Volodymyr EMETS, Jan ROGOWSKI

Institute of Information Technology, Lodz University of Technology

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Solution of the Plane Wave Diffraction Problem by a Strip with Different Boundary Conditions on its Faces

Abstract. *A two-dimensional problem of electromagnetic wave diffraction by a plane strip with different boundary conditions on its sides is in the focus of our study. The diffracted field is expressed by an integral in terms of the induced electric and magnetic current densities. Applying the representation theorem for the scattering field, the problem is reduced to the solutions of the integral equations. Numerical solutions and high-frequency asymptotics for the scattering field in the wave zone of the inclusion are presented.*

Streszczenie. *W artykule przedstawiono dwuwymiarowy problem dyfrakcji elektromagnetycznych fal na płaskim włączeniu z różnymi warunkami brzegowymi na jego powierzchniach. Pole dyfrakcji przedstawiono w postaci całki indukowanych elektrycznych i magnetycznych gęstości prądu. Stosując twierdzenie o reprezentacji dla rosproszonego pola problem sprowadza się do rozwiązań układu równań całkowych. Przedstawiono numeryczne oraz asymptotyczne rozwiązania dla zakresu wysokich częstotliwości dla rozproszonego pola w strefie falowej włączenia.**(Rozwiązanie problemu dyfrakcji elektromagnetycznej fali na płaskim włączeniu z różnymi warunkami brzegowymi na jego powierzchniach)***

Keywords: Wave diffraction, system of integral equations, analytical-numerical solutions, high-frequency asymptotics.
Słowa kluczowe: Dyfrakcja falowa, układ równań całkowych, rozwiązania analityczno-numeryczne, asymptotyka dla zakresu wysokich częstotliwości.

Introduction

A large number of practical problems can be simulated by conducting, resistive or impedance strips. Diffraction of waves by strips in an infinite conducting plane have been extensively investigated by many authors by using different analytical and numerical methods [1-9]. It is well known that the electrical size of the body restricts the tractability of the numerical methods while the geometrical complexity of the object limits the applicability of the analytical methods. In the high-frequency domain, for the asymptotic solution, the hybrid methods are used besides techniques based upon the extension of the Kirchhoff approximation. The hybrid methods based on the numerical and high frequency asymptotic techniques have the potential to enlarge the class of electromagnetic scattering problems that can be treated. It is noted that the hybrid method presented in [10] uses the solution of a boundary value problem while the method presented in this paper should be based on the solution of the mixed boundary value problem.

Problem Formulation

Let us assume that an E-polarized plane wave, described by the function $u^i(\mathbf{x}) = E^i_{x_2}(\mathbf{x})$ is an incident field being scattered by a strip located at the plane $x_3 = 0$ for $(|x_1| \le a)$ and $(|x_2| < \infty)$ as shown in Fig. 1, where (x_1, x_2, x_3) are the Cartesian coordinates. Since the strip is uniform along the x_2 axis, the problem can be reduced to a two dimensional scalar problem. The incident field $u^i(\mathbf{x})$ is

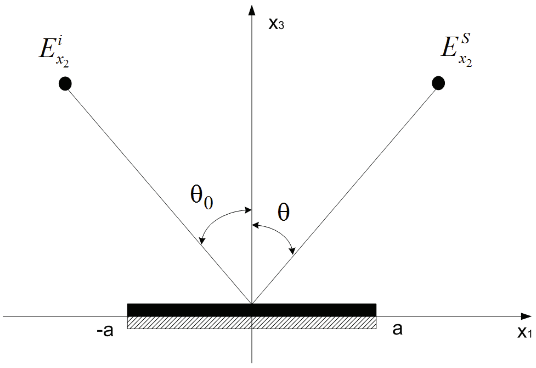

Fig. 1. Geometry of the problem.

given as a linearly polarised plane wave

$$u^i(\mathbf{x}) = A_0 e^{ik(\mathbf{l},\mathbf{x})},\ \mathbf{x} = (x_1, x_3), \tag{1}$$

$$\mathbf{l} = (\sin\theta_0, -\cos\theta_0),$$

where θ_0 denotes the incident angle, k is the wave number and A_0 is the amplitude. A time factor $e^{-i\omega t}$ is assumed and suppressed throughout. The total field $u(\mathbf{x}) = u^i(\mathbf{x}) + u^s(\mathbf{x})$, $u^s(\mathbf{x}) = E^s_{x_2}(\mathbf{x})$ is decomposed into the given incident wave $u^i(\mathbf{x})$ and the unknown scattered field $u^s(\mathbf{x})$ which is required to satisfy the Sommerfeld radiation condition at infinity, from which it follows that

$$u^s(\mathbf{x}) = \frac{e^{ik\mathbf{x}+\mathrm{i}\pi/4}}{\sqrt{8\pi k|\mathbf{x}|}} f(\omega;\mathbf{l},\boldsymbol{\nu}),\ |\mathbf{x}| \to \infty, \tag{2}$$

where $f(\omega, \mathbf{l}, \boldsymbol{\nu})$ is the far-field pattern of the scattering wave, $\boldsymbol{\nu} = x/|x| = (\sin\theta, \cos\theta)$ with θ denoting the observation angle.

The total field satisfies in the surrounding medium the Helmholtz equation

$$(\Delta + k^2)u(\mathbf{x}) = 0, \tag{3}$$

and the following boundary conditions:

$$u^+(x_1) = 0,\ \frac{\partial u^-}{\partial x_3} = 0,\ |x_1| < a, \tag{4}$$

where $u^\pm = u(x_1, x_3)$ for $x_3 \to \pm 0$. The upper (+) and lower (-) sides of the interface $|x_1| < a$ in presentation (4) correspond to a perfect electric and magnetic conductor materials, respectively.

By considering the densities of surface electric and magnetic currents which are denoted by $\Phi_1(x_1)$ and $\Phi_3(x_1)$ respectively, and using Green's theorem, the integral representation of the scattered field can be obtained as

(5)

$$u^s(\mathbf{x}) = k\int_{-a}^{a}\left[g(\mathbf{x},\mathbf{y})\Phi_1(y_1) - \frac{\Phi_3(y_1)}{k}\frac{\partial g(\mathbf{x},\mathbf{y})}{\partial y_3}\right]_{y_3=0} dy_1,$$

$$g(\mathbf{x},\mathbf{y}) = -\frac{i}{4}H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|), \mathbf{y} = (y_1, y_3),$$

$$k\Phi_1(x_1) = \left(\frac{\partial u^+(\mathbf{x})}{\partial x_3} - \frac{\partial u^-(\mathbf{x})}{\partial x_3}\right)\Bigg|_{x_3=0},$$

$$\Phi_3(x_1) = u^+(x_1) - u^-(x_1),$$

where $H_0^{(1)}$ is the Hankel function of the first kind.

As a middle surface of the scatterer is a plane, let us use the expansion of fundamental solution of the Helmholtz equation $g(\mathbf{x},\mathbf{y})$ (cylindrical wave) through plane waves. This will allow to deal with symbols of corresponding pseudo-differential operators only. As a result the following integral equations for $\Phi_\beta,\ \beta=1,3$ are obtained from Eqs. (1)-(5):

(6) $$\Phi_1(x_1)+k\int_{-a}^{a}\Phi_3(p)K_1\left(k\left|x_1-p\right|\right)dp=q_1e^{ikl_1x_1},$$

$$\Phi_3(x_1)+k\int_{-a}^{a}\Phi_1(p)K_3\left(k\left|x_1-p\right|\right)dp=q_3e^{ikl_1x_1},$$

$$|x_1|<a,$$

$$K_1(|z|)=\frac{1}{2\pi}\int_\Gamma\gamma(\alpha)e^{\pm i\alpha z}d\alpha,$$

$$K_3(|z|)=-\frac{1}{2\pi}\int_\Gamma\gamma^{-1}(\alpha)e^{\pm i\alpha z}d\alpha,$$

$$q_1=-2A_0i\cos(\theta_0),\ q_3=-2A_0,\ \gamma(\alpha)=\sqrt{\alpha^2-1}.$$

Herein the contour Γ coincides with the real axis everywhere except for the branching points $\alpha=\pm1$ and passes these points below in the right-hand half-plane of complex variable α and above in the left-hand one according to the limiting absorption principle. The point $\alpha=0$ is situated below the contour Γ and for $|\alpha|<1$ the radical $\gamma(\alpha)$ is defined by the condition $\mathrm{Im}\sqrt{\alpha^2-1}<0$.

Analytical Solution

The principal terms of the asymptotic expansion (as $x=ka\gg1$) of the solution to the equations (6) are represented in the form

(7) $$\Phi_\beta(x_1)=\Phi_\beta^+(x_1)+\Phi_\beta^-(x_1)-\vartheta_\beta(x_1),\ |x_1|<a,$$

where the functions $\Phi_\beta^\pm(\eta)=\Phi_\beta\left(\mp a\pm\eta k^{-1}\right)\exp\left(\pm ixl_1\right)$ satisfy the corresponding convolution-type integral equations

(8)
$$\Phi_1^\pm(\eta)+\int_0^\infty\Phi_3^\pm(\zeta)K_1\left(|\eta-\zeta|\right)d\zeta=q_1\exp\left(\pm i\eta l_1\right),$$

$$\Phi_3^\pm(\eta)+\int_0^\infty\Phi_1^\pm(\zeta)K_3\left(|\eta-\zeta|\right)d\zeta=q_3\exp\left(\pm i\eta l_1\right),$$

$$0<\eta<\infty,$$

$$\vartheta_\beta(\eta)=\Phi_\beta(\eta k^{-1}),\ \vartheta_1(\eta)=q_1\exp(i\eta l_1),\ \vartheta_3(\eta)=0.$$

The Fourier transform

$$\varphi_\beta^\pm(\alpha)=\int_0^\infty\Phi_\beta^\pm(\eta)\exp(i\alpha\eta)\,d\eta,$$

$$Y_\beta^-(\alpha)=\int_{-\infty}^0\Phi_\beta^\pm(\eta)\exp(i\alpha\eta)\,d\eta$$

can be employed to reduce the integral equations (8) to a matrix Wiener-Hopf equation

(9)
$$H(\alpha)\begin{pmatrix}\varphi_3^\pm(\alpha)\\ \varphi_1^\pm(\alpha)\end{pmatrix}=\frac{i}{\alpha\pm l_1}\begin{pmatrix}q_1\\ q_3\end{pmatrix}+H(\alpha)\begin{pmatrix}Y_3^-(\alpha)\\ Y_1^-(\alpha)\end{pmatrix},$$

$$\alpha\in\Gamma$$

in which the matrix kernel has the form

$$H(\alpha)=\begin{pmatrix}1&\gamma\\ -\gamma^{-1}&1\end{pmatrix}.$$

In order to obtain an unique solution of (9) it is necessary to take into account the following edge conditions:

$$u^s(r,\phi)=\left(\cos\frac{\phi}{4}+\sin\frac{\phi}{4}\right)r^{1/4}+o\left(r^{1/4}\right),$$

$$-\pi\le\phi\le\pi,\ r\to0,$$

where r is the distance from inhomogeneity tip and ϕ is the polar angle. Now, it can be shown that

(10)
$$\varphi_1^\pm(\alpha)=O\left(\alpha^{-1/4}\right),\ \varphi_3^\pm(\alpha)=O\left(\alpha^{-5/4}\right),\ |\alpha|\to\infty.$$

The first step in the factorization procedure is to rearrange the matrix kernel of (9) into the Khrapkov-like form [11]:

(11) $$H(\alpha)=\mathrm{I}+\frac{1}{\gamma}J(\alpha),\ J(\alpha)=\begin{pmatrix}0&\gamma^2\\ -1&0\end{pmatrix},$$

$$J^2(\alpha)=-\gamma^2\mathrm{I},$$

where I is the identity matrix. The matrix given by (11) is of a special form which can be factorized through the Khrapkov method. The result is

$$H(\alpha)=H_+(\alpha)H_-(\alpha)=H_-(\alpha)H_+(\alpha),$$

$$H_\pm(\alpha)=\sqrt{2}\cos\left[\gamma(\alpha)\theta_\pm(\alpha)\right]\mathrm{I}+$$

$$+\frac{\sqrt{2}}{\gamma(\alpha)}\sin\left[\gamma(\alpha)\theta_\pm(\alpha)\right]J(\alpha),$$

$$\theta_\pm(\alpha)=-\frac{i}{4\gamma(\alpha)}\ln\left[\gamma(\alpha)\mp\alpha\right].$$

Now, it can be shown that a solution of (9) reads

$$\begin{pmatrix}\varphi_1^\pm\\ \varphi_3^\pm\end{pmatrix}(\alpha)=A_0H_+^{-1}(\alpha)\begin{pmatrix}I_1^\pm+c\\ I_3^\pm\end{pmatrix}$$

with

$$\begin{pmatrix}I_1^\pm\\ I_3^\pm\end{pmatrix}=\frac{\sqrt{2}}{\alpha\pm\sin\theta_0}\begin{pmatrix}\cos\theta_0\left(\cos\varphi_0^\pm+\sin\varphi_0^\pm\right)\\ i\left(\cos\varphi_0^\pm-\sin\varphi_0^\pm\right)\end{pmatrix},$$

$$\varphi_0^\pm=\frac{1}{4}\left(\pi/2\pm\theta_0\right).$$

The unknown constant c can be specified using the relation (10). The correct behavior of $\varphi_{1,3}^\pm(\alpha)$ is recovered if we choose $c=-i\tilde{I}_3^\pm,\ \tilde{I}_3^\pm=\lim\limits_{\alpha\to\infty}\alpha I_3^\pm(\alpha)$. This completes the exact explicit factorization of (9). So, the explicit expressions for $\Phi_{1,3}(x_1)$ in the domain $k(x_1\pm a)>>1$ for $|\theta_0|<\frac{\pi}{2}$ can be written in the form

(12) $$\Phi_1(x_1)=q_1e^{ikx_1\sin\theta_0}+O\left(x^{-3/2}\right),x=ka,$$

$$\Phi_3(x_1)=q_3\sum_\pm D^\pm\frac{e^{ik(a\pm x_1)}}{\sqrt{k(a\pm x_1)}}+O\left(x^{-3/2}\right),$$

$$D^\pm=-\frac{1}{2\sqrt{2\pi}}e^{i\pi/4}e^{\mp ix\sin\theta_0}\left[\frac{\cos\varphi_0^\mp}{1\mp\sin\theta_0}\cos\theta_0+\sin\varphi_0^\mp\right],$$

where $D^\pm$ is the diffraction coefficients at the left and right scatterer ends correspondingly.

Numerical Solution

The solution of the integral equations (8) for electric and magnetic current can be reduced to the solution of two coupled systems of linear equations. By using the integral representation of the Hankel function, the integral equations (8) can be rewritten as follows:

(13)
$$\Phi_1(p)-\frac{1}{\pi x}\int_{-1}^{1}\left[\frac{1}{(t-p)^2}+K_0(p,t)\right]\Phi_3(t)dt=q_1e^{ixl_1p},$$

$$\Phi_3\left(p\right)-\frac{ix}{2}\int_{-1}^{1}H_0^{(1)}(x\left|p-t\right|)\Phi_1\left(t\right)dt=q_3e^{ixl_1p},$$

$$K_0\left(p,t\right)=i\pi x\frac{H_1^{(1)}\left(x\left|p-t\right|\right)}{2\left|p-t\right|}-\frac{1}{\left(p-t\right)^2},\ p=x_1/a,$$

where $H_1^{(1)}\left(x\left|p-t\right|\right)$ is the Hankel function of the first order.

Applying a direct collocation technique to solve hypersingular integral equations (13) we have

(14)
$$\Phi_1(p_i)-\frac{1}{\pi x}\sum_{j=1}^{N}\left[\frac{1}{p_i-t_j}-\frac{1}{p_i-t_{j-1}}+\right.$$

$$\left.+hK_0(p_i,t_j)\right]\Phi_3(t_j)=q_1e^{ixl_1p_i},$$

$$\Phi_3\left(t_j\right)-\frac{ix}{2}h\sum_{i=1}^{N}H_0^{(1)}\left(x\left|p_i-t_j\right|\right)\Phi_1\left(p_i\right)=q_3e^{ixl_1t_j},$$

$$t_j=-1+jh,\ j=1,.....,N,\ p_i=-1+\left(i-1/2\right)h,$$

$$i=1,2,...,N,h=2/N,\ \Phi_3\left(t_0\right)=\Phi_3\left(t_N\right)=0.$$

The difference between the solutions of the linear algebraic system (14) and the solutions of integral equations (13) tends to zero when truncation parameter N tends to infinity [12].

Results

The computational algebra package Mathematica has been used for obtaining the numerical results.

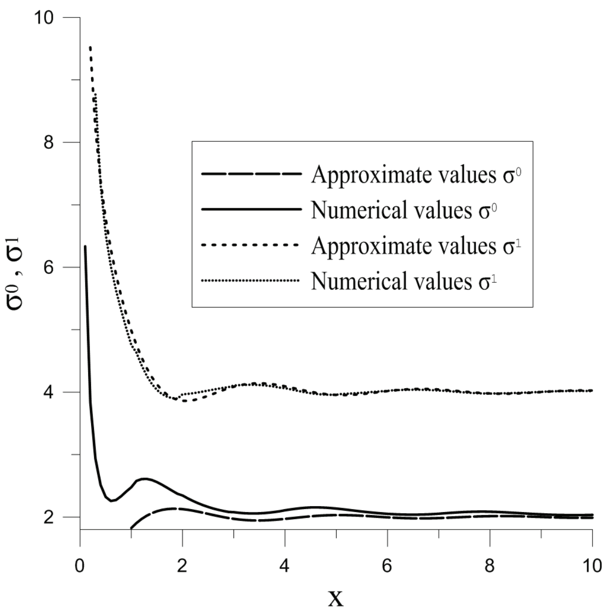

Fig. 2. The dimensionless total σ^0 and monostatic radar σ^1 cross sections versus x for $\theta_0=0$

The dimensionless total $\sigma^0=\mathrm{Im}f(\omega,\mathbf{l},\mathbf{l})/2aA_0x$ and monostatic radar $\sigma^1=|f(\omega,\mathbf{l},-\mathbf{l})|/aA_0x$ cross sections are plotted as functions of x in Fig. 2 for $\theta_0=0$.

In the numerical procedure the parameter N is chosen such that it is the smallest integer not less than $4x$, that gives the sufficient calculation accuracy. As is seen in Fig. 2 the approximate solutions are in good agreement with the exact solutions for $x>1$.

Conclusion

We have constructed an analytical and numerical solutions for the scattering and diffraction of E-polarized plane waves by a strip with different boundary conditions on its faces. The related mixed boundary-value problem in the domain of the short-wavelength is formulated as a matrix Wienner-Hopf equation which is solved explicitly through the Khrapkov method. This analytical solution is a powerful tool to study the effects of a single strip under the incidence of plane waves. It can be used to validate the numerical methods. The direct numerical treatment of this problem is presented also. The obtained numerical results are in a good agreement with corresponding analytical results.

***Autorzy**: prof. dr hab. Włodzimierz Jemec*
Politechnika Łódzka, Instytut Informatyki
ul. Wólczańska 215, 93-005 Łódź
email: wfemets@ics.p.lodz.pl
dr inż. Jan Rogowski
Politechnika Łódzka, Instytut Informatyki
ul. Wólczańska 215, 93-005 Łódź
email: jasiorog@ics.p.lodz.pl

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