s_α, ρ_α, k_α, μ_α, i_α, $\vec{u}_\alpha$, I_α are the pressure, saturation, density, relative permeability, viscosity, enthalpy, velocity, and phase transition rate of the phase α, respectively. For saturations, the following constraint holds:

$$s_w+s_o+s_g=1. \tag{5}$$

The difference in pressures across the interface between two phases is expressed by the relations:

$$p_{ow}=p_o-p_w,\quad p_{go}=p_g-p_o \tag{6}$$

where capillary pressure functions p_{ow} and p_{go} depend on saturations and temperature and are assumed to be known. For simplicity, following [23, 24, 25, 26], we neglect the influence of temperature on capillary pressures in the present work.

We now define initial and boundary conditions for the system of equations (1)-(6). At the initial time, the reservoir temperature and pressure are known, and the porous media is saturated with oil and water:

$$T\,(x,0)=T_0,\;\; p_w\,(x,0)=p_0,\;\; s_\alpha\,(x,0)=s_{\alpha 0},\;\; \alpha=w,\,o,$$

$$s_g\,(x,0)=0,\;\; x\in\Omega. \tag{7}$$

Depending on the specific problem, the system is complemented with appropriate boundary conditions, cf. [27, 4].

Derivation of the fractional flow formulation for the non-isothermal problem

Derivation of the global formulation for the non-isothermal problem (1)-(7) is close to the presentation given in [13]. For convenience, we introduce the functions

$$\theta_\alpha\left(s_w,s_g,p_o,T\right)=\lambda_\alpha\lambda^{-1},$$

$$\lambda_\alpha\left(s_w,s_g,p_o,T\right)=\rho_\alpha c_\alpha k_\alpha\mu_\alpha^{-1}, \tag{8}$$

$$\lambda\left(s_w,s_g,p_o,T\right)=\lambda_w+\lambda_o+\lambda_g$$

where c_α is the specific heat capacity of the phase α. Let us introduce the vector

$$\vec{u}=\rho_w c_w\vec{u}_w+\rho_o c_o\vec{u}_o+\rho_g c_g\vec{u}_g. \tag{9}$$

Using the relations (2) and (6), one can easily show that the vector $\vec{u}$ can be expressed in terms of the phase pressure p_o and capillary pressure functions p_{ow} and p_{go} as follows:

$$\vec{u}=-k\lambda\left(\nabla p_o-\theta_w\nabla p_{ow}+\theta_g\nabla p_{go}\right). \tag{10}$$

The idea of the fractional flow approach is based on the introduction of a function $p=p\left(s_w,s_g,p_o,T\right)$ which is determined from the following differential equation:

$$\nabla p=\nabla p_o-\theta_w\nabla p_{ow}+\theta_g\nabla p_{go}. \tag{11}$$

It is known [9] that the gradients of capillary pressure functions ∇p_{ow} and ∇p_{go} increase unboundedly when saturations s_α approach corresponding residual values $s_{r\alpha}$. In order to eliminate these terms from (11), we will seek a function $p_c=p_c\left(s_w,s_g,p,T\right)$ such that

$$\nabla p_c=-\theta_w\nabla p_{ow}+\theta_g\nabla p_{go}+\frac{\partial p_c}{\partial p}\nabla p+\frac{\partial p_c}{\partial T}\nabla T. \tag{12}$$

This holds if and only if the following conditions are satisfied:

$$\frac{\partial p_c}{\partial s_w}=-\theta_w\frac{\partial p_{ow}}{\partial s_w}+\theta_g\frac{\partial p_{go}}{\partial s_w},$$

$$\frac{\partial p_c}{\partial s_g}=-\theta_w\frac{\partial p_{ow}}{\partial s_g}+\theta_g\frac{\partial p_{go}}{\partial s_g}. \tag{13}$$

A necessary and sufficient condition for the existence of the function p_c satisfying (13) is the equality of the mixed derivatives:

$$\frac{\partial^2 p_c}{\partial s_g\partial s_w}=\frac{\partial^2 p_c}{\partial s_w\partial s_g}$$

which leads to the condition

$$-\frac{\partial\theta_w}{\partial s_g}\frac{\partial p_{ow}}{\partial s_w}+\frac{\partial\theta_g}{\partial s_g}\frac{\partial p_{go}}{\partial s_w}=-\frac{\partial\theta_w}{\partial s_w}\frac{\partial p_{ow}}{\partial s_g}+\frac{\partial\theta_g}{\partial s_w}\frac{\partial p_{go}}{\partial s_g}. \tag{14}$$

Obviously, the condition (14), which is referred to as *the total differential condition* in [13], limits the choice of the functions p_{ow}, p_{go}, k_α, ρ_α and μ_α. When the condition (14) holds, the function p_c is defined as follows [13]:

$$p_c\left(s_w,s_g,p,T\right)=\int_1^{s_w}\left[-\theta_w\left(\eta,0,p,T\right)\frac{\partial p_{ow}}{\partial s_w}\left(\eta,0\right)+\right.$$

$$\left.+\theta_g\left(\eta,0,p,T\right)\frac{\partial p_{go}}{\partial s_w}\left(\eta,0\right)\right]d\eta+$$

$$+\int_0^{s_g}\left[-\theta_w\left(s_w,\eta,p,T\right)\frac{\partial p_{ow}}{\partial s_g}\left(s_w,\eta\right)+\right.$$

$$\left.+\theta_g\left(s_w,\eta,p,T\right)\frac{\partial p_{go}}{\partial s_g}\left(s_w,\eta\right)\right]d\eta \tag{15}$$

where p and T are considered as parameters. A direct check shows that the function p_c defined in (15) satisfies the condition (12) under the condition (14). Now we define the sought function p as

$$p=p_o+p_c. \tag{16}$$

From (10) and (11), we have

$$\vec{u}=-k\lambda\left(\gamma\nabla p-\xi\nabla T\right) \tag{17}$$

where $\gamma=1-\frac{\partial p_c}{\partial p}$, $\xi=\frac{\partial p_c}{\partial T}$. For simplicity, we assume that the density (as a function of pressure and temperature) changes slowly with the change of pressure, and hence [10, 13, 27],

$$\rho_\alpha = \rho_\alpha\left(p, T\right). \tag{18}$$

As in [13], the function p defined in (16) will be referred to as *the global pressure* in this work.

Using the definition of the global pressure (16) and the relation (17), the equations (1)-(6) reduce to the following set of equations for the global pressure p, saturations s_w, s_g and temperature T:

$$a_1\frac{\partial p}{\partial t} + b_1\frac{\partial T}{\partial t} - \nabla\cdot\left(a_2\nabla p\right) + \nabla\cdot\left(b_2\nabla T\right) = f_1, \tag{19}$$

$$\phi\rho_\alpha\frac{\partial s_\alpha}{\partial t} + \phi\rho_\alpha s_\alpha\left(\beta_{p_\alpha}\frac{\partial p}{\partial t} + \beta_{T_\alpha}\frac{\partial T}{\partial t}\right) - \nabla\cdot\left(\omega_\alpha\nabla s_\alpha\right) -$$

$$-\nabla\cdot\left(k\lambda_\alpha\nabla\left(p - p_c\right)\right) + I_\alpha = q_\alpha,\ \ \alpha = w,\, g, \tag{20}$$

$$a_3\frac{\partial T}{\partial t} + \vec{u}\cdot\nabla T - \nabla\cdot\left(k_T\nabla T\right) = f_2 \tag{21}$$

where

$$a_1\left(s_w, s_g, p, T\right) = \sum_\alpha \phi c_\alpha s_\alpha \rho_\alpha \beta_{p\alpha},$$

$$b_1\left(s_w, s_g, p, T\right) = \sum_\alpha \phi c_\alpha s_\alpha \rho_\alpha \beta_{T\alpha},$$

$$a_2\left(s_w, s_g, p, T\right) = k\lambda\gamma,\quad b_2\left(s_w, s_g, p, T\right) = k\lambda\xi, \tag{22}$$

$$a_3\left(s_w, s_g, p, T\right) = \phi\sum_\alpha \rho_\alpha s_\alpha c_\alpha + \left(1-\phi\right)\rho_r c_r,$$

$$f_1 = \sum_\alpha c_\alpha\left(q_\alpha - I_\alpha\right),\quad f_2 = q_T - \sum_\alpha q_\alpha i_\alpha,$$

$$\beta_{p_\alpha} = \frac{1}{\rho_\alpha}\frac{\partial\rho_\alpha}{\partial p},\ \ \beta_{T_\alpha} = \frac{1}{\rho_\alpha}\frac{\partial\rho_\alpha}{\partial T},\ \ \omega_w = \left|\frac{\partial p_{ow}}{\partial s_w}\right|,$$

$$\omega_g = \left|\frac{\partial p_{go}}{\partial s_g}\right|,\quad s_o = 1 - s_w - s_g.$$

The equation (19) was obtained from the equations (1) where the chain rule was applied to the first term, and the ratio (18) was used. Further, the resulting equations were multiplied by c_α and summed. The equation (21) was obtained by subtracting from the equation (3) the equations (1), multiplied by the corresponding enthalpy i_α. To derive the equations (20), the chain rule was applied to the first term of (1), and (18), (16) were used.

To determine the initial and boundary values for the global pressure, (16) is used. The initial and boundary conditions for s_w, s_g and T remain unchanged.

Numerical implementation of the problem

In this section, we present a method for solving the three-phase non-isothermal flow problem in the *global pressure - saturations - temperature* formulation on the example of a one-dimensional model problem. We consider a problem of displacement of oil by steam in the segment $\Omega = [0,\,1]$, at the ends of which injection and production wells are placed. In this case, the initial and boundary conditions can be defined as follows:

$$p\left(x,0\right) = p_0,\ \ T\left(x,0\right) = T_0,\ \ s_w\left(x,0\right) = s_{w0}, \tag{23}$$

$$p\left(0,t\right) = p_1,\ \ p\left(1,t\right) = p_0,\ \ T\left(0,t\right) = T_1,\ \ \frac{\partial}{\partial x}T\left(1,t\right) = 0,$$

$$s_w\left(0,t\right) = s_{w1},\ \ s_g\left(0,t\right) = s_{g1}. \tag{24}$$

For the numerical integration of the equations (19)-(21), we use the finite difference method in the present work. Let us introduce a uniform difference grid $\Omega_h = \left\{x_{(i)} = i\Delta x,\ i = 0,\,1,...,N,\ x_{(0)} = 0,\ x_{(N)} = 1\right\}$ in the domain Ω, and let the time segment $[0, t_1]$ be divided into discrete steps $t^n = n\tau$, $n = 0,\,1,\,...,\,M$, $t^0 = 0$, $t^M = t_1$. Then, we associate the following difference problem with the boundary value problem (19)-(24): find grid functions $p,\,T,\,s_w,\,s_g$ satisfying the set of the following grid equations in Ω_h:

$$a_1^n p_{\bar{t}}^{n+1} + b_1^n T_{\bar{t}}^{n+1} - \left(a_2^n p_{\bar{x}}^{n+1}\right)_x + \left(b_2^n T_{\bar{x}}^{n+1}\right)_x = f_1^n, \tag{25}$$

$$\phi\rho_\alpha^{n+1} s_{\alpha,\bar{t}}^{n+1} + \phi\rho_\alpha^{n+1} s_\alpha^{n+1}\left(\beta_{p\alpha}^n p_{\bar{t}}^{n+1} + \beta_{T\alpha}^n T_{\bar{t}}^{n+1}\right) -$$

$$-\left(\omega_\alpha^n s_{\alpha,\bar{x}}^{n+1}\right)_x - \left(k\lambda_\alpha^n\left(p_{\bar{x}}^{n+1} - p_{c,\bar{x}}^n\right)\right)_x + I_\alpha^n = q_\alpha^n,\ \ \alpha = w,\,g, \tag{26}$$

$$a_3^n T_{\bar{t}}^{n+1} - k\lambda^n\left(\gamma^n p_{\bar{x}}^n - \xi^n T_{\bar{x}}^n\right)T_x^{n+1} -$$

$$-\left(k_T T_{\bar{x}}^{n+1}\right)_x = f_2^n \tag{27}$$

and the boundary and initial conditions:

$$p_{(i)}^0 = p_0,\ \ T_{(i)}^0 = T_0,\ \ s_{w,(i)}^0 = s_{w0},\ \ 0 \le i \le N, \tag{28}$$

$$p_{(0)}^n = p_1,\ \ p_{(N)}^n = p_0,\ \ T_{(0)}^n = T_1,\ \ T_{\bar{x},(N)}^n = 0,$$

$$s_{w,(0)}^n = s_{w1},\ \ s_{g,(0)}^n = s_{g1},\ \ 1 \le\ n \le M. \tag{29}$$

Here we use the notations

$$u_{x,(i)} = \frac{u_{(i+1)} - u_{(i)}}{\Delta x},\ \ u_{\bar{x},(i)} = \frac{u_{(i)} - u_{(i-1)}}{\Delta x}.$$

The equations (25)-(27) are solved using the sweep method. The transition from the n-th time layer to the $(n+1)$-th time layer is carried out using the following algorithm. Using the values of p^n, s_w^n, s_g^n, T^n, the temperature field T^{n+1} is determined from the equation (27). Then, the global pressure p^{n+1}, water saturation s_w^{n+1} and steam saturation s_g^{n+1} are calculated from the equations (25) and (26), respectively. The obtained values are used to determine the phase transitions rate I^{n+1}. Iterations are conducted until the condition $\left|s_g^{n+1} - s_g^n\right| < \varepsilon$ is satisfied for some predetermined number $\varepsilon > 0$.

In the section entitled Numerical Results, we present the results of computational experiments carried out for the problem above.

A priori estimate

In this section, we derive an a priori estimate for the solution of the difference problem (25)-(29). We consider the space H of grid functions $u, v, ...$ with the scalar product and norm of the form

$$(u,\,v) = \sum_{i=1}^{N-1} u_{(i)}v_{(i)}\Delta x,\quad \|u\|^2 = \sum_{i=1}^{N-1} u_{(i)}^2\Delta x.$$

We also introduce the notations

$$(u,\, v] = \sum_{i=1}^{N} u_{(i)} v_{(i)} \Delta x, \quad ||u]|^2 = \sum_{i=1}^{N} u_{(i)}^2 \Delta x,$$

$$\|u\|_C = \max_{\Omega_h} |u| \,.$$

In the analysis of the difference scheme (25)-(29), we omit the phase transition terms and assume that the following conditions hold:

$$\lambda_\alpha \in C\,[0,\,1]\,, \;\; \rho_\alpha \in C\,[0,\,1]\,, \;\; p_c \in C^1\,[0,\,1]\,, \tag{30}$$

$$c_1^{-1} \le (\phi,\, k,\, \lambda,\, \rho_\alpha,\, c_\alpha,\, k_T,\, i) \le c_1, \tag{31}$$

$$c_2^{-1} \le (|\beta_{p_\alpha}|\,,\, |\beta_{T_\alpha}|\,,\, |\gamma|\,,\, |\xi|) \le c_2, \tag{32}$$

$$0 \le \omega_0 \le \omega_\alpha, \;\; (|q_\alpha|\,,\, |q_T|) \le q_0 < \infty,$$

$$(p_0,\, T_0,\, s_{\alpha 0},\, p_1,\, T_1) < \infty \tag{33}$$

where $c_1,\, c_2,\, \omega_0,\, q_0$ are some positive numbers and $\alpha = w,\, o,\, g$. Note that the conditions (31)-(32) imply that

$$a_0^{-1} \le a_k \le a_0, \;\; b_0^{-1} \le b_l \le b_0, \;\; k = 1,\, 2,\, 3, \;\; l = 1,\, 2$$

for some $a_0,\, b_0 > 0$.

Theorem. Let the assumptions (30)-(33) be satisfied. Then, there are numbers $M_0,\, \tau_0$ depending only on initial data and right-hand sides of the grid equations (25)-(27) such that for any $\tau \le \tau_0$, the inequality $E^n \le M_0$ holds for all $n > 0$ where

$$E^n = \|p^n\|^2 + \|T^n\|^2 + \|s_w^n\|^2 + \left\|s_g^n\right\|^2 .$$

Proof. Multiply the equation (25) by $2\tau p^{n+1}$:

$$\sum_{i=1}^{4} \zeta_i^{(1)} = \left(a_1^n p_{\bar t}^{n+1},\, 2\tau p^{n+1}\right) + \left(b_1^n T_{\bar t}^{n+1},\, 2\tau p^{n+1}\right) -$$

$$- \left(\left(a_2^n p_{\bar x}^{n+1}\right)_x,\, 2\tau p^{n+1}\right) + \left(\left(b_2 T_{\bar x}^{n+1}\right)_x,\, 2\tau p^{n+1}\right) =$$

$$= \left(f_1^n,\, 2\tau p^{n+1}\right). \tag{34}$$

Let us estimate the scalar products in (34) using the assumptions (30)-(33), the formula for summation by parts, and the Cauchy inequality with ε:

$$\left|\zeta_1^{(1)}\right| \ge 2\tau a_0^{-1} \left|\left(p_{\bar t}^{n+1},\, p^{n+1}\right)\right| =$$

$$= a_0^{-1} \left(\left\|p^{n+1}\right\|^2 - \|p^n\|^2 + \tau^2 \left\|p_{\bar t}^{n+1}\right\|^2\right),$$

$$\left|\zeta_2^{(1)}\right| \le 2\tau^2 b_0 \varepsilon_1 \left\|T_{\bar t}^{n+1}\right\|^2 + \frac{b_0}{16\varepsilon_1} \left|\left|p_{\bar x}^{n+1}\right]\right|^2,$$

$$\left|\zeta_3^{(1)}\right| = 2\tau \left(a_2^n p_{\bar x}^{n+1},\; p_{\bar x}^{n+1}\right] - 2\tau p_{(N)}^{n+1} a_{2,(N)}^n p_{\bar x,(N)}^{n+1} +$$

$$+ 2\tau p_{(0)}^{n+1} a_{2,(1)}^n p_{\bar x,(1)}^{n+1} \ge$$

$$\left|2\tau a_0^{-1} \left|\left|p_{\bar x}^{n+1}\right]\right|^2 - 2\tau p_1 a_0^{-1} \left\|p_{\bar x}^{n+1}\right\|_C\right|,$$

$$\left|\zeta_4^{(1)}\right| = -2\tau \left(b_2 T_{\bar x}^{n+1},\, p_{\bar x}^{n+1}\right] + 2\tau p_{(N)}^{n+1} b_{2,(N)}^n T_{\bar x,(N)}^{n+1} -$$

$$- 2\tau p_{(0)}^{n+1} b_{2,(1)}^n T_{\bar x,(1)}^{n+1}$$

$$\le 2\tau^2 b_0 \varepsilon_2 \left|\left|T_{\bar x}^{n+1}\right]\right|^2 + \frac{b_0}{2\varepsilon_2} \left|\left|p_{\bar x}^{n+1}\right]\right|^2 +$$

$$+ 2\tau p_1 b_0 \left\|T_{\bar x}^{n+1}\right\|_C ,$$

$$\left|\zeta_5^{(1)}\right| \le 2\tau\varepsilon_3 \|f_1^n\|^2 + \frac{\tau}{16\varepsilon_3} \left|\left|p_{\bar x}^{n+1}\right]\right|^2 .$$

Substituting these inequalities into (34), we obtain

$$\left\|p^{n+1}\right\|^2 - \|p^n\|^2 + \tau^2 \left\|p_{\bar t}^{n+1}\right\|^2 + \nu_1 \left|\left|p_{\bar x}^{n+1}\right]\right|^2 \le$$

$$\le 2\tau^2 a_0 b_0 \varepsilon_1 \left\|T_{\bar t}^{n+1}\right\|^2 + 2\tau^2 a_0 b_0 \varepsilon_2 \left|\left|T_{\bar x}^{n+1}\right]\right|^2 +$$

$$+ 2\tau p_1 a_0 b_0 \left\|T_{\bar x}^{n+1}\right\|_C + 2\tau a_0 \varepsilon_3 \|f_1^n\|^2 \tag{35}$$

where the positive numbers $\varepsilon_1,\, \varepsilon_2,\, \varepsilon_3$ are chosen such that

$$\nu_1 \equiv 2\tau - \frac{a_0 b_0}{16\varepsilon_1} - \frac{a_0 b_0}{2\varepsilon_2} - \frac{a_0 \tau}{16\varepsilon_3} > 0,$$

$$\nu_1' \equiv 1 - 2a_0 b_0 \varepsilon_1 > 0.$$

We now multiply the equation (27) by $2\tau T^{n+1}$:

$$\sum_{i=1}^{4} \zeta_i^{(2)} \equiv \left(a_3^n T_{\bar t}^{n+1},\, 2\tau T^{n+1}\right) -$$

$$- \left(k\lambda^n \gamma^n p_{\bar x}^n T_{\bar x}^{n+1},\, 2\tau T^{n+1}\right) +$$

$$+ \left(k\lambda^n \xi^n T_{\bar x}^n T_{\bar x}^{n+1},\, 2\tau T^{n+1}\right) -$$

$$- \left(\left(k_T T_{\bar x}^{n+1}\right)_x,\, 2\tau T^{n+1}\right) = \left(f_2^n,\, 2\tau T^{n+1}\right). \tag{36}$$

Let us dwell on the evaluation of the second, third and fourth terms in the left-hand side of (36). Under the same assumptions, we have

$$\left|\zeta_2^{(2)}\right| \le 2\tau c_1^2 c_2 \left\|p_{\bar x}^n T_{\bar x}^{n+1}\right\| \left\|T^{n+1}\right\| \le$$

$$\le 2\tau^2 \varepsilon_4 c_1^2 c_2 \|p_{\bar x}^n\|^2 \left\|T_{\bar x}^{n+1}\right\|^2 + \frac{c_1^2 c_2}{2\varepsilon_4} \left(\left|\left|T_{\bar x}^{n+1}\right]\right|^2 + T_1^2\right),$$

$$\left|\zeta_3^{(2)}\right| \le 2\tau^2 \varepsilon_5 c_1^2 c_2 \|T_{\bar x}^n\|^2 \left\|T_{\bar x}^{n+1}\right\|^2 +$$

$$+ \frac{c_1^2 c_2}{2\varepsilon_5} \left(\left|\left|T_{\bar x}^{n+1}\right]\right|^2 + T_1^2\right),$$

$$\left|\zeta_4^{(2)}\right| \ge -2\tau c_1^{-1} \left|\left|T_{\bar x}^{n+1}\right]\right|^2 - 2\tau T_1 \left\|T_{\bar x,(1)}^{n+1}\right\|_C .$$

Applying the obtained inequalities, the equation (36) reduces to the form

$$\left\|T^{n+1}\right\|^2 - \|T^n\|^2 + \tau^2 \left\|T_{\bar t}^{n+1}\right\|^2 + 2\tau a_0 c_1^{-1} \left|\left|T_{\bar x}^{n+1}\right]\right|^2 +$$

$$+ 2\tau a_0 T_1 \left\|T_{\bar x,(1)}^{n+1}\right\|_C \le \tau^2 \varepsilon_4 a_0 c_1^2 c_2 \|p_{\bar x}^n\|^2 \left\|T_{\bar x}^{n+1}\right\|^2 +$$

$$+ \frac{a_0 c_1^2 c_2}{2\varepsilon_4} \left(\left|\left|T_{\bar x}^{n+1}\right]\right|^2 + T_1^2\right) +$$

$$+2\tau^2\varepsilon_5 a_0 c_1^2 c_2 \left\|T_{\overline{x}}^n\right\|^2 \left\|T_{\overline{x}}^{n+1}\right\|^2 +$$

$$+\frac{a_0 c_1^2 c_2}{2\varepsilon_5}\left(\left|\left|T_{\overline{x}}^{n+1}\right]\right|^2 + T_1^2\right) +$$

(37) $$+2\tau a_0 \varepsilon_6 \left\|f_2^n\right\|^2 + \frac{a_0\tau}{16\varepsilon_6}\left|\left|T_{\overline{x}}^{n+1}\right]\right|^2 .$$

Now, we add the inequalities (35) and (37) and drop the last term on the left-hand side of (37). Choosing the positive numbers ε_4, ε_5, ε_6 and τ_1 such that for all $\tau \le \tau_1$

$$\nu_2' = 2\tau a_0\left(c_1^{-1} - \tau c_1^2 c_2\left(\varepsilon_4\left\|p_{\overline{x}}^n\right\|^2 + \varepsilon_5\left\|T_{\overline{x}}^n\right\|^2\right) - \tau b_0\varepsilon_2\right) -$$

$$-2\tau a_0 p_1 b_0 - \frac{\tau a_0}{16\varepsilon_6} - \frac{a_0 c_1^2 c_2}{2}\left(\varepsilon_4^{-1} + \varepsilon_5^{-1}\right),$$

we obtain:

$$\left\|p^{n+1}\right\|^2 - \left\|p^n\right\|^2 + \tau^2\left\|p_{\overline{t}}^{n+1}\right\|^2 + \nu_1\left|\left|p_{\overline{x}}^{n+1}\right]\right|^2 + \left\|T^{n+1}\right\|^2 -$$

$$-\left\|T^n\right\|^2 + \tau^2\nu_1'\left\|T_{\overline{t}}^{n+1}\right\|^2 + \nu_2'\left|\left|T_{\overline{x}}^{n+1}\right]\right|^2 \le$$

(38)
$$\le \frac{a_0 c_1^2 c_2 T_1^2}{2}\left(\varepsilon_4^{-1} + \varepsilon_5^{-1}\right) + 2\tau a_0\varepsilon_3\left\|f_1^n\right\|^2 + 2\tau a_0\varepsilon_6\left\|f_2^n\right\|^2 .$$

Now, we multiply (26) by $2\tau s_\alpha^{n+1}$:

$$\sum_{i=1}^{6}\zeta_i^{(3)} \equiv \left(\phi\rho_\alpha^{n+1} s_{\alpha,\overline{t}}^{n+1},\, 2\tau s_\alpha^{n+1}\right) +$$

$$+\left(\phi\rho_\alpha^{n+1} s_\alpha^{n+1}\beta_{p\alpha}^n p_{\overline{t}}^{n+1},\, 2\tau s_\alpha^{n+1}\right) +$$

$$+\left(\phi\rho_\alpha^{n+1} s_\alpha^{n+1}\beta_{T\alpha}^n T_{\overline{t}}^{n+1},\, 2\tau s_\alpha^{n+1}\right) +$$

$$-\left(\left(\omega_\alpha^n s_{\alpha,\overline{x}}^{n+1}\right)_x,\, 2\tau s_\alpha^{n+1}\right) - \left(\left(k\lambda_\alpha^{n+1} p_{\overline{x}}^{n+1}\right)_x,\, 2\tau s_\alpha^{n+1}\right) +$$

(39) $$+\left(\left(k\lambda_\alpha^{n+1} p_{c,\overline{x}}^n\right)_x,\, 2\tau s_\alpha^{n+1}\right) = \left(q_\alpha^n,\, 2\tau s_\alpha^{n+1}\right).$$

We consider the second, third and fourth terms in detail. The reasoning is similar for the rest terms.

$$\left|\zeta_2^{(3)}\right| \le 2\tau c_1^2 c_2\left|\left(s_\alpha^{n+1} p_{\overline{t}}^{n+1},\, s_\alpha^{n+1}\right)\right| \le$$

$$\le 2c_1^2 c_2\left\|\tau s_\alpha^{n+1} p_{\overline{t}}^{n+1}\right\|\left\|s_\alpha^{n+1}\right\| \le$$

$$\le 2\tau^2 c_1^2 c_2\varepsilon_{7,\alpha}\left\|s_\alpha^{n+1}\right\|^2\left\|p_{\overline{t}}^{n+1}\right\|^2 + \frac{c_1^2 c_2}{2\varepsilon_{7,\alpha}}\left\|s_\alpha^{n+1}\right\|^2,$$

$$\left|\zeta_3^{(3)}\right| \le 2\tau^2 c_1^2 c_2\varepsilon_{8,\alpha}\left\|s_\alpha^{n+1}\right\|^2\left\|T_{\overline{t}}^{n+1}\right\|^2 + \frac{c_1^2 c_2}{2\varepsilon_{8,\alpha}}\left\|s_\alpha^{n+1}\right\|^2,$$

$$\left|\zeta_4^{(3)}\right| = \left|2\tau\left(\omega_\alpha s_{\alpha,\overline{x}}^{n+1},\, s_{\alpha,\overline{x}}^{n+1}\right] - 2\tau s_{\alpha,(N)}^{n+1}\omega_{(N)}^n s_{\alpha,\overline{x}(N)}^{n+1} +\right.$$

$$\left.+2\tau s_{\alpha,(0)}^{n+1}\omega_{(1)}^n s_{\alpha,\overline{x}(1)}^{n+1}\right| \ge$$

$$\ge \left|2\tau\omega_0\left\|s_{\alpha,\overline{x}}^{n+1}\right\|^2 - 2\tau s_{\alpha 1}\left|\omega_{(1)}^n\right|\left\|s_{\alpha,\overline{x}(1)}^{n+1}\right\|_C\right|.$$

Substituting these inequalities into (39), we obtain:

$$\left\|s_\alpha^{n+1}\right\|^2 - \left\|s_\alpha^n\right\|^2 + \tau^2\left\|s_{\alpha,\overline{t}}^{n+1}\right\|^2 + 2\tau c_1^2\omega_0\left\|s_{\alpha,\overline{x}}^{n+1}\right\|^2 +$$

$$+2\tau c_1^2 s_{\alpha 1}\left|\omega_{(1)}^n\right|\left\|s_{\alpha,\overline{x}(1)}^{n+1}\right\|_C \le 4\tau^2 c_1^4 c_2\varepsilon_{7,\alpha}\left\|s_{\alpha,\overline{x}}^{n+1}\right\|^2 \times$$

$$\times\left\|p_{\overline{t}}^{n+1}\right\|^2 + \frac{c_1^4 c_2}{\varepsilon_{7,\alpha}}\left\|s_{\alpha,\overline{x}}^{n+1}\right\|^2 + 4\tau^2 c_1^4 c_2\varepsilon_{8,\alpha}\left\|s_{\alpha,\overline{x}}^{n+1}\right\|^2 \times$$

$$\times\left\|T_{\overline{t}}^{n+1}\right\|^2 + \frac{c_1^4 c_2}{\varepsilon_{8,\alpha}}\left\|s_\alpha^{n+1}\right\|^2 + 2c_1^4\tau^2\varepsilon_{9,\alpha}\left\|p_{\overline{x}}^{n+1}\right\|^2 +$$

$$+\frac{c_1^4}{2\varepsilon_{9,\alpha}}\left\|s_{\alpha,\overline{x}}^{n+1}\right\|^2 + 4c_1^4\tau^2\left\|s_{\alpha,\overline{x}}^{n+1}\right\|^2 + 6s_{\alpha,(0)}^2 + c_1^2\nu_{8,\alpha} +$$

$$+2\tau c_1^2\varepsilon_{11,\alpha}\left\|q_\alpha^n\right\|^2 + \frac{\tau c_1^2}{2\varepsilon_{11,\alpha}}\left\|s_{\alpha,\overline{x}}^{n+1}\right\|^2 .$$

Choosing the positive numbers ε_k, $k = 7, 8, ..., 11$ and τ_2 such that for all $\tau \le \tau_2$:

$$\nu_3 = 2\tau c_1^2\left(\omega_0 - 2\tau c_1^2 c_2\left(\varepsilon_{7,\alpha}\left\|p_{\overline{t}}^{n+1}\right\|^2 + \varepsilon_{8,\alpha}\left\|T_{\overline{t}}^{n+1}\right\|^2 +\right.\right.$$

$$\left.\left.+\frac{\varepsilon_{10,\alpha}}{c_2}\right) - \frac{1}{4}\varepsilon_{11,\alpha}^{-1}\right) - c_1^4 c_2\left(\varepsilon_{7,\alpha}^{-1} + \varepsilon_{8,\alpha}^{-1} + \frac{1}{2}\varepsilon_{9,\alpha}^{-1}\right)$$

and dropping the last term in the left-hand side of the inequality, we have:

$$\left\|s_\alpha^{n+1}\right\|^2 - \left\|s_\alpha^n\right\|^2 + \tau^2\left\|s_{\alpha,\overline{t}}^{n+1}\right\|^2 + \nu_3\left\|s_{\alpha,\overline{x}}^{n+1}\right\|^2 \le$$

(40)
$$\le 2c_1^4\tau^2\varepsilon_{9,\alpha}\left\|p_{\overline{x}}^{n+1}\right\|^2 + 6s_{\alpha 1}^2 + c_1^2\nu_{8,\alpha} + 2\tau c_1^2\varepsilon_{11,\alpha}\left\|q_\alpha^n\right\|^2 .$$

Multiply the inequalities (40) by $\eta_w > 0$ and $\eta_g > 0$, respectively, and sum the resulting inequalities with (38) to obtain

$$\left\|p^{n+1}\right\|^2 - \left\|p^n\right\|^2 + \tau^2\left\|p_{\overline{t}}^{n+1}\right\|^2 + \nu_1\left|\left|p_{\overline{x}}^{n+1}\right]\right|^2 + \left\|T^{n+1}\right\|^2 -$$

$$-\left\|T^n\right\|^2 + \tau^2\nu_1'\left\|T_{\overline{t}}^{n+1}\right\|^2 + \nu_2'\left|\left|T_{\overline{x}}^{n+1}\right]\right|^2 +$$

$$+\sum_{\alpha=w,g}\eta_\alpha\left(\left\|s_\alpha^{n+1}\right\|^2 - \left\|s_\alpha^n\right\|^2 + \tau^2\left\|s_{\alpha,\overline{t}}^{n+1}\right\|^2 + \nu_3\left\|s_{\alpha,\overline{x}}^{n+1}\right\|^2\right) \le$$

$$\le \sum_{\alpha=w,g}\eta_\alpha\left(2c_1^4\tau^2\varepsilon_{9,\alpha}\left\|p_{\overline{x}}^{n+1}\right\|^2 + 6s_{\alpha 1}^2 + c_1^2\nu_{8,\alpha} +\right.$$

$$\left.+\,2\tau c_1^2\varepsilon_{11,\alpha}\left\|q_\alpha^n\right\|^2\right) + \frac{a_0 c_1^2 c_2 T_1^2}{2}\left(\varepsilon_4^{-1} + \varepsilon_5^{-1}\right) +$$

(41) $$+2\tau a_0\varepsilon_3\left\|f_1^n\right\|^2 + 2\tau a_0\varepsilon_6\left\|f_2^n\right\|^2 .$$

Choosing η_w, η_g such that

$$\nu_4 \equiv \nu_1 - \sum_{\alpha=w,g}\eta_\alpha 2c_1^4\tau^2\varepsilon_{9,\alpha} > 0,$$

we obtain:

$$\left\|p^{n+1}\right\|^2 - \left\|p^n\right\|^2 + \tau^2\left\|p_{\overline{t}}^{n+1}\right\|^2 + \nu_4\left|\left|p_{\overline{x}}^{n+1}\right]\right|^2 + \left\|T^{n+1}\right\|^2 -$$

$$-\|T^n\|^2+\tau^2\nu_1'\left\|T_{\bar{t}}^{n+1}\right\|^2+\nu_2'\left\|T_{\overline{x}}^{n+1}\right\|^2+$$

$$+\sum_{\alpha=w,g}\eta_\alpha\left(\left\|s_\alpha^{n+1}\right\|^2-\|s_\alpha^n\|^2+\tau^2\left\|s_{\alpha,\bar{t}}^{n+1}\right\|^2+\nu_3\left\|s_{\alpha,\overline{x}}^{n+1}\right\|^2\right)\leq$$

$$\leq\sum_{\alpha=w,g}\eta_\alpha\left(6s_{\alpha 1}^2+c_1^2\nu_{8,\alpha}+2\tau c_1^2\varepsilon_{11,\alpha}\|q_\alpha^n\|^2\right)+$$

(42)
$$+\frac{a_0c_1^2c_2T_1^2}{2}\left(\varepsilon_4^{-1}+\varepsilon_5^{-1}\right)+2\tau a_0\varepsilon_3\|f_1^n\|^2+2\tau a_0\varepsilon_6\|f_2^n\|^2.$$

Introducing the notations

$$\nu=\sum_{\alpha=w,g}\eta_\alpha\left(6s_{\alpha 1}^2+c_1^2\nu_{8,\alpha}\right)+\frac{a_0c_1^2c_2T_1^2}{2}\left(\varepsilon_4^{-1}+\varepsilon_5^{-1}\right)<\infty,$$

$$\tau_0=\min\{\tau_1,\,\tau_2\},\quad \nu_5=\max\left\{2c_1^2\varepsilon_{11,\alpha},\ 2a_0\varepsilon_3,\ 2a_0\varepsilon_6\right\}$$

and dropping the last term in the left-hand side of (42), we obtain the following inequality for all $\tau<\tau_0$:

$$\left\|p^{n+1}\right\|^2+\left\|T^{n+1}\right\|^2+\eta_w\left\|s_w^{n+1}\right\|^2+\eta_g\left\|s_g^{n+1}\right\|^2\leq\|p^n\|^2+$$

$$+\|T^n\|^2+\eta_w\|s_w^n\|^2+\eta_g\left\|s_g^n\right\|^2+\nu+$$

(43) $$+\nu_5\tau\left(\|f_1^n\|^2+\eta_w\|q_w^n\|^2+\eta_g\left\|q_g^n\right\|^2+\|f_2^n\|^2\right).$$

Applying (43) $n+1$ times, we obtain the inequality:

$$\|p^n\|^2+\|T^n\|^2+\eta_w\|s_w^n\|^2+\eta_g\left\|s_g^n\right\|^2\leq\|p_0\|^2+\|T_0\|^2+$$

$$+\eta_w\|s_{w0}\|^2+\eta_g\|s_{g0}\|^2+\nu_5t_1\left(\|f_1\|_*^2+\eta_w\|q_w\|_*^2+\right.$$

$$\left.+\eta_g\|q_g\|_*^2+\|f_2\|_*^2\right)+M\nu$$

where

$$\|u\|_*=\max_{0\leq n\leq M}\|u^n\|.$$

Then, denoting

$$M_0=\left(\|p_0\|^2+\|T_0\|^2+\sum_{\alpha=w,g}\eta_\alpha\left(\|s_{\alpha 0}\|^2+\nu_5t_1\|q_\alpha\|_*^2\right)+\right.$$

$$\left.+\nu_5t_1\sum_{k=1}^{2}\|f_k\|_*^2+M\nu\right)\nu_6^{-1},$$

$$\nu_6=\min\{1,\,\eta_w,\,\eta_g\},$$

we arrive at the inequality $E^n\leq M_0$. The theorem is proved.

Numerical results

To test the adequacy of the proposed model, numerical experiments were conducted. We accept the following values of the input parameters: $p_1=T_1=1$, $p_0=T_0=0$, $s_{w0}=s_{w1}=0.3$, $s_{g1}=0.7$, $s_{rw}=0.29$, $N=10^3$, $\Delta t=10^{-5}$. For the simulation of phase transitions, the methodology proposed in [4] is used. For determination of relative permeabilities and capillary pressures, the following simplified relations are used:

(44) $$k_w=s_w,\quad k_o=1-s_w-s_g,\quad k_g=s_g,$$

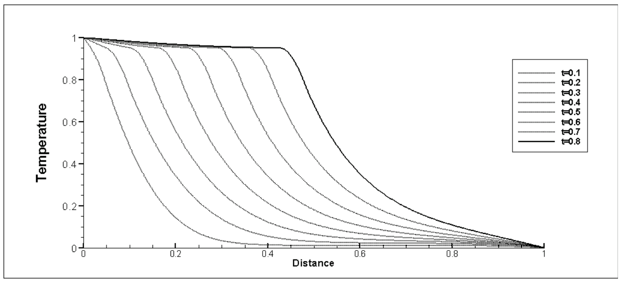

Fig. 1. Temperature distribution

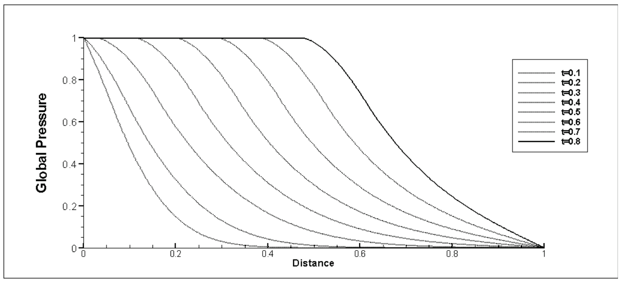

Fig. 2. Distribution of the global pressure

(45)
$$p_{ow}=-0.01\cdot\ln s_w,\quad p_{go}=0.1+0.01\cdot\ln\left(0.0004\cdot s_g\right).$$

To determine the density, viscosity and specific heat capacity of the phases, the following dimensionless relations are used:

$$\rho_w=1-0.1\cdot T^2,\ \rho_g=0.08\cdot T^2+1.568\cdot 10^{-3},$$

(46) $$\frac{\rho_o}{\rho_{o,\mathrm{ref}}}=1+\beta_{p,o}\left(p-p_{\mathrm{ref}}\right)+\beta_{T,o}\left(T-T_{\mathrm{ref}}\right),$$

$$\mu_w=0.005\cdot\left(1-0.1\cdot T^2\right),\ \ \mu_o=0.2+10\cdot(1-T)^3,$$

(47) $$\mu_g=2.4\cdot 10^{-5}\left(T^2+0.02\right),$$

(48) $$c_w=1,\ \ c_o=0.56,\ \ c_g=0.6$$

where the index ref indicates the value calculated at the initial conditions. A direct check shows that with this choice of data, the condition (14) is satisfied identically.

The process of oil recovery by steam is usually characterized by the development of three zones differing in temperature, filtration properties and the nature of saturation, namely, steam zone, variable temperature zone and the ambient temperature zone. As shown in Figure 1, in a neighbourhood of the injection well, the reservoir temperature is equal to the temperature of steam injected, and it then slowly decreases due to heat loss to the surrounding rocks. In the second zone, the temperature varies from the condensation

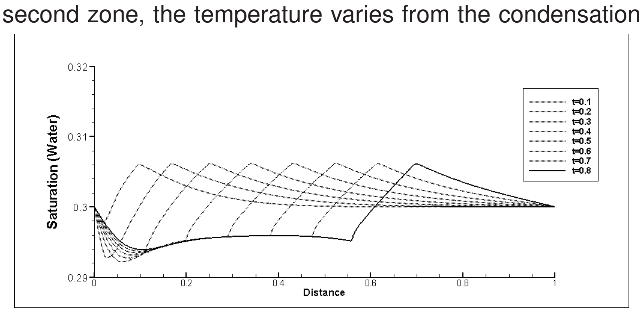

Fig. 3. Distribution of water saturation

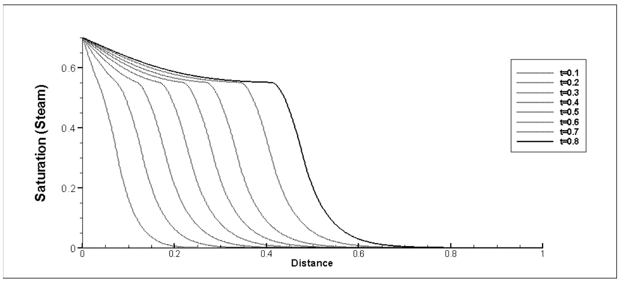

Fig. 4. Distribution of steam saturation

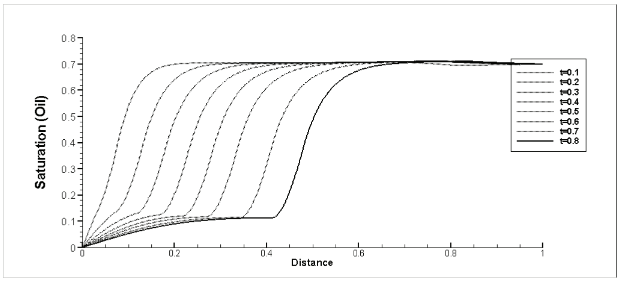

Fig. 5. Distribution of oil saturation

temperature to the initial reservoir temperature, wherein heating of the reservoir is carried out by the hot condensate. The third zone is an area not covered by steam; oil displacement in this zone is carried out by water at a temperature equal to the temperature of the surrounding rocks. With the growth of the injected agent, the first and second zones are gradually expanding, and the third one is shortened.

Figure 2 shows the dynamics of changes in the global pressure at regular intervals $t = 0.1, 0.2, ..., 0.8$. Obviously, the greatest intensity of fluids occurs near the injection well followed by the decrease in the direction of heat transfer agent flow. In this regard, the steam zone and the variable temperature zone are characterized by the maximum change in the global pressure.

Figures 3, 4 and 5 show the profiles of the water, steam and oil saturation. Near the injection well, saturation of steam increases due to evaporation of water, initially saturated the reservoir. When passing to the variable temperature zone, condensation of injected steam takes place, so water saturation increases dramatically in this area.

Conclusion

Thus, in this work, we obtained a new fractional flow formulation for the three-phase non-isothermal compressible flow problem taking into account capillary forces and phase transitions between the phases of water and heat transfer agent. Unlike the classical phase formulation, the gradients of capillary pressure functions are eliminated from the equations for calculating the pressure and temperature, which lead to unbounded growth of the solution at residual saturations. Research in this area could be improved in several ways. Firstly, in this paper, simplified capillary pressure functions, relative permeability and viscosity functions are used to identically satisfy the total differential condition (14). Secondly, a simplifying assumption was made that the densities of the phases depend on the global pressure, and not the pressure of corresponding phases. However, the simulation results obtained by solving a one-dimensional problem using simplified curves of relative permeabilities and capillary pressures reproduce the characteristic features of the process of oil displacement by steam that allows the use of the proposed approach in more complex non-isothermal flow problems.

Authors: *Prof. Waldemar Wójcik, Institute of Electronics and Information Technology, Electrical Engineering and Computer Science Faculty, Lublin University of Technology, ul. Nadbystrzycka 38A, 20-618 Lublin, Poland, email: waldemar.wojcik@pollub.pl, Prof. Nurlan Temirbekov, D.Serikbayev East Kazakhstan State Technical University, Ust-Kamenogorsk, Kazakhstan , email: temirbekov@rambler.ru PhD student Dossan Baigereyev, D.Serikbayev East Kazakhstan State Technical University, Ust-Kamenogorsk, Kazakhstan , email: dbaigereyev@gmail.com*

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