Tadeusz KACZOREK

Bialystok University of Technology

doi:10.15199/48.2016.06.14

Characteristic polynomials of positive and minimal-phase electrical circuits

***Abstract**. Characteristic polynomials of positive and minimal-phase electrical circuits are addressed. It is shown that the characteristic polynomials of the electrical circuits are independent of the choice of their reference mesh and of their reference node. Conditions are established under which the positive electrical circuits have real eigenvalues and are minimal-phase linear systems. Sufficient conditions for cancelation of zeros and poles of minimal-phase electrical circuits are given.*

***Streszczenie.** W pracy wykazano, że wielomiany charakterystyczne obwodów elektrycznych są niezależne od wyboru oczka odniesienia w metodzie oczkowej i węzła odniesienia w metodzie węzłowej analizy tych obwodów. Podano warunki przy spełnieniu których dodatnie obwody elektryczne mają rzeczywiste wartości własne i są minimalnofazowymi obwodami elektrycznymi. Podano również warunki wystarczające skracania zer i biegunów w minimalnofazowych obwodach elektrycznych. Rozważania zostały zilustrowane przykładami dodatnich i minimalnofazowych obwodów elektrycznych. (**Wielomiany charakterystyczne dodatnich i minimalnofazowych obwodów elektrycznych**).*

Keywords: minimal-phase, positive, electrical circuit, cancelation, pole, zero, independence of characteristic polynomial.
Słowa kluczowe: minimalnofazowość, dodatniość, obwody elektryczne, zera, bieguny, niezależność wielomianu charakterystycznego.

Introduction

In positive electrical circuits the state variables and outputs take only non-negative values for any non-negative initial conditions and inputs. The positive standard and fractional order electrical circuits have been investigated in many papers and books [2, 4, 10, 13, 21, 22, 31, 35]. A new class of normal electrical circuits has been introduced in [18]. The minimum energy control of electrical circuits has been investigated in [17]. Positive linear systems consisting of n subsystems with different fractional orders have been addressed in [24, 30]. Decoupling zeros of positive linear systems have been introduced in [11].

Determination of the state space equations for given transfer matrices is a classical problem, called the realization problem, which has been addressed in many papers and books [1, 3, 14, 34, 36]. An overview of the positive realization problem is given in [1, 3, 19, 34]. The realization problem for positive continuous-time and discrete-time linear system has been considered in [6-9, 16, 20, 23, 27, 29, 33, 34] and for linear systems with delays in [6, 12, 23, 25, 33, 34]. The realization problem for fractional linear systems has been analyzed in [26, 28, 32, 34] and for positive 2D hybrid linear systems in [25]. A new modified state variable diagram method for determination of positive realizations with reduced number of delays for given proper transfer matrices has been proposed in [5]. The minimal-phase positive electrical circuits have been analyzed in [15].

In this paper the characteristic polynomials of positive and minimal-phase electrical circuits will be investigated.

The paper is organized as follows. In section 2 some preliminaries on positivity and asymptotic stability of continuous-time linear systems are recalled. In section 3 it is shown that the characteristic polynomial of electrical circuits is independent of the choice of reference mesh and of the choice of reference node of the electrical circuits. The asymptotic stability and eigenvalues of the Metzler matrices of the positive electrical circuits are also analyzed in section 4. The minimal-phase positive electrical circuits are addressed in section 5. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ real matrices with nonnegative entries, $\Re^{n\times m}(s)$ - the set of $n\times m$ rational matrices in s with real coefficients, I_n - the $n\times n$ identity matrix.

Preliminaries

Consider the continuous-time linear system

$$\dot{x} = Ax + Bu, \tag{1a}$$

$$y = Cx + Du, \tag{1b}$$

where $x = x(t) \in \Re^n$, $u = u(t) \in \Re^m$, $y = y(t) \in \Re^p$ are the state, input and output vectors and $A \in \Re^{n\times n}$, $B \in \Re^{n\times m}$, $C \in \Re^{p\times n}$, $D \in \Re^{p\times m}$.

Definition 1. [19] The system (1) is called (internally) positive if $x = x(t) \in \Re_+^n$ and $y = y(t) \in \Re_+^p$, $t \in [0,+\infty]$ for all $x_0 = x(0) \in \Re_+^n$ and $u = u(t) \in \Re_+^m$, $t \in [0,+\infty]$.

Theorem 1. [19] The system (1) is positive if and only if

$$A \in M_n,\ B \in \Re_+^{n\times m},\ C \in \Re_+^{p\times n},\ D \in \Re_+^{p\times m}, \tag{2}$$

where M_n is the set of $n\times n$ Metzler matrices, i.e. the matrices with nonnegative off-diagonal entries.

The transfer matrix of (1) is given by

$$T(s) = C[I_n s - A]^{-1}B + D = \frac{N(s)}{d(s)} \in \Re^{p\times m}(s), \tag{3}$$

where $N(s)$ is the polynomial matrix and $d(s)$ is the polynomial.

For single-input single-output (SISO, $m = p = 1$) linear system the transfer function can be written in the form

$$T(s) = \frac{n(s)}{d(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + ... + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0}. \tag{4}$$

Definition 2. The roots s_1, s_2,…, s_n of the equation

$$\begin{aligned} d(s) &= s^n + a_{n-1}s^{n-1} + ... + a_1 s + a_0 \\ &= (s - s_1)(s - s_2)...(s - s_n) = 0 \end{aligned} \tag{5}$$

are called the poles of the linear system.

Definition 3. The roots s_1^0, s_2^0,…, s_n^0 of the equation

$$\begin{aligned} n(s) &= b_n s^n + b_{n-1}s^{n-1} + ... + b_1 s + b_0 \\ &= b_n(s - s_1^0)(s - s_2^0)...(s - s_n^0) = 0 \end{aligned} \tag{6}$$

are called the zeros of the linear system.

The poles s_1, s_2,..., s_n and the zeros s_1^0, s_2^0,..., s_n^0 are called distinct if $s_i \neq s_j$ for $i \neq j$ and $s_i^0 \neq s_j^0$ for $i \neq j$, $i, j = 1,...,n$, respectively.

Definition 4. The linear system is called minimal-phase if

(7) $$\mathrm{Re}\, s_k < 0 \text{ and } \mathrm{Re}\, s_k^0 < 0 \text{ for } k = 1,...,n,$$

where Re denotes the real part of the complex number.

Definition 5. [19] The positive system (1) is called asymptotically stable if

(8) $$\lim_{t \to \infty} x(t) = 0 \text{ for all } x_0 \in \Re_+^n.$$

Theorem 2. [19] The positive system (1) is asymptotically stable if and only if

(9) $$\mathrm{Re}\, \lambda_k < 0 \text{ for } k = 1,...,n,$$

where λ_k is the eigenvalue of the matrix $A \in M_n$ and

(10) $$\det[I_n \lambda - A] = (\lambda - \lambda_1)(\lambda - \lambda_2)...(\lambda - \lambda_n).$$

Note that the set of poles {s_1, s_2,..., s_n} in general case is the subset of the set of eigenvalues {λ_1, λ_2,..., λ_n} [14].

Independence of characteristic polynomial of the choice of reference mesh and reference node

First we shall show the independence of the characteristic polynomial of linear electrical circuits of the choice of reference mesh and of reference node on the following simple electrical circuit.

Example 1. Consider the electrical circuit shown in Figure 1 with given resistances R_1, R_2, R_3, inductances L_1, L_2 and source voltages e_1, e_2.

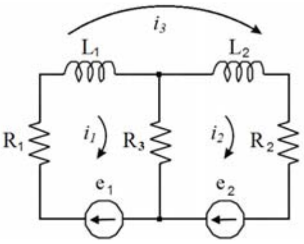

Fig. 1. Electrical circuit of Example 1.

The electrical circuit has three meshes but only two of them are linearly independent [2, 13]. Using the mesh method we may write the matrix equation

(11) $$\begin{bmatrix} R_1+R_3+sL_1 & -R_3 & -(R_1+sL_1) \\ -R_3 & R_2+R_3+sL_2 & -(R_2+sL_2) \\ -(R_1+sL_1) & -(R_2+sL_2) & R_1+R_2+s(L_1+L_2) \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \\ e_1+e_2 \end{bmatrix}$$

where i_1, i_2, i_3 are the mesh currents.

Note that the sum of entries of each row and of each column of the matrix

(12) $$\begin{bmatrix} R_1+R_3+sL_1 & -R_3 & -(R_1+sL_1) \\ -R_3 & R_2+R_3+sL_2 & -(R_2+sL_2) \\ -(R_1+sL_1) & -(R_2+sL_2) & R_1+R_2+s(L_1+L_2) \end{bmatrix}$$

is zero. Choosing as the reference mesh the third mesh we obtain

(13) $$\begin{bmatrix} R_1+R_3+sL_1 & -R_3 \\ -R_3 & R_2+R_3+sL_2 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Similarly, for the choice as the reference mesh of the second mesh we obtain

(14) $$\begin{bmatrix} R_1+R_3+sL_1 & -(R_1+sL_1) \\ -(R_1+sL_1) & R_1+R_2+s(L_1+L_2) \end{bmatrix} \begin{bmatrix} i_1 \\ i_3 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_1+e_2 \end{bmatrix}$$

and for the choice as the reference mesh of the first mesh

(15) $$\begin{bmatrix} R_2+R_3+sL_2 & -(R_2+sL_2) \\ -(R_2+sL_2) & R_1+R_2+s(L_1+L_2) \end{bmatrix} \begin{bmatrix} i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} e_2 \\ e_1+e_2 \end{bmatrix}.$$

It is easy to verify that the characteristic polynomial of the matrices of the equations (13), (14) and (15) is the same, i.e.

(16) $$\begin{aligned} p(s) &= \det \begin{bmatrix} R_1+R_3+sL_1 & -R_3 \\ -R_3 & R_2+R_3+sL_2 \end{bmatrix} \\ &= \det \begin{bmatrix} R_1+R_3+sL_1 & -(R_1+sL_1) \\ -(R_1+sL_1) & R_1+R_2+s(L_1+L_2) \end{bmatrix} \\ &= \det \begin{bmatrix} R_2+R_3+sL_2 & -(R_2+sL_2) \\ -(R_2+sL_2) & R_1+R_2+s(L_1+L_2) \end{bmatrix} \\ &= s^2 L_1 L_2 + s[L_1(R_2+R_3) + L_2(R_1+R_3)] \\ &\quad + R_1(R_2+R_3) + R_2 R_3. \end{aligned}$$

Therefore, we have the following important conclusion.

Conclusion 1. The characteristic polynomial of the electrical circuit is independent of the choice of the reference mesh.

In general case we have the following theorem.

Theorem 3. The characteristic polynomial of any linear electrical circuit is independent of the choice of the reference mesh.

Proof. The proof follows immediately from Theorem A.1 given in the Appendix.

Dual results can be obtained for the node method of analysis of the linear electrical circuits.

Theorem 4. The characteristic polynomial of any linear electrical circuit is independent of the choice of the reference node.

Positive electrical circuits

Example 2. Consider the electrical circuit shown in Figure 2 with given resistances R_k, $k = 1,...,8$, inductances L_1, L_2 and source voltages e_1, e_2.

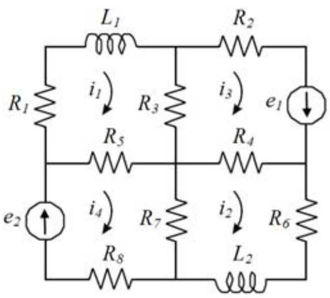

Fig. 2. Positive electrical circuit of Example 2.

Using the mesh method we obtain the equations

(17a)
$$L_1\frac{di_1}{dt}=-R_{11}i_1+R_3i_3+R_5i_4,$$
$$L_2\frac{di_2}{dt}=-R_{22}i_2+R_4i_3+R_7i_4,$$

(17b)
$$0=R_3i_1+R_4i_2-R_{33}i_3+e_1,$$
$$0=R_5i_1+R_7i_2-R_{44}i_4+e_2,$$

where $R_{11}=R_1+R_3+R_5$, $R_{22}=R_4+R_6+R_7$, $R_{33}=R_2+R_3+R_4$, $R_{44}=R_5+R_7+R_8$ and i_k, $k=1,...,4$ are the mesh currents.

From (17b) we have

(18)
$$\begin{bmatrix} i_3 \\ i_4 \end{bmatrix}=\begin{bmatrix} \frac{R_3}{R_{33}} & \frac{R_4}{R_{33}} \\ \frac{R_5}{R_{44}} & \frac{R_7}{R_{44}} \end{bmatrix}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}+\begin{bmatrix} \frac{1}{R_{33}} & 0 \\ 0 & \frac{1}{R_{44}} \end{bmatrix}\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}.$$

Substitution of (18) into (17a) yields

(19a)
$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}=A\begin{bmatrix} i_1 \\ i_2 \end{bmatrix}+B\begin{bmatrix} e_1 \\ e_2 \end{bmatrix},$$

where

(19b)
$$A=\begin{bmatrix} \frac{-R_{11}R_{33}R_{44}+R_3^2R_{44}+R_5^2R_{33}}{L_1R_{33}R_{44}} & \frac{R_3R_4R_{44}+R_5R_7R_{33}}{L_1R_{33}R_{44}} \\ \frac{R_3R_4R_{44}+R_5R_7R_{33}}{L_2R_{33}R_{44}} & \frac{-R_{22}R_{33}R_{44}+R_4^2R_{44}+R_7^2R_{33}}{L_2R_{33}R_{44}} \end{bmatrix},$$
$$B=\begin{bmatrix} \frac{R_3}{L_1R_{33}} & \frac{R_5}{L_1R_{44}} \\ \frac{R_4}{L_2R_{33}} & \frac{R_7}{L_2R_{44}} \end{bmatrix}.$$

Note that the matrix A is a Metzler matrix and the matrix B has positive entries. Therefore, the electrical circuit is a positive one with real negative eigenvalues (poles). The electrical circuit is positive and asymptotically stable and its matrix A satisfies the condition [13, 19]

(20)
$$-A^{-1}\in\Re_+^{n\times n}.$$

In general case we have the following theorem [13].

Theorem 5. The electrical circuit composed of resistances, inductances and source voltages is positive and asymptotically stable for positive values of the resistances and inductances if and only if the number of the inductances is less or equal to the number of linearly independent meshes and each independent mesh contains at least one positive resistance. The matrix $A\in M_n$ of the asymptotically stable positive electrical circuit satisfies the condition (4.4).

Dual results hold for positive asymptotically stable electrical circuits with given resistances, capacitances and source voltages [2, 13].

Theorem 6. [19] The Metzler matrix $A\in M_n$ of positive electrical circuit is asymptotically stable if and only if all coefficients of the characteristic polynomial

(21)
$$\det[I_n\lambda-A]=\lambda^n+a_{n-1}\lambda^{n-1}+...+a_1\lambda+a_0$$
$$=(\lambda-\lambda_1)(\lambda-\lambda_2)...(\lambda-\lambda_n)$$

are positive, i.e. $a_k>0$ for $k=0,1,...,n-1$.

Theorem 7. The Metzler matrix $A\in M_n$ of the positive electrical circuit composed of resistances and inductances or of resistances and capacitances has non-positive real eigenvalues. The eigenvalues are real negative if the electrical circuit has not independent meshes containing at least one positive resistance or at least one node with branches containing only capacitors and current sources.

Proof. Proof follows immediately from Theorem 5 and dual results hold for positive asymptotically stable electrical circuits composed of resistances, capacitances and source currents [2, 13].

Remark 1. The Metzler matrix $A\in M_n$ may have a pair of complex conjugate eigenvalues only if the positive electrical circuit is composed of resistances, inductances and capacitances.

Remark 2. [19] The Metzler matrix $A\in M_n$ of positive electrical circuit has at least one real eigenvalue $\lambda_1=\alpha$ satisfying the condition

(22)
$$\mathrm{Re}\,\lambda_k<\alpha \text{ for } k=2,...,n.$$

Remark 3. The Metzler matrix $A\in M_n$ of positive electrical circuit

1) for $n=1,2$ has only real eigenvalues;
2) for $n=3$ may have a pair of complex conjugate eigenvalues only if the matrix is not symmetric;
3) for $n=4$ may have only one pair of complex conjugate eigenvalues only if the matrix is not symmetric.

Proof.

1) For $n=1$ the proof is evident since $A\in\Re^{n\times n}$.
For $n=2$ let the matrix have the form
$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$
Then
$$\det[I_2\lambda-A]=\begin{vmatrix} \lambda-a_{11} & -a_{12} \\ -a_{21} & \lambda-a_{22} \end{vmatrix}$$
$$=\lambda^2-(a_{11}+a_{22})\lambda+a_{11}a_{22}-a_{12}a_{21}$$
and
$$(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})$$
$$=(a_{11}-a_{22})^2+4a_{12}a_{21}\ge 0$$
for $a_{12}\ge 0$, $a_{21}\ge 0$.

2) For $n=3$ by Remark 2 one eigenvalue is always real. If the matrix $A\in\Re^{3\times 3}$ is symmetric then all eigenvalues are real. Therefore, the matrix may have one pair of complex conjugate eigenvalues only if A is not symmetric.

3) For $n=4$ the matrix $A\in M_4$ may have at least one pair of complex conjugate eigenvalues since by Remark 2, A has at least one real eigenvalue. The coefficients of the characteristic polynomial of the matrix A are real. Therefore, the matrix has two real eigenvalues. □

Example 3. Consider the electrical circuit shown in Figure 3 with given resistances R_1, R_2, R_3, inductances L_1, L_2, L_3 and source voltage e.

Using the Kirchhoff's laws we may write the equations

$$\begin{aligned}&\begin{bmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{bmatrix}\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2\end{bmatrix} \\ &= \begin{bmatrix} -(R_1+R_3) & -R_3 \\ -R_3 & -(R_2+R_3)\end{bmatrix}\begin{bmatrix} i_1\\ i_2\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}e\end{aligned} \quad (23)$$

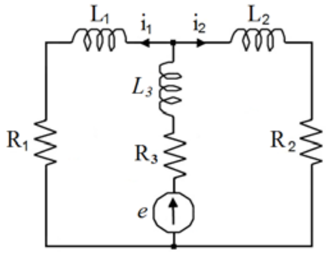

Fig. 3. Electrical circuit of Example 3.

From (23) we obtain

$$\frac{d}{dt}\begin{bmatrix} i_1 \\ i_2\end{bmatrix} = A\begin{bmatrix} i_1 \\ i_2\end{bmatrix} + Be, \quad (24)$$

where

$$\begin{aligned}A&=\begin{bmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{bmatrix}^{-1}\begin{bmatrix} -(R_1+R_3) & -R_3 \\ -R_3 & -(R_2+R_3)\end{bmatrix}\\ &=\frac{1}{\Delta L}\begin{bmatrix} -[L_2(R_1+R_3)+L_3R_1] & L_3R_2-L_2R_3 \\ L_3R_1-L_1R_3 & -[L_1(R_2+R_3)+L_3R_2]\end{bmatrix},\\ B&=\begin{bmatrix} L_1+L_3 & L_3 \\ L_3 & L_2+L_3 \end{bmatrix}^{-1}\begin{bmatrix}1\\1\end{bmatrix}=\frac{1}{\Delta L}\begin{bmatrix}L_2\\L_1\end{bmatrix},\\ \Delta L&=L_1(L_2+L_3)+L_2L_3.\end{aligned} \quad (25)$$

From (25) it follows that the matrix A is a stable Metzler matrix if and only if

$$L_3R_2 > L_2R_3 \text{ and } L_3R_1 > L_1R_3. \quad (26)$$

Note that if $R_3 = 0$ then the conditions (26) are satisfied and the electrical circuit is positive and asymptotically stable for all positive values of the resistances R_1, R_2 and inductances L_1, L_2, L_3.

Therefore, the electrical circuit is a positive asymptotically stable if the conditions (26) are satisfied.

In the general case we have the following conclusion.

Conclusion 2. The electrical circuit consisting of resistors, inductances and source voltages with at least one node with branches containing inductances is positive and asymptotically stable only for some particular values of the resistances and inductances.

Positive minimal-phase electrical circuits

In this section following [15] the positive minimal-phase electrical circuits will be analyzed.

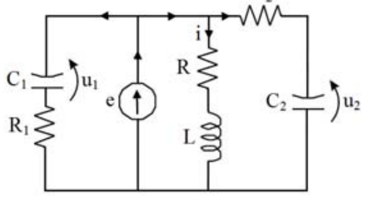

Fig. 4. Positive electrical circuit of Example 4.

Example 4. Consider the positive electrical circuit shown in Figure 4 with given positive resistances R_1, R_2, R, inductance L, capacitances C_1, C_2 and source voltage e. Using Kirchhoff's laws we may write the equations

$$e = R_1C_1\frac{du_1}{dt}+u_1, \quad (27a)$$

$$e = Ri + L\frac{di}{dt}, \quad (27b)$$

$$e = R_2C_2\frac{du_2}{dt}+u_2, \quad (27c)$$

which can be written in the form

$$\frac{d}{dt}\begin{bmatrix}u_1\\u_2\\i\end{bmatrix} = A\begin{bmatrix}u_1\\u_2\\i\end{bmatrix}+Be, \quad (28a)$$

where

$$A=\begin{bmatrix} -\frac{1}{R_1C_1} & 0 & 0 \\ 0 & -\frac{1}{R_2C_2} & 0 \\ 0 & 0 & -\frac{R}{L}\end{bmatrix},\ B=\begin{bmatrix}\frac{1}{R_1C_1}\\ \frac{1}{R_2C_2}\\ \frac{1}{L}\end{bmatrix}. \quad (28b)$$

As the output *y* we choose

$$y=u_1+Ri=C\begin{bmatrix}u_1\\u_2\\i\end{bmatrix},\ C=[1 \quad 0 \quad R]. \quad (29)$$

The transfer function of the electrical circuit has the form (30)

$$\begin{aligned}T(s)&=C[I_3s-A]^{-1}B\\ &=[1\quad 0\quad R]\begin{bmatrix} s+\frac{1}{R_1C_1} & 0 & 0 \\ 0 & s+\frac{1}{R_2C_2} & 0 \\ 0 & 0 & s+\frac{R}{L}\end{bmatrix}^{-1}\begin{bmatrix}\frac{1}{R_1C_1}\\ \frac{1}{R_2C_2}\\ \frac{1}{L}\end{bmatrix}\\ &=[1\quad 0\quad R]\frac{1}{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)\left(s+\frac{R}{L}\right)}\\ &\times\begin{bmatrix} \left(s+\frac{1}{R_2C_2}\right)\left(s+\frac{R}{L}\right) & 0 & 0 \\ 0 & \left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{R}{L}\right) & 0 \\ 0 & 0 & \left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)\end{bmatrix}\\ &\times\begin{bmatrix}\frac{1}{R_1C_1}\\ \frac{1}{R_2C_2}\\ \frac{1}{L}\end{bmatrix}=\frac{n(s)}{d(s)},\end{aligned}$$

where

(31)
$$n(s)=\left(s+\frac{1}{R_2C_2}\right)\left(s+\frac{R}{L}\right)\frac{1}{R_1C_1}+\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)\frac{R}{L}$$
$$=\left(s+\frac{1}{R_2C_2}\right)\left[\left(s+\frac{R}{L}\right)\frac{1}{R_1C_1}+\left(s+\frac{1}{R_1C_1}\right)\frac{R}{L}\right],$$

(32) $$d(s)=\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_2C_2}\right)\left(s+\frac{R}{L}\right).$$

The poles of the electrical circuit are

(33) $$s_1=-\frac{1}{R_1C_1},\ s_2=-\frac{1}{R_2C_2},\ s_3=-\frac{R}{L}$$

and its zeros are

(34) $$s_1^0=-\frac{1}{R_2C_2},\ s_2^0=-\frac{2R}{RR_1C_1}.$$

If $R_1C_1 \ge R_2C_2$ and $\frac{R}{L} \ge \frac{1}{R_2C_2}$, then the poles and zeros satisfy the conditions $s_k \le s_k^0 \le s_{k+1}$ for $k=1,...,n-1$.
Therefore, the positive electrical circuit is asymptotically stable and minimal-phase.

Note that the zero s_1^0 is equal to the pole s_2 since the matrix A is diagonal and after the cancelation of the zero and pole the transfer function has the form

(35) $$T(s)=\frac{\left(\frac{1}{R_1C_1}+\frac{R}{L}\right)s+\frac{2R}{R_1C_1L}}{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{R}{L}\right)}.$$

In general case we have the following theorem.

Theorem 8. If $A=\mathrm{diag}[-a_1 \quad -a_2 \quad \cdots \quad -a_n]\in M_n$ and at least one entry in the matrix $B=\begin{bmatrix}b_1 & b_2 & \cdots & b_n\end{bmatrix}^T\in\Re_+^n$ or in the matrix $C=\begin{bmatrix}c_1 & c_2 & \cdots & c_n\end{bmatrix}\in\Re_+^{1\times n}$ is zero, then at least one zero of the electrical circuit is equal to one of its poles.
Proof. Let $c_2=0$, then the transfer function of the electrical circuit has the form (36).
Therefore, the pole $s_2=-a_2$ is also the zero of the electrical circuit. The proof if one entry of the matrix B is zero is similar. □
Theorem 8 can be easily extended to MIMO positive asymptotically stable electrical circuits.

(36)
$$T(s)=C[I_ns-A]^{-1}B=[c_1 \quad 0 \quad c_3 \quad \cdots \quad c_n][\mathrm{diag}(s+a_1 \quad s+a_2 \quad \cdots \quad s+a_n)]^{-1}\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}$$
$$=[c_1 \quad 0 \quad c_3 \quad \cdots \quad c_n]\frac{1}{(s+a_1)(s+a_2)...(s+a_n)}\mathrm{diag}[(s+a_2)(s+a_3)...(s+a_n)$$
$$(s+a_1)(s+a_3)...(s+a_n) \quad \cdots \quad (s+a_1)(s+a_2)...(s+a_{n-1})]\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_n\end{bmatrix}$$
$$=\frac{(s+a_2)[c_1b_1(s+a_3)...(s+a_n)+c_3b_3(s+a_1)(s+a_4)...(s+a_n)+c_nb_n(s+a_1)(s+a_3)...(s+a_{n-1})]}{(s+a_1)(s+a_2)...(s+a_n)}$$
$$=\frac{c_1b_1(s+a_3)...(s+a_n)+c_3b_3(s+a_1)(s+a_4)...(s+a_n)+...+c_nb_n(s+a_1)(s+a_3)...(s+a_{n-1})}{(s+a_1)(s+a_3)...(s+a_n)}.$$

Example 5. Consider the positive electrical circuit shown in Figure 5 for $n_1=3$, $n_2=4$ with given positive resistances R_1, R_2, R_3, R_4, inductances L_2, L_4, capacitances C_1, C_3 and source voltages e_0, e_2, e_4.
In this case the state equations have the form

(37a) $$\frac{d}{dt}\begin{bmatrix}u_1\\ u_3\\ i_2\\ i_4\end{bmatrix}=A\begin{bmatrix}u_1\\ u_3\\ i_2\\ i_4\end{bmatrix}+B\begin{bmatrix}e_0\\ e_2\\ e_4\end{bmatrix},$$

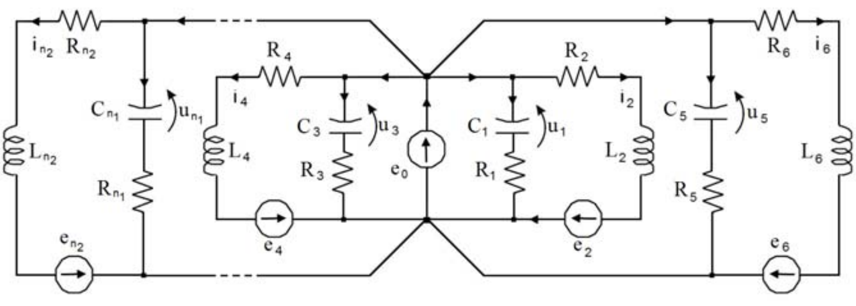

Fig. 5. Positive electrical circuit of Example 5.

where

(37b) $$A = \operatorname{diag}\begin{bmatrix} -\frac{1}{R_1C_1} & -\frac{1}{R_3C_3} & -\frac{R_2}{L_2} & -\frac{R_4}{L_4} \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 \\ \frac{1}{R_3C_3} & 0 & 0 \\ \frac{1}{L_2} & \frac{1}{L_2} & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} \end{bmatrix}.$$

As the output of the electrical circuit we choose

(38) $$y = u_3 + i_4 = C\begin{bmatrix} u_1 \\ u_3 \\ i_2 \\ i_4 \end{bmatrix}, \; C = [0 \;\; 1 \;\; 0 \;\; 1].$$

The transfer matrix of the electrical circuit has the form

$$T(s) = C[I_4 s - A]^{-1} B$$

$$= [0 \;\; 1 \;\; 0 \;\; 1]\left\{\operatorname{diag}\left[\left(s+\frac{1}{R_1C_1}\right) \; \left(s+\frac{1}{R_3C_3}\right) \; \left(s+\frac{R_2}{L_2}\right) \; \left(s+\frac{R_4}{L_4}\right)\right]\right\}^{-1}\begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 \\ \frac{1}{R_3C_3} & 0 & 0 \\ \frac{1}{L_2} & \frac{1}{L_2} & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} \end{bmatrix}$$

$$= \frac{1}{\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_3C_3}\right)\left(s+\frac{R_2}{L_2}\right)\left(s+\frac{R_4}{L_4}\right)}\left[0 \;\; \left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{R_2}{L_2}\right)\left(s+\frac{R_4}{L_4}\right) \;\; 0\right.$$

(39) $$\left.\left(s+\frac{1}{R_1C_1}\right)\left(s+\frac{1}{R_3C_3}\right)\left(s+\frac{R_2}{L_2}\right)\right]\begin{bmatrix} \frac{1}{R_1C_1} & 0 & 0 \\ \frac{1}{R_3C_3} & 0 & 0 \\ \frac{1}{L_2} & \frac{1}{L_2} & 0 \\ \frac{1}{L_4} & 0 & \frac{1}{L_4} \end{bmatrix} = \begin{bmatrix} \frac{1}{R_3C_3s+1}+\frac{1}{L_4s+R_4} & 0 & \frac{1}{L_4s+R_4} \end{bmatrix}.$$

From (39) it follows that in this case all zeros of the electrical circuit are equal to the corresponding poles.

Theorem 9. In SISO positive asymptotically stable electrical circuits the distinct negative zeros s_k^0, $k=1,...,n$ and the distinct negative poles s_j, $j=1,...,n$ satisfy the condition

(40) $$s_{k-1} < s_k^0 < s_{k+1} \text{ for } k=1,...,n-1.$$

Proof. The proof follows from Theorem 7 in [8] and its proof. By this theorem there exists a minimal-phase realization if and only if the poles and zeros are distinct and negative and satisfies the condition (40). □

Theorem 9 can be easily extended to MIMO positive asymptotically stable electrical circuits.

Theorem 10. In MIMO positive asymptotically stable electrical circuits the distinct negative zeros $s_{ij}^{0,k}$, $i=1,...,p$, $j=1,...,m$, $k=1,...,n_{ij}$ and the distinct negative poles s_k, $k=1,...,n$ satisfy the conditions $s_k \le s_{ij}^{0,k} \le s_{k+1}$ for $i=1,...,p$, $j=1,...,m$, $k=1,...,n_{ij}$.

6. Concluding remarks

The characteristic polynomials of positive and minimal-phase electrical circuits have been addressed. It has been shown that the characteristic polynomials of electrical circuits are independent of the choice of their reference mesh and of their reference node. Conditions have been established under which the positive electrical circuits have real eigenvalues and are minimal-phase linear systems. Sufficient conditions have been given for the cancelation of zeros and poles of minimal-phase electrical circuits (Theorem 8). It has been shown that the stable electrical circuits with distinct poles and zeros are minimal-phase (Theorems 9 and 10). The considerations have been illustrated by examples of positive and minimal-phase electrical circuits. The considerations can be extended to fractional positive electrical circuits.

Appendix

Theorem A.1. If the matrix $A=[a_{ij}]\in\Re^{n\times n}$ satisfies the conditions

(A.1) $$\sum_{j=1}^{n} a_{ij} = 0, \; i=1,...,n$$

then

(A.2) $$A_{kl} = A_{i_0,j_0} \text{ for } k,l=1,...,n \text{ and } i_0, j_0 \in [1,...,n],$$

where A_{kl} is the cofactor of the matrix A ($A_{ad} = [A_{kl}]$ is the adjoint matrix of A).

Proof. If $A_{kl} = 0$ for $k, l = 1,...,n$ then the hypothesis holds. Assume that $A_{i_0, j_0} \neq 0$ for some $i_0, j_0 \in [1,...,n]$ then from (A.1) we have

$$\text{(A.3)} \begin{bmatrix} a_{11} & \cdots & a_{1,j_0-1} & a_{1,j_0+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{i_0-1,1} & \cdots & a_{i_0-1,j_0-1} & a_{i_0-1,j_0+1} & \cdots & a_{i_0-1,n} \\ a_{i_0+1,1} & \cdots & a_{i_0+1,j_0-1} & a_{i_0+1,j_0+1} & \cdots & a_{i_0+1,n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,j_0-1} & a_{n,j_0+1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = - \begin{bmatrix} a_{1,j_0} \\ \vdots \\ a_{i_0-1,j_0} \\ a_{i_0+1,j_0} \\ \vdots \\ a_{n,j_0} \end{bmatrix}.$$

Applying to (A.3) the Cramer formula we obtain

$$\text{(A.4)} \quad 1 = \frac{1}{(-1)^{i_0+j_0} A_{i_0,j_0}} \times \begin{bmatrix} a_{11} & \cdots & a_{1,j_0-1} & -a_{1,j_0} & a_{1,j_0+1} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{i_0-1,1} & \cdots & a_{i_0-1,j_0-1} & -a_{i_0-1,j_0} & a_{i_0-1,j_0+1} & \cdots & a_{i_0-1,n} \\ a_{i_0+1,1} & \cdots & a_{i_0+1,j_0-1} & -a_{i_0+1,j_0} & a_{i_0+1,j_0+1} & \cdots & a_{i_0+1,n} \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{n,j_0-1} & -a_{n,j_0} & a_{n,j_0+1} & \cdots & a_{nn} \end{bmatrix}$$

and $A_{i_0,j_0} = A_{i_0,k}$ for $k = 1,...,n$ since $(-1)^{i_0+j_0} A_{i_0,j_0} = -(-1)^{j_0-k-1}(-1)^{i_0+k} A_{i_0,k}$. □

Author: *prof. dr hab. inż. Tadeusz Kaczorek, Politechnika Białostocka, Wydział Elektryczny, ul. Wiejska 45D, 15-351 Białystok, E-mail:* kaczorek@isep.pw.edu.pl.

REFERENCES

[1] Benvenuti L., Farina L., A tutorial on the positive realization problem. *IEEE Trans. on Automatic Control*, 49 (2004), no. 5, 651-664.
[2] Cholewicki T., Matrix Analysis of Electrical Circuits. *PWN*, Warszawa-Wrocław, 1958.
[3] Farina L., Rinaldi S., Positive Linear Systems; Theory and Applications. *J. Wiley*, New York, 2000.
[4] Kaczorek T., A class of positive and stable time-varying electrical circuits. *Electrical Review*, 91 (2015), no. 5, 121-124.
[5] Kaczorek T., A modified state variables diagram method for determination of positive realizations of linear continuous-time systems with delays. *Int. J. Appl. Math. Comput. Sci.*, 22 (2012), no. 4, 897-905.
[6] Kaczorek T., A realization problem for positive continuous-time linear systems with reduced numbers of delays. *Int. J. Appl. Math. Comput. Sci.*, 16 (2006), no. 3, 325-331.
[7] Kaczorek T., Computation of positive stable realizations for discrete-time linear systems. *Computational Problems of Electrical Engineering*, 2 (2012), no. 1, 41-48.
[8] Kaczorek T., Computation of positive stable realizations for linear continuous-time systems. *Bull. Pol. Acad. Techn. Sci.*, 59 (2011), no. 3, 273-281.
[9] Kaczorek T., Computation of realizations of discrete-time cone systems. *Bull. Pol. Acad. Sci. Techn.*, 54 (2006), no. 3, 347-350.
[10] Kaczorek T., Constructability and observability of standard and positive electrical circuits. *Electrical Review*, 89 (2013), no. 7, 132-136.
[11] Kaczorek T., Decoupling zeros of positive continuous-time linear systems and electrical circuits. *Advances in Systems Science. Advances in Intelligent Systems and Computing*, 240 (2014), Springer, 1-15.
[12] Kaczorek T., Determination of positive realizations with reduced numbers of delays or without delays for discrete-time linear systems. *Archives of Control Sciences*, 22 (2012), no. 4, 371-384.
[13] Kaczorek T., Rogowski K., Fractional Linear Systems and Electrical Circuits. *Studies in Systems, Decision and Control*, 13, 2015.
[14] Kaczorek T., Linear Control Systems, vol. 1. *Research Studies Press*, J. Wiley, New York, 1992.
[15] Kaczorek T., Minimal-phase positive electrical circuits, Submitted to Electrical Review, 2015.
[16] Kaczorek T., Minimal-phase realizations for positive linear systems, Proc. of Conf. TransComp, Zakopane, 2015.
[17] Kaczorek T., Minimum energy control of positive electrical circuits. *Proc. of Conf. MMAR, Miedzyzdroje*, Aug. 2-5, 2014, 2-9.
[18] Kaczorek T., Normal positive electrical circuits. *IET Circuits Theory and Applications*, 9 (2015), no. 5, 691-699.
[19] Kaczorek T., Positive 1D and 2D Systems. *Springer-Verlag*, London, 2002.
[20] Kaczorek T., Positive and asymptotically stable realizations for descriptor discrete-time linear systems. *Bull. Pol. Acad. Sci. Techn.*, 61 (2013), no. 1, 229-237.
[21] Kaczorek T., Positive electrical circuits and their reachability. *Archives of Electrical Engineering*, 60 (2011), no. 3, 283-301.
[22] Kaczorek T., Positive fractional linear electrical circuits. *Proceedings of SPIE*, 8903, Bellingham WA, USA, Art. No 3903-35.
[23] Kaczorek T., Positive minimal realizations for singular discrete-time systems with delays in state and delays in control. *Bull. Pol. Acad. Sci. Techn.*, 53 (2005), no. 3, 293-298.
[24] Kaczorek T., Positive linear systems with different fractional orders. *Bull. Pol. Acad. Sci. Techn.*, 58 (2010), no. 3, 453-458.
[25] Kaczorek T., Positive realizations with reduced numbers of delays for 2-D continuous-discrete linear systems. *Bull. Pol. Acad. Sci. Techn.*, 60 (2012), no. 4, 835-840.
[26] Kaczorek T., Positive stable realizations for fractional descriptor continuous-time linear systems. *Archives of Control Sciences*, 22 (2012), no. 3, 255-265.
[27] Kaczorek T., Positive stable realizations of discrete-time linear systems. *Bull. Pol. Acad. Sci. Techn.*, 60 (2012), no. 3, 605-616.
[28] Kaczorek T., Positive stable realizations of fractional continuous-time linear systems. *Int. J. Appl. Math. Comput. Sci.*, 21 (2011), no. 4, 697-702.
[29] Kaczorek T., Positive stable realizations with system Metzler matrices. *Archives of Control Sciences*, 21 (2011), no. 2, 167-188.
[30] Kaczorek T., Positive systems consisting of n subsystems with different fractional orders. *IEEE Trans. Circuits and Systems – regular paper*, 58 (2011), no. 6, 1203-1210.
[31] Kaczorek T., Positive unstable electrical circuits. *Electrical Review*, 88 (2012), no. 5a, 187-192.
[32] Kaczorek T., Realization problem for fractional continuous-time systems. *Archives of Control Sciences*, 18 (2008), no. 1, 43-58.
[33] Kaczorek T., Realization problem for positive multivariable discrete-time linear systems with delays in the state vector and inputs. *Int. J. Appl. Math. Comput. Sci.*, 16 (2006), no. 2, 101-106.
[34] Kaczorek T., Sajewski Ł., Realization Problem for Positive and Fractional Systems, Springer, 2014.
[35] Kaczorek T., Zeroing of state variables in descriptor electrical circuits by state-feedbacks. *Electrical Review*, 89 (2013), no. 10, 200-203.
[36] Shaked U., Dixon M., Generalized minimal realization of transfer-function matrices. *Int. J. Contr.*, 25 (1977), no. 5, 785-803.