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Adaptive Motion Control with State Constraints Using Barrier Lyapunov Functions

***Abstract**. A servo control with unknown system parameters and the constraints imposed on the maximal position and velocity is considered. The barrier Lyapunov functions approach is applied to assure the preservation of bounds in any conditions. The system performance is compared for three cases of the controller design: based on quadratic Lyapunov functions, based on barrier Lyapunov functions if only position constraints are imposed and based on barrier Lyapunov functions if both position and velocity bounds are present. The tuning rules are discussed and several numerical experiments demonstrating features of the proposed control and the influence of the parameters are presented.*

Streszczenie.** Opisano problem sterowania napędowym układem nadążnym z nieznanymi parametrami i ograniczeniami nałożonymi na maksymalne wartości położenia i prędkości. Porównano właściwości trzech układów regulacji: ze sterowaniem zaprojektowanym na podstawie kwadratowych funkcji Lapunowa, ze sterowaniem zaprojektowanym na podstawie barierowych funkcji Lapunowa i ograniczeniem na położenie, oraz ze sterowaniem zaprojektowanym na podstawie barierowych funkcji Lapunowa przy ograniczeniach na położenie i prędkość. Opisano szereg eksperymentów, które ilustrują charakterystyczne właściwości układu regulacji i dostarczają wniosków co do wyboru parametrów algorytmu sterowania. (**Adaptacyjne sterowanie nadążnego układu napędowego z ograniczeniami z zastosowaniem barierowych funkcji Lapunowa)

Keywords: nonlinear control, adaptive control, servo control.
Słowa kluczowe: sterowanie nieliniowe, sterowanie adaptacyjne, sterowanie napędem nadążnym.

Introduction

Servo systems are commonly used in various branches of industrial automation, robotics, motion control etc. It is well known that the acceptable and save operation of servo drives requires not only a precise tracking of a reference position but also the rigorous handling of constraints imposed on a position and/or speed during any dynamic transient. Any violation of the constraints can lead to performance degradation, hazards, or system damage. Nonlinear adaptive control is widely applied to design high-performance servo systems in the presence of unknown plant parameters. Usually the controller design is based on control Lyapunov functions (CLF) and backstepping techniques. Quadratic Lyapunov functions (QLF) are commonly used to assure the system stability, but unfortunately such approach does not guarantee that the constraints may be imposed a priori and fulfilled during any transient conditions.

Several approaches to control nonlinear systems with constraints were investigated. Among them, the nonlinear model predictive control seems to be promising. Recently, the use of the so called barrier Lyapunov functions (BLF) in control synthesis has been proposed for constraint handling in Brunovsky type systems [1], nonlinear systems in strict feedback form [2], adaptive control etc. [3,4]. The BLF approach applies the backstepping technique but allows to keep the system output (or all state variables) inside the predefined constraints. Although the theory of stability investigation by BLF is well established, only a few practical applications are reported [5,6]. The obtained control laws are more complex than those resulting from quadratic Lyapunov functions. Applying BLF with adaptive backstepping leads to three groups of design parameters: the bounds imposed on the output or the state variables, the gains that influence the speed of the error system convergence to zero, and the adaptive loop gains that affect the adaptive parameter behaviour. The interaction among these parameters as well as the influence of the design parameters on the maximal control value remain important problems in the controller design.

The aim of the presented brief is to demonstrate the possibility of BLF applications in servo systems design, to provide the systematic description of the design procedure and to formulate some rules for the design parameters selection. The considered combination of assumptions are: output constraint + unknown model parameters and full state constraint + unknown model parameters. The proposed approach will enable the designer to impose the constraints that will be preserved during any transient conditions.

Plant model and control objectives

The same approach may be used for rotational or linear motors, but without the loss of generality the liner motion notation is applied. Therefore a linear servo is considered and the model is described by

$$\frac{d}{dt}x = v \,, \tag{1}$$

$$m\frac{d}{dt}v = \varphi \cdot i - F_o \,, \tag{2}$$

where x, v is the forcer position and velocity, m – the forcer mass, φ represents the coefficient converting the motor current i into the thrust force, and F_o is an external load force, acting against the motion. The motor current i is supplied by a PWM inverter working in a current control mode and it is assumed that this loop is much faster than mechanical dynamics, so the motor current i is considered as the control input.

It is assumed that the parameters $m>0$, $\varphi>0$, are unknown, constant or slowly varying. Although the constant φ is usually provided by the motor manufacturer, this information must not be trusted. This constant may vary with the motor temperature, PWM conduction mode, or, for some tubular linear motors with permanent magnets build in the inner part, it may be noticeably lower if the forcer operates at the ends of the inner part. It is assumed that the load may be modelled as a nonlinear, memoryless function of the position and the velocity and that this model may be represented as a linear combination of known nonlinear functions ξ with unknown parameters A:

$$F_o(x,v) = A^T \xi(x,v) \,. \tag{3}$$

Such models are natural if the load is approximated using any approximation technique: artificial neural networks, fuzzy modelling, polynomial approximation etc. The number of unknown parameters and the approximation basis ξ may be decided for the particular application. For the sake of brevity it is assumed here that the model (3) is accurate, but it is possible to consider inaccurate approximation with a bounded approximation error ε:

(4) $$F_o(x,v) = A^T \xi(x,v) + \varepsilon \, .$$

In order to get rid of the difficulties caused by the unknown control gain φ, the motion equation (2) is transferred into

(5) $$\mu \frac{d}{dt} v = i - A_o^T \xi(x,v) \, ,$$

where

(6) $$\mu = \frac{m}{\varphi}, \quad A_o = \frac{1}{\varphi} A \, .$$

The control objective is that the motor position has to follow a smooth reference x_d. It is assumed that the reference is bounded

(7) $$\left| x_d(t) \right| \le x_{\max} \, ,$$

the reference derivatives are bounded as well, and the motor position must be constrained for any t by a pre-defined bound $\Delta_x > x_{\max}$

(8) $$\left| x(t) \right| < \Delta_x \, .$$

If the tracking error is denoted by

(9) $$e_x = x_d - x \, ,$$

the constraint (8) is equivalent to

(10) $$\left| e_x(t) \right| < \Delta_{ex}, \quad \Delta_{ex} = \Delta_x - x_{\max} \, .$$

Lyapunov functions

Lyapunov stability theory will be used to construct the stabilizing control for the discussed problem. For the sake of completeness, some preliminaries will be given in this section.

Definition 1: Let $V : R^n \to R$ be a continuously differentiable, proper, and positive definite function defined with respect to the nonlinear system $\dot{x} = f(x,u)$. Let us denote $\dot{V}(x,u) = V_x^T f(x,u)$. $V(x)$ is a control Lyapunov function (CLF) for the system $\dot{x} = f(x,u)$ if, for all $x \neq 0$, there exists a u such that $\dot{V}(x,u) < 0$. If $V(x) = x^T P x$ for some positive definite P, it is called a quadratic Lyapunov function (QLF).

Definition 2. [2] A Barrier Lyapunov Function (BLF) is a scalar function $V(x)$, defined with respect to the system $\dot{x} = f(x)$ on an open region D containing the origin, that is continuous, positive definite, has continuous first-order partial derivatives at every point of D, has the property $V(x) \to \infty$ as x approaches the boundary of D, and satisfies $\exists M, \ \forall t > 0 \ V(x(t)) < M$ along any system trajectory starting inside D.

Several functions may be considered as candidates for BLF, providing symmetric or asymmetric domain D, but a commonly accepted form of a single variable BLF corresponding to the interval $D = (-\Delta, \Delta)$ is

(11) $$V(x) = \frac{1}{2} \log \frac{\Delta^2}{\Delta^2 - x^2} \, .$$

It is straightforward to obtain

(12) $$\dot{V}(x) = \frac{x\dot{x}}{\Delta^2 - x^2} \, .$$

The application of BLFs to proving that the system fulfils the output or state constraints follows from the lemma below. It is a modified version of lemma 1 in [4] and the prove is omitted for brevity.

Lemma 1

Consider a smooth dynamical system $\dot{z} = f(t,x,w)$, with the state variables $z = [x,w]^T$. Let $V_i(x_i)$ be a BLF satisfying $V_i(x_i) \to \infty \ \ if \ \ x_i \to \pm\Delta_{xi}$, let $Q(w)$ be a QLF. Let $V = \sum_{i=1}^{\dim(x)} V_i(x_i) + Q(w)$. If the inequality $\dot{V} = \frac{\partial V}{\partial z}^T f \le 0$ holds anywhere in the set $S = \{(x,w) : |x_i| < \Delta_{xi}\}$, then any trajectory which fulfills the initial constraints $\forall i \ \ |x_i(0)| < \Delta_{xi}$ remains in S for any t.

In lemma 1 the state is split into the constrained variables x and the unconstrained variables w. For each x_i a BLF is constructed, while a QLF may be used for w.

Several techniques of proving stability with the use of BLF may be suggested [2,3,4]. Below, the approach proposed in [4] is investigated.

QLF control design

Let us forget for a moment about the constraints (8) and design the controller using QLFs. The adaptive backstepping scheme [7] will be used to design the controller. The velocity will be the 'virtual control' for the position tracking. Let us consider the error equation

(13) $$\dot{e}_x = \dot{x}_d - v \, ,$$

and the desired 'virtual control' trajectory v_d with the tracking error defined as

(14) $$e_v = v_d - v \, .$$

The desired 'virtual control' $v_d \neq \dot{x}_d$ will be designed to guarantee the wanted convergence of the error e_x. Considering the following QLF

(15) $$V_1 = \frac{1}{2} e_x^2$$

allows to conclude that the desired 'virtual control' v_d

(16) $$v_d = \dot{x}_d + k_x \cdot e_x \, ,$$

where $k_x > 0$ is a design parameter, will generate the tracking error dynamics

(17) $$\dot{e}_x = \dot{x}_d - \dot{x}_d - k_x \cdot e_x + e_v = -k_x \cdot e_x + e_v \, ,$$

and

(18) $$\dot{V}_1 = -k_x \cdot e_x^2 + e_x e_v \, .$$

During the second stage of the backstepping procedure the velocity error is considered:

(19) $$\mu \cdot \dot{e}_v = \mu \cdot \dot{v}_d - \mu \cdot \dot{v} = \mu \cdot \dot{v}_d - i + A_0^T \xi = -i + A_1^T \xi_1 \, ,$$

where new variables are defined as

(20) $$A_1^T = [\mu, A_0^T], \quad \xi_1^T = [\dot{v}_d, \xi^T] \, .$$

The derivative of the reference speed is given by

(21) $$\dot{v}_d = \ddot{x}_d + k_x \cdot \left(-k_x \cdot e_x + e_v\right),$$

so, fortunately, it is available for the control algorithm and ξ_1 in (20) is the known function. Parameters A_1 in (20) are not known, therefore they will be replaced by adaptive parameters $\hat{A}_1^T = [\hat{\mu}, \hat{A}_0^T]$.

The control variable *i* will be designed using the QLF

(22) $$V_2 = V_1 + \frac{1}{2}\mu e_v^2 + \frac{1}{2}\widetilde{A}_1^T \Gamma^{-1} \widetilde{A}_1,$$

where

(23) $$\widetilde{A}_1 = A_1 - \hat{A}_1$$

denotes the adaptation error and positive definite Γ is the matrix of the design parameters of appropriate dimensions. Plugging in (17,18,19) into

(24) $$\dot{V}_2 = \dot{V}_1 + e_v \mu \dot{e}_v + \widetilde{A}_1^T \Gamma^{-1} \dot{\widetilde{A}}_1$$

allows to calculate the Lyapunov function derivative

(25) $$\dot{V}_2 = -k_x \cdot e_x^2 + e_x e_v + e_v(-i + A_1^T \xi_1) + \widetilde{A}_1^T \Gamma^{-1} \dot{\widetilde{A}}_1.$$

The control variable *i* will be designed to compensate the unnecessary components in (25) and to introduce the stabilizing component, so

(26) $$i = e_x + \hat{A}_1^T \xi_1 + k_v e_v,$$

where $k_v > 0$ is a design parameter. Such control allows to describe the tracking error by

(27) $$\mu \cdot \dot{e}_v = -k_v e_v - e_x - \widetilde{A}_1^T \xi_1,$$

and to represent the Lyapunov function derivative as

(28) $$\dot{V}_2 = -k_x \cdot e_x^2 - k_v \cdot e_v^2 + \widetilde{A}_1^T (e_v \xi_1 + \Gamma^{-1} \dot{\widetilde{A}}_1).$$

As $\dot{\widetilde{A}}_1 = -\dot{\hat{A}}_1$, the differential rule describing the adaptation may be used to guarantee that (28) is non-positive for any, unknown $\widetilde{A}_1$. The simplest way is to cancel the last component in (28):

(29) $$\dot{\hat{A}}_1 = e_v \Gamma \xi_1$$

By using the Lasalle's-Yoshizawa theorem [7], (27, 28) guarantees that all errors $e_v, e_x, \widetilde{A}_1$ are uniformly bounded and e_v, e_x are regulated to zero. Since the reference x_d is bounded, x is bounded as well. The boundedness of v_d follows from the boundedness of $\dot{x}_d$ and $\dot{v}_d$ in (16). Combining this with (26), we find that the control is also bounded. Although the boundedness of state variables is proven using QLF, it is impossible to define the constraints *a priori*. The maximal value of each state variable depends on the design parameters and initial conditions.

Remark 1: It is well known that similar results may be obtained with some other adaptation rules. For example,

(30) $$\dot{\hat{A}}_1 = proj_\rho(\hat{A}_1, e_v \Gamma \xi_1),$$

where $proj_\rho(*,\bullet)$ is a projection operator assuring that $\|*\| \le \rho$ [7]. Although (30) allows to influence the bounds for adaptive parameters, it will not provide *a priori* constraints for the state variables.

Remark 2: The design parameter k_x influences not only the values of e_x, but also the 'virtual control' v_d in (16), and so the error e_v in (19). Therefore the maximal value of the current (26) depends on both design parameters k_x and k_v, although only k_v is explicitly visible in (26).

Remark 3: State variables may be constrained by the initial value of the Lyapunov function. As $\dot{V}_2 \le 0$, $V_2(t) \le V_2(0)$ along any trajectory of the system (17,27). Therefore, $\frac{1}{2}e_x^2 \le V_2(0)$, so $\left|e_x\right| \le \sqrt{2V_2(0)}$. Unfortunately, $V_2(0) = \left[\frac{1}{2}e_x^2 + \frac{1}{2}\mu e_v^2 + \frac{1}{2}\widetilde{A}_1^T \Gamma^{-1} \widetilde{A}_1\right]_{t=0}$ depends on the initial guess of the unknown parameters and the obtained bound is not effective.

Remark 4: The error dynamics (17) and (27) may be described together as:

(31) $$\begin{bmatrix} \dot{e}_x \\ \dot{e}_v \end{bmatrix} = \begin{bmatrix} -k_x & 1 \\ -\frac{1}{\mu} & -\frac{k_v}{\mu} \end{bmatrix} \begin{bmatrix} e_x \\ e_v \end{bmatrix} - \begin{bmatrix} 0 \\ \widetilde{A}_1^T \xi_1 \end{bmatrix}$$

and the design coefficients may be easily selected to assure predefined eigenvalues s_1, s_2 of the state matrix in (31):

(32) $$\frac{k_v^2}{\mu} + k_v(s_1 + s_2) - 1 + \mu s_1 s_2 = 0,$$

(33) $$k_x = -\frac{k_v}{\mu} - (s_1 + s_2).$$

It is also easy to notice that complex eigenvalues (so oscillatory behavior of the system (31)) will be observed if and only if

(34) $$\left(k_x - \frac{k_v}{\mu}\right)^2 < \frac{4}{\mu}.$$

Remark 5: Having positive position tracking error, it is easy to observe that it will increase (tending to violate the bound) if

(35) $$\dot{e}_x = -k_x \cdot e_x + e_v > 0 \quad \Rightarrow \quad e_v > k_x \cdot e_x.$$

Taking into account that

(36) $$e_v = \dot{x}_d - v + k_x \cdot e_x,$$

this is equivalent to

(37) $$\dot{x}_d - v > 0,$$

so the increase of position error cannot be avoided by any selection of the design parameters.

Similarly, a positive velocity tracking error will be increased if

(38) $$\mu \cdot \dot{e}_v > 0 \quad \Rightarrow \quad -\widetilde{A}_1^T \xi_1 > k_v e_v + e_x.$$

Summing up, although QLF design allows to influence the error system dynamics by a proper selection of design parameters, it will not provide any tool to impose constraints for position or velocity a priori.

BLF design with position constraints

In order to satisfy the position error constraints (10), the BLF will be applied during the first stage of backstepping:

(39) $$V_1 = \frac{1}{2}\log\frac{\Delta_{ex}^2}{\Delta_{ex}^2 - e_x^2}.$$

The derivative of the BLF is given by

(40) $$\dot{V}_1(x) = \frac{e_x \dot{e}_x}{\Delta_{ex}^2 - e_x^2},$$

hence plugging (17) into (40) gives

(41) $$\dot{V}_1(x) = \frac{e_x(\dot{x}_d - v_d + e_v)}{\Delta_{ex}^2 - e_x^2}.$$

The application of the 'virtual control' defined by (16) will result in the error equations (17) and (19) and will provide

(42) $$\dot{V}_1(x) = -k_x \frac{e_x^2}{\Delta_{ex}^2 - e_x^2} + \frac{e_x e_v}{\Delta_{ex}^2 - e_x^2}.$$

As the constrains are imposed only on the first state variable (position), the control variable *i* will be designed using the Lyapunov function

(43) $$V_2 = V_1 + \frac{1}{2}\mu e_v^2 + \frac{1}{2}\tilde{A}_1^T \Gamma^{-1}\tilde{A}_1,$$

where V_1 defined in (39) is a BLF.
Substitution of (42, 19) allows to calculate the Lyapunov function derivative as

(44) $$\dot{V}_2(x) = -k_x \frac{e_x^2}{\Delta_{ex}^2 - e_x^2} + \frac{e_x e_v}{\Delta_{ex}^2 - e_x^2} + \\ + e_v(-i + A_1^T \xi_1) + \tilde{A}_1^T \Gamma^{-1}\dot{\tilde{A}}_1$$

The control variable *i* will be designed to compensate the unnecessary components in (34) and to introduce the stabilizing component, so

(45) $$i = \frac{e_x}{\Delta_{ex}^2 - e_x^2} + \hat{A}_1^T \xi_1 + k_v e_v,$$

where $k_v > 0$ is a design parameter. This control gives the velocity tracking error

(46) $$\mu \cdot \dot{e}_v = -k_v e_v + \tilde{A}_1^T \xi_1 - \frac{e_x}{\Delta_{ex}^2 - e_x^2}$$

and the Lyapunov function derivative fulfills

(47) $$\dot{V}_2(x) = -k_x \frac{e_x^2}{\Delta_{ex}^2 - e_x^2} - k_v e_v^2 + \tilde{A}_1^T\left(e_v \xi_1 + \Gamma^{-1}\dot{\tilde{A}}_1\right).$$

Any of the adaptation rules (29) or (30) assures that

(48) $$\dot{V}_2 \le -k_x \cdot \frac{e_x^2}{\Delta_{ex}^2 - e_x^2} - k_v \cdot e_v^2 \le 0.$$

The following corollary abstracts the main features of the obtained system.

Corollary 1.
Consider the closed loop system (17), (46) with any of the adaptation laws (29) or (30) and the reference position trajectory, under all assumptions formulated above. Consider any trajectory with initial conditions fulfilling $|e_x(0)| < \Delta_{ex}$, then the following properties hold along this trajectory:
1. The variables $e_x, e_v, \tilde{A}_1^T$ remain inside a compact set and the output fulfils the constraint (8).
2. All closed loop signals are bounded.
3. The tracking errors e_x, e_v converge to zero asymptotically.

Sketch of the proof:
1. $V_2(0)$ is bounded and as $\dot{V}_2 \le 0$, $V_2 \le V_2(0)$ along the considered trajectory. The lemma 1 yields that $|e_x(t)| < \Delta_{ex}$. Another constraint for the tracking error may be obtained noticing that $\frac{1}{2}\log\frac{\Delta_{ex}^2}{\Delta_{ex}^2 - e_x^2} \le V_2(0)$ and thus $|e_x| \le \Delta_{ex}\sqrt{1 - e^{-2V_2(0)}}$. Finally, because of (7) and (9), we get (8). Similarly we may derive that $|e_v| \le \sqrt{\frac{2}{\mu}V_2(0)}$ and $\|\tilde{A}_1\| \le \sqrt{\frac{2V_2(0)}{\lambda_{\min}(\Gamma^{-1})}}$, where $\lambda_{\min}(*)$ denotes the smallest eigenvalue of the symmetric matrix $*$.
2. As $e_x, e_v, \tilde{A}_1^T$ are bounded, $\hat{A}_1$ are bounded also. From (16), (21) the desired 'virtual control' and it's derivative are bounded as well. Therefore the functions ξ_1 are bounded and from (45) the control is bounded.
3. The tracking error asymptotic convergence may be obtained by demonstrating that $\ddot{V}_2$ is bounded and making use of Barbalat's lemma [8].

Remark 6: The component $\frac{e_x}{\Delta_{ex}^2 - e_x^2}$ in (45) suggests that the control variable increases if $e_x \to \pm\Delta_{ex}$. Although the inequality $|e_x(t)| < \Delta_{ex}$ is always fulfilled and the control is bounded, the maximal value of the control variable depends on all design parameters (similarly as it was explained in Remark 2) and requires careful investigation. Usually the initial jump of the current is the critical one, and this may happen to be $|i(0)| \ge \left(\frac{1}{\Delta_{ex}^2 - e_x^2(0)} + k_v k_x\right)|e_x(0)|$. It is also obvious that the 'adaptive' component $\hat{A}_1^T \xi_1$ contributes to the current values.

BLF design with position and velocity constraints

The constraints imposed on the velocity may be preserved if a BLF will be used also during the second stage of the backstepping design. Therefore,

(49) $$V_2 = V_1 + \frac{\mu}{2}\log\frac{\Delta_{ev}^2}{\Delta_{ev}^2 - e_v^2} + \frac{1}{2}\tilde{A}_1^T \Gamma^{-1}\tilde{A}_1,$$

where V_1 defined in (31) is a BLF and Δ_{ev} is a constraint imposed on the tracking error e_v. The Lyapunov function derivative may be represented as

(50) $$\dot{V}_2(x) = -k_x e_x^2 + \frac{e_x e_v}{\Delta_{ex}^2 - e_x^2} + \frac{e_v \mu \dot{e}_v}{\Delta_{ev}^2 - e_v^2} + \tilde{A}_1^T \Gamma^{-1}\dot{\tilde{A}}_1$$

and after plugging in (19)

(51) $$\dot{V}_2(x) = -k_x \frac{e_x^2}{\Delta_{ex}^2 - e_x^2} + \frac{e_x e_v}{\Delta_{ex}^2 - e_x^2} + \\ + \frac{e_v(-i + A_1^T \xi_1)}{\Delta_{ev}^2 - e_v^2} + \tilde{A}_1^T \Gamma^{-1}\dot{\tilde{A}}_1.$$

Once again, the control variable i will be designed to compensate the unnecessary components in (51) and to introduce the stabilizing component, so

(52) $$i = \frac{e_x(\Delta_{ev}^2 - e_v^2)}{\Delta_{ex}^2 - e_x^2} + \hat{A}_1^T \xi_1 + k_v e_v ,$$

where $k_v > 0$ is a design parameter. This control gives the velocity tracking error

(53) $$\mu \cdot \dot{e}_v = -k_v e_v + \tilde{A}_1^T \xi_1 - \frac{e_x(\Delta_{ev}^2 - e_v^2)}{\Delta_{ex}^2 - e_x^2} ,$$

and the Lyapunov function derivative fulfills

(54) $$\begin{aligned} \dot{V}_2 &= -k_x \cdot \frac{e_x^2}{\Delta_{ex}^2 - e_x^2} - k_v \cdot \frac{e_v^2}{\Delta_{ev}^2 - e_v^2} + \\ &+ \tilde{A}_1^T \Big(\frac{e_v}{\Delta_{ev}^2 - e_v^2} \xi_1 + \Gamma^{-1} \dot{\tilde{A}}_1 \Big) \end{aligned} .$$

As $\dot{\tilde{A}}_1 = -\dot{\hat{A}}_1$, the differential rule describing the adaptation may be used to guarantee that (54) is non-positive for any, unknown $\tilde{A}_1$. The simplest way is to cancel the last component in (54) by selecting:

(55) $$\dot{\hat{A}}_1 = \frac{e_v}{\Delta_{ev}^2 - e_v^2} \Gamma \xi_1 ,$$

which results in

(56) $$\dot{V}_2 = -k_x \cdot \frac{e_x^2}{\Delta_{ex}^2 - e_x^2} - k_v \cdot \frac{e_v^2}{\Delta_{ev}^2 - e_v^2} .$$

Therefore,

(57) $\dot{V}_2 \le 0$ in $S = \left\{ (e_x, e_v, \tilde{A}_1) : |e_x| < \Delta_{ex}, |e_v| < \Delta_{ev} \right\}$.

Note that not any velocity constraint is applicable. Let us consider the maximal value of the desired 'virtual control':

(58)
$$v_{d\max} = \max_{|e_x| \le \Delta_{ex}, t>0} |v_d| = \max_{|e_x| \le \Delta_{ex}, t>0} |\dot{x}_d + k_x e_x| \le \bar{v}_{d\max} ,$$

(59) $$\bar{v}_{d\max} = \max_{t>0} |\dot{x}_d| + k_x \Delta_{ex}$$

and assume that the motor velocity must be constrained for any t by a pre-defined bound $\Delta_v > \bar{v}_{d\max}$

(60) $$|v(t)| < \Delta_v .$$

This constraint will be satisfied if

(61) $$|e_v(t)| < \Delta_v - \bar{v}_{d\max} =: \Delta_{ev} .$$

A designer may be interested in constraining the gap between the desired position derivative and the actual velocity:

(62) $$E(t) = \dot{x}_d(t) - v(t) .$$

As

(63) $$|E| = |\dot{x}_d - v| = |v_d - v - k_x e_x| = |e_v - k_x e_x| ,$$

it may be bounded by a proper choice of Δ_{ex} and Δ_{ev}:

(64) $$|E| \le \Delta_{ev} + k_x \Delta_{ex} .$$

The following corollary summaries the main features of the obtained system.

Corollary 2.

Consider the closed loop system (19), (53) with any of the adaptation laws (55), and the reference position trajectory, under all assumptions formulated above and the given bounds Δ_x, Δ_v. Assume that the system design parameters k_x, k_v, Γ are selected such that there exists a trajectory with initial conditions fulfilling $|e_x(0)| < \Delta_{ex}$, $|e_v(0)| < \Delta_{ev}$ (defined according to (10) and (61)), then the following properties hold along this trajectory:

1. The variables e_x, e_v, $\tilde{A}_1^T$ remain inside a compact set and the output fulfils the constraints (8) and (61).
2. All closed loop signals are bounded.
3. The tracking errors e_x, e_v converge to zero asymptotically.

Sketch of the proof:

1. $V_2(0)$ is bounded and as $\dot{V}_2 \le 0$, $V_2 < V_2(0)$ along the considered trajectory. Lemma 1 yields that $|e_x(t)| < \Delta_{ex}$ and $|e_v(t)| < \Delta_{ev}$. Another constraint for the tracking error may be obtained noticing that $\frac{1}{2} \log \frac{\Delta_{ex}^2}{\Delta_{ex}^2 - e_x^2} \le V_2(0)$, thus $|e_x| \le \Delta_{ex} \sqrt{1 - e^{-2V_2(0)}}$. Similarly, it may be proved that $|e_v| \le \Delta_{ev} \sqrt{1 - e^{-\frac{2}{\mu} V_2(0)}}$. Finally, (8) is obtained because of (7) and (9) and (61) yields (60). Using analogical reasoning, it can be derived that $|e_v| \le \sqrt{\frac{2}{\mu} V_2(0)}$ and $\|\tilde{A}_1\| \le \sqrt{\frac{2V_2(0)}{\lambda_{\min}(\Gamma^{-1})}}$, where $\lambda_{\min}(*)$ denotes the smallest eigenvalue of the symmetric matrix $*$.
2. As $e_x, e_v, \tilde{A}_1^T$ are bounded, $\hat{A}_1$ are also bounded. The desired 'virtual control' and it's derivative are bounded as well. Therefore the functions ξ_1 are bounded and from (52) the control is bounded.
3. The tracking error asymptotic convergence may be obtained by demonstrating that $\ddot{V}_2$ is bounded and making use of the Barbalat's lemma [8].

Remark 7: As it is visible in (59), (61) there exists an implicit relation among the design parameters, initial conditions and the imposed constraints. The imposed velocity constraint Δ_v must be bigger than the maximal value of the desired 'virtual control' (59), which depends on the imposed constraints Δ_{ex} and the gain k_x used in the position control loop. All parameters influence the maximal value of the control variable, hence the feasibility conditions have to be checked carefully as it will be demonstrated by examples.

Remark 8: The property (57), and therefore Corollary 2 will hold also with other adaptation rules, corresponding to (30), for example:

(65) $$\dot{\hat{A}}_1 = proj_\rho(\hat{A}_1 , \frac{e_v}{\Delta_{ev}^2 - e_v^2} \Gamma \xi_1) ,$$

where $proj_\rho(*,\bullet)$ is a projection operator assuring that $\|*\| \le \rho$.

Numerical experiments

The simulated liner actuator with the parameters m=8kg, and φ=39N/A is supposed to track the desired position provided by the filtered sinusoid:

(66) $$x_d(t) = \mathbf{L}^{-1}\left\{\frac{1}{T^2 s^2 + 2T + 1}\mathbf{L}\{0.3\sin(3t)\}\right\} \quad [m],$$

where T=0.1s and denotes the Laplace transform. The desired trajectory is presented in fig.1.

It is assumed that the actuator works against the load described by

(67) $$F_o(x, v) = Av^2 sign(v) + B|v|\sin(2\pi x) \quad [N],$$

with unknown coefficients A and B, hence, according to (20)

(68) $$A_1^T = \frac{1}{\varphi}[m, A, B], \qquad \xi_1^T = [\dot{v}_d, v^2 sign(v), |v|\sin(2\pi x)]^T .$$

It is assumed that none of the motor or load parameters are known. Therefore the initial values of adaptive parameters are selected as

(69) $$\hat{A}_1^T(0) = [0.5\frac{m}{\varphi}, 0, 0] .$$

The command (66) is presented at the moment when $x(0)$=-0.03 [m] and $v(0)$=-0.1 [m/s], so $e_x(0) = 0.03, \quad E(0) = 0.1$.

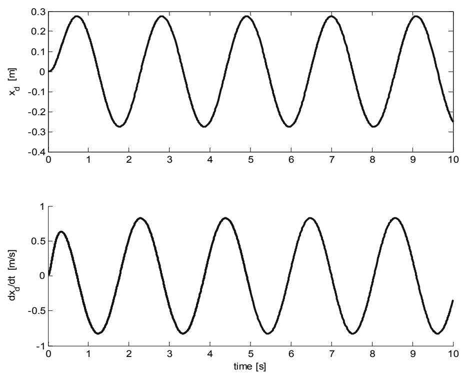

Fig.1. Desired position trajectory x_d and its derivative $\dot{x}_d$

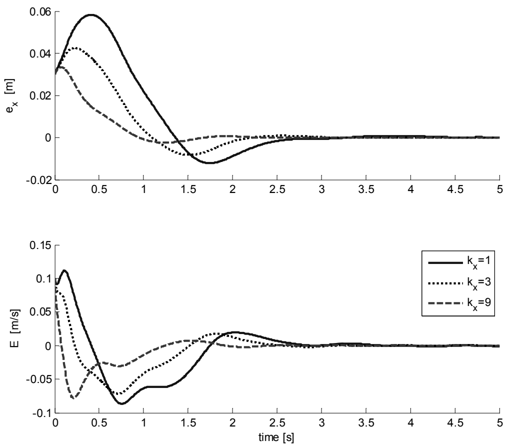

Fig.2. Position tracking error e_x and 'velocity gap' E for different gains k_x (k_v=1)

QLF based control

The control (26) and the adaptive law (29) were applied with design coefficients k_v=1 and k_x=1;3;9. In all cases the errors tend to zero. The initial part of system responses are plotted in fig. 2, 3. It is noticeable that the position tracking error increases at the beginning of the transient and it is impossible to impose a predefined constraint on the error maximal value. For the smallest gain k_x=1 the biggest value of the position error is observed. Increasing the gain k_x allows to decrease the position error, but the 'virtual control' tracking error the maximal current are increased. Also manipulations with the gain k_v do not guarantee that the position tracking error will remain in the predefined bounds, as it is presented in fig. 4,5. In any case the bigger gain implies the bigger values of the motor current.

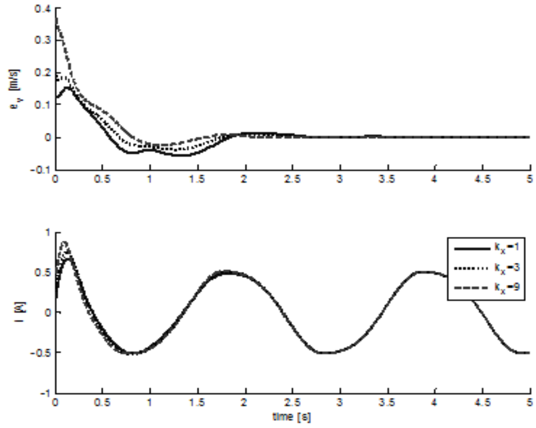

Fig.3. 'Virtual control' tracking error e_v and control i for different gains k_x (k_v=1)

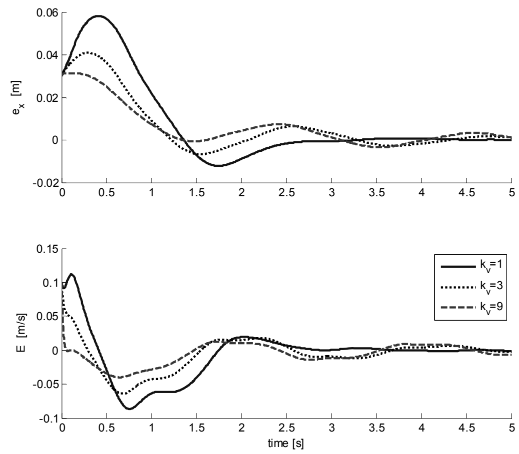

Fig.4. Position tracking error e_x and 'velocity gap' E for different gains k_v (k_x=1)

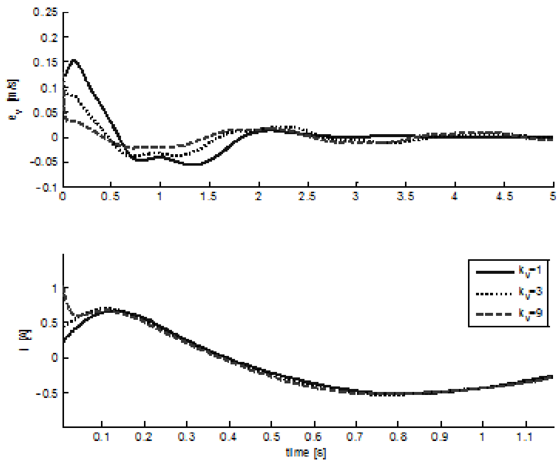

Fig.5. 'Virtual control' tracking error e_v and control i for different gains k_v (k_x=1). Note different time-scale for the last plot

BLF based control with position constraints

The control (45) and the adaptive law (29) were applied with design coefficients k_x=1; k_v=1. Trajectories for QLF-designed control with these parameters are plotted as continuous lines in fig. 2-5. The influence of the imposed constraint on the system performance was tested in three cases: $\Delta_{ex} = 0.2$; these constraint was fulfilled by the QLF-designed system, $\Delta_{ex} = 0.05$ – violated by the QLF-designed system but relatively far from the initial condition $e_x(0) = 0.03$ and $\Delta_{ex} = 0.04$ – violated by the QLF-designed system but closer to the initial condition. The asymptotic stability of all errors is observed. The starting part of the transients are plotted in fig. 6,7. Even the introduction of relatively big constraint 0.1 (that was fulfilled by the QLF-designed system with the same parameters) makes the position tracking error smaller and smoother, although the regulation time remains relatively long. The imposed constrained is preserved in all cases with a sufficient, save distant between the extremal value and the bound. If the constraint gets tighter the maximal 'virtual control' tracking error and the 'velocity gap' increase.

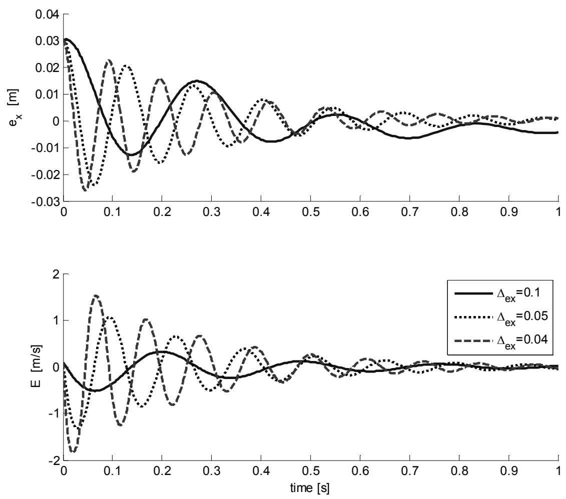

Fig.6. Position tracking error e_x and 'velocity gap' E for different constraints Δ_{ex} (k_x=1, k_v=1)

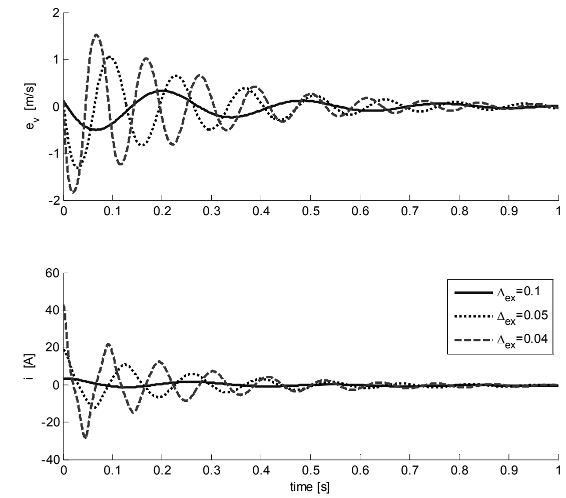

Fig.7. 'Virtual control' tracking error e_v and control i for different constraints Δ_{ex} (k_x=1, k_v=1)

The maximal current increases significantly, what is not surprising as the first component in the control (45) is

$$\frac{e_x(0)}{\Delta_{ex}^2 - e_x^2(0)} = \begin{cases} 18{,}75\ [A]\ \ for\ \ \Delta_{ex} = 0.05\,[m] \\ 42{,}86\ [A]\ \ for\ \ \Delta_{ex} = 0.04\,[m] \end{cases} \tag{70}$$

Because of large oscillations of e_v observed, this system requires a careful tuning of adaptation gains in (29). Therefore, considering all this, the application of a velocity constraint that will contribute to current restriction and will damp the oscillations is highly recommended.

BLF based control with position and velocity constraints

The control (52) and the adaptive law (55) were applied with design coefficients k_x=1; k_v=1 and the constraints Δ_{ex}=0.05 [m] and Δ_{ev} = 1; 0.5; 0.3 [m/s]. In any case the imposed bounds are preserved during the transient. The initial parts of system history is presented in fig. 8 and 9.

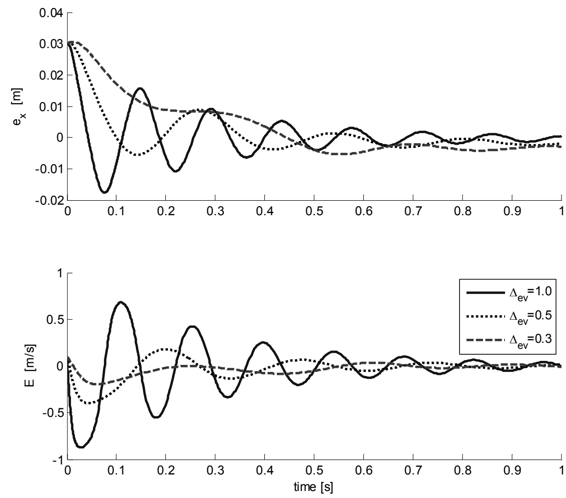

Fig.8. Position tracking error e_x and 'velocity gap' E for different constraints Δ_{ev} (k_x=1, k_v=1, Δ_{ex}=0.05)

It is noticeable, that the bound $\Delta_{ev} = 1$ is too large to assure the acceptable system behavior. Tightening the bound allows to decrease the 'velocity gap' and the maximal current as well.

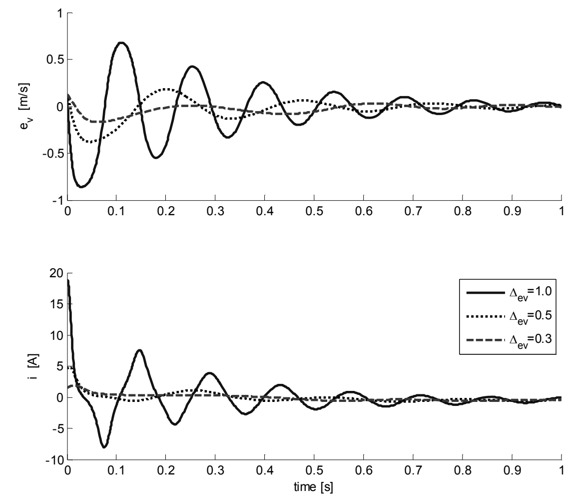

Fig.9. 'Virtual control' tracking error e_v and control i for different constraints Δ_{ev} (k_x=1, k_v=1, Δ_{ex}=0.05)

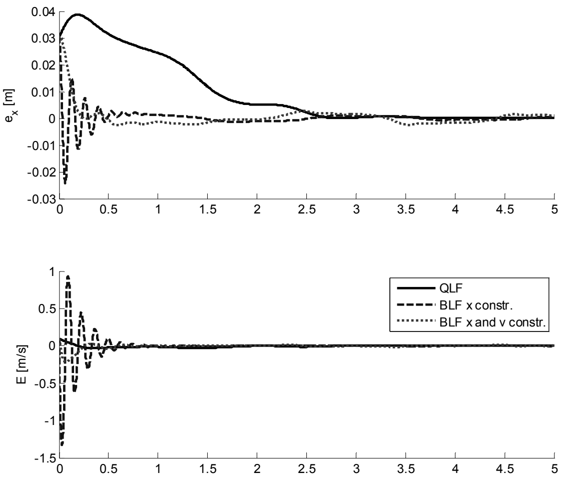

Fig.10. Position tracking error e_x and 'velocity gap E for different design methods

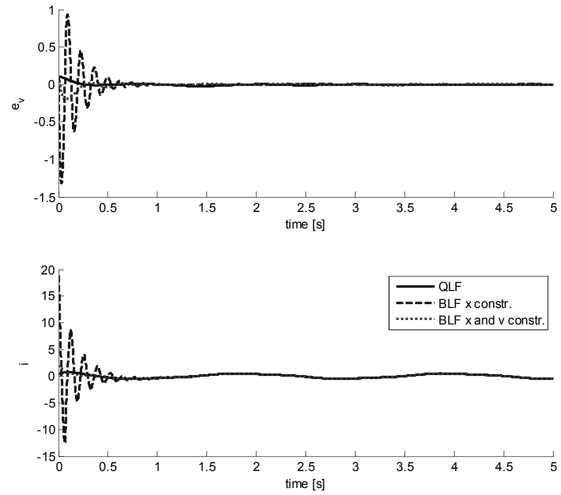

Fig.11. 'Virtual control' tracking error e_v and control i for different design methods

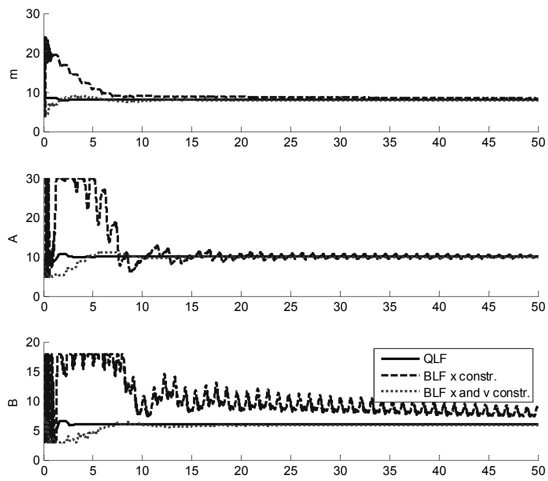

Fig.12. Estimates of parameters *m*, *A*, *B* for different design methods

System performance with bounded adaptive parameters

All systems tested above demonstrate robustness against imperfect reconstruction of the load. Tuning of the adaptive gains in (29) was not difficult and the maximal values of adaptive parameters were reasonable. In spite of this, the system performance with a priori constraints imposed on adaptive parameters was tested. The control strategies (26), (45), (52) with appropriate adaptive laws (30), (62) were applied with constraints Δ_{ex}=0.05 [m] and Δ_{ev} =0.3 [m/s]. The design coefficients k_v and k_x were selected according to (32, 33) to provide eigenvalues of the state matrix in (31) are $s_1 = -2, s_2 = -4$. Estimated parameters were bounded from 50% to 300% of their exact value. The speed of estimation (parameters Γ) was selected individually for the best performance. The results are plotted in fig. 10-12. The transient of the position error is the worse for the QLF-designed system and the best for the BLF-designed with position and velocity constraints.

Conclusions

Numerical experiments have proven that the described modification of adaptive backstepping algorithm based on barrier Lyapunov functions may be effectively implemented for servo systems control, where some physical quantities like position, speed and motor current have to be bounded. The BLF-designed system with a position constraint only demonstrates oscillatory behavior and this is also visible in transients of adaptive parameters in fig. 12. The proper choice of control system parameters was not difficult, especially in the case of coexisting position and velocity constraints. This set of constraints is highly recommended and it outperforms a QLF-designed one and a BLF-designed with position constraint only. The derived system stability and convergence was confirmed by simulations. State variables (position and velocity) remain bounded, the motor current was acceptable.

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