Adam NIKOŁAJEW

Bialystok University of Technology, Faculty of Electrical Engineering

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Effect of Disturbances on Sigma Undersampling of Periodical Signals

Streszczenie**. Zaprezentowano metodę pomiaru sygnałów okresowych wykorzystującą podpróbkowanie Σ. Przedstawiono algorytm przetwarzania sygnałów, umożliwiający redukcję błędów pomiarowych, spowodowanych przez wahania częstotliwości próbkowania i mierzonego sygnału oraz szumy odbiornika. Na przykładzie przebiegu trójkątnego przeanalizowano zależność błędów metody od współczynnika podpróbkowania, liczby próbek w okresie sygnału, mocy szumów oraz wahań częstotliwości.* ***Metoda pomiaru sygnałów okresowych wykorzystującą podpróbkowanie Σ

***Abstract**. A method of measurements of periodical signals, applying Σ undersampling is presented. An algorithm of signal processing, enabling a reduction of errors is proposed. A discussion of errors, caused by fluctuations of sampling and signal's frequencies and a noise of a detector is performed. A dependence of these errors on a number of samples per period, an undersampling factor and a relation between sampling and signal's frequencies for the triangle signal was considered.*

Keywords: a periodical signal, undersampling, frequency fluctuations, Gaussian noise
Słowa kluczowe: podpróbkowanie, szumy Gaussa

An introduction

A continuous progress in a digital signal processing technology enables applications of DSP in wider range of measurements. However even the most modern DSP are too slow for some applications. For example, to measure an output signal of a single-mode optical fiber, a sampling frequency in range of several GHz is necessary [1]. In these cases an undersampling, i.e. a sampling of the measured signal with frequency lower than Nyquist's frequency may be a solution of this problem. An analysis of errors, caused by fluctuations of signal's and sampling frequencies as the function of an undersampling factor and a number of samples per period is a subject of this work. An influence of errors of the measurement, caused by an inaccuracy of a detector is also taken into considerations.

A description of the method

Every periodical signal $x_0(t)$ can be expressed in form of Fourier's series.

$$x_0(t) = A_0 + \sum_{m=1}^{\infty} A_m \cdot \cos\left(2 \cdot \pi \cdot m \cdot f_0 \cdot t + \varphi_m\right) \tag{1}$$

A_0 is the mean value of the signal, A_m and φ_m - the amplitude and the phase of its m-th harmonic component and f_0 - a frequency of the signal.

To apply any digital signal processing algorithm, the signal should be sampled. Its period ought to be represented by N samples. To use FFT algorithm the number of samples must be a power of 2. To realize an undersampling of the signal, it must be sampled once during M+1 periods, where M is an integer number (the undersampling factor), and the phase difference between two successive samples must equal to $(f_0 \cdot N)^{-1}$. The sampling period T_{d0} must satisfy a following condition [2-6]

$$T_{d0} = f_0^{-1} \cdot \left(M + N^{-1}\right). \tag{2}$$

This method would operate correctly, when the condition, expressed by (2) is satisfied exactly. In an opposite case phases of succeeding samples differ, what generates additional errors. This factor ought to be taken into considerations. Assuming that relative stabilities of signal's frequency and sampling period are respectively δf and δT_d, the real values of the frequency of the signal f and the sampling period T_d can be expressed as [7]:

$$f = f_0 \cdot \left(1 + \delta f\right), \tag{3}$$

$$T_d = T_{d0} \cdot \left(1 + \delta T_d\right). \tag{4}$$

Taking (3) and (4) into account, k-th sample of the measured signal can be written as

$$x(k) = A_0 + \sum_{m=1}^{\infty} A_m \cdot \cos\left[\frac{2\pi m k(MN+1)}{N}(1+\delta\varphi) + \varphi_m\right], \tag{5}$$

where

$$\delta\varphi = \delta f + \delta T_d + \delta f \cdot \delta T_d. \tag{6}$$

A sigma undersampling is based on an integrating of the signal before the sampling. When the integration of the output signal is performed during the sampling period T_d, the output signal of the integrator y(t) can be expressed as [7]

$$y(t) = \int_{t-T/2\cdot N}^{t+M\cdot T+T/2\cdot N} x(t)\cdot dt = \frac{A_0 \cdot T \cdot (M \cdot N + 1)}{N} + \frac{T}{N} \cdot \sum_{m=1}^{\infty} A_m \cdot \frac{\sin\left(\frac{m\cdot\pi}{N}\right)}{\frac{m\cdot\pi}{N}} \cdot \cos(2\cdot m\cdot\pi\cdot f\cdot t + \varphi_m) \tag{7}$$

In (7), m-th harmonic component of the signal is attenuated with factor sin(mπ/N)/(mπ/N). It delimits the bandwidth of the signal according to Shannon's theorem, what enables an avoidance of an aliasing effect. The rise of the mean value of the signal is undesirable, because it might saturate the integrator. Nevertheless, it can be easily reduced by an appropriate initial value of the integrator. To retrieve the original signal a digital filter should be applied. Its transfer function H(f) must satisfy a following condition

$$X(m \cdot f_0) = H(m \cdot f_0) \cdot Y(m \cdot f_0). \tag{8}$$

where X(f) and Y(f) are the spectra of the signals x(t) and y(t), respectively.

This filter can be realized as the finite response filter, which transfer function H(z) is given as below:

$$H(z) = \sum_{n=0}^{N-1} h(n) \cdot z^{-n}, \tag{9}$$

(10) $$h(n)=\frac{1}{T\cdot(M\cdot N+1)}+(-1)^n\cdot\frac{\pi}{2\cdot T}+ +\frac{2}{T}\cdot\sum_{m=1}^{N/2-1}\frac{\frac{m\cdot\pi}{N}\cdot\cos\frac{2\cdot m\cdot n\cdot\pi}{N}}{\sin\frac{m\cdot\pi}{N}}.$$

When the initial value is applied in the integrator, the first component of this sum must be properly corrected.

In order to avoid accidental errors of the measurement a multiple repetition of the measurements is required. It is realized by means of a specific algorithm of calculations [7], which operates in few steps. A measurement of single period is repeated P times. In the first step FFT of every period is calculated. Afterwards, a geometrical mean value of all FFTs gives the mean spectrum and enables synchronization of the signal. In the third step - the values of the samples are obtained as IFFT of the geometric mean. In the end, signal is filtered by the filter H(z). This algorithm is described by (11)-(14).

(11) $$Y(p,n)=\sum_{k=0}^{N-1}y(k)\cdot\exp\left(-j\cdot\frac{2\cdot\pi\cdot k\cdot n}{N}\right)$$

(12) $$\overline{Y}(n)=\left[\prod_{p=1}^{P}Y(p,n)\right]^{\frac{1}{P}}$$

(13) $$\overline{y}(k)=\frac{1}{N}\cdot\sum_{n=0}^{N-1}\overline{Y}(n)\cdot\exp\left(j\cdot\frac{2\cdot\pi\cdot k\cdot n}{N}\right)$$

(14) $$\overline{x}(k)=\sum_{i=0}^{N-1}\overline{y}(i)\cdot h(k-i)$$

In these formulas y(k) denotes the value of y(t) for $t=k\cdot T_d$, Y(p,n) - the n-th value of FFT of the integrator's output signal, $\overline{Y}(n)$ - the n-th value of the mean of all FFTs, $\overline{y}(k)$ - the mean value of the k-th sample of the integrator's output signal and $\overline{x}(k)$ - the mean value of the kth sample of the signal, obtained at the output of the filter h(n).

When the measured signal is comparable to an accuracy of the deceiver, it may lead to additional errors. To estimate errors caused by this effect, Gaussian noise of the mean value 0 and variance σ^2 is added to signal x(t).

An estimation of an efficiency of the method

The efficiency of the method, described in the previous section is estimated basing on values of the error δ, defined as

(15) $$\delta=\frac{\sum_{p=1}^{P}\sum_{k=0}^{N-1}\left\|\left[\overline{x}(k)-x_0\left(\frac{k}{f_0\cdot N}\right)\right]\right\|}{x_{0\max}\cdot P\cdot N}$$

where x_{0max} is the maximum value of the signal. Simulations were performed for probability density p(f), given as [7]

(16) $$p(f)=\frac{1}{f_0\cdot\delta f}\cdot rect\left(\frac{1}{f_0\cdot\delta f}-f_0\right),$$

because in this case the largest values of errors were obtained. Therefore this criterion of the accuracy of this method is chosen. Additionally, it was assumed, that the time of the measurement is short enough in order that both frequencies remain constant. In this case the errors of both phases accumulate and the errors maximize.

Simulations were performed simultaneously for the sigma undersampling and the sampling in the same conditions. Calculations were repeated $P=10^5$ times. The mean values of the spectra and the samples in the cases of the sampling and the undersampling were calculated according to the same algorithm, described above. Both methods are compared, using values of errors, defined by (15). The simulations were performed for a triangle signal and their results are divided into 3 cases.

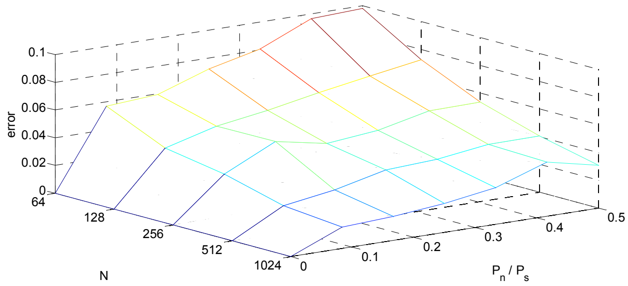

Fig. 1. A dependence of the error δ on N and P_n/P_s for the sampling ($\delta_f=\delta_{Td}=0$)

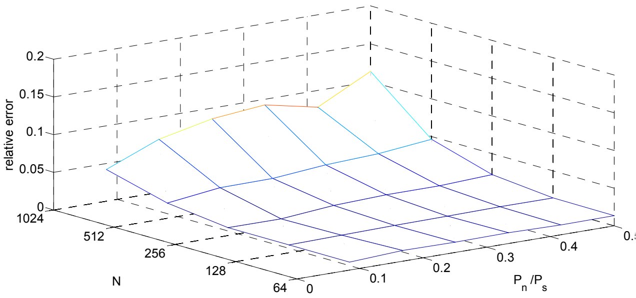

Fig. 2. The quotient of the errors δ for the sigma undersampling (M=100) and the sampling as the function of N and P_n/P_s, $\delta_f=\delta_{Td}=0$

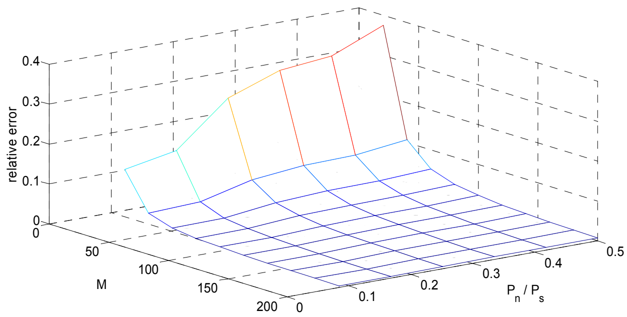

Fig. 3. The quotient of the errors δ for the sigma undersampling (N=256) and the sampling as the function of M and P_n/P_s, $\delta_f=\delta_{Td}=0$

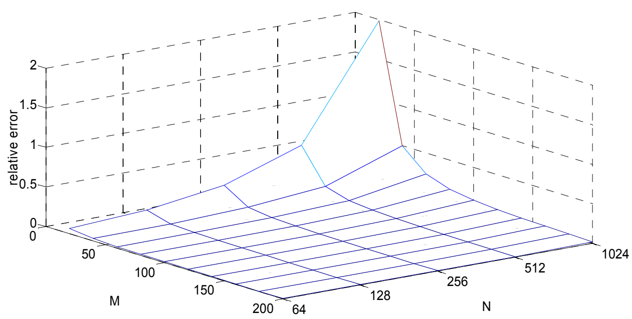

Fig. 4. The quotient of the errors δ for the sigma undersampling and the sampling as the function of M and N, P_n/P_s=0,25, $\delta_f=\delta_{Td}=0$

The results of calculations, presented on Figures 1-4, refer to a situation, when errors of both frequencies are omitted and only errors of a deceiver influence the results of the measurement. The relative error denotes the quotient of the errors δ for Σ undersampling and the sampling. The parameter P_n / P_s is the quotient of the power of the deceiver's noise (for Gaussian noise $P_n = \sigma^2$) and the power of measured signal. The results exhibite, that in this case Σ undersampling is an effective method of a significant reduction of the errors of the measurement. An extension of time of the measurement substantially reduces the mean value of the noise.

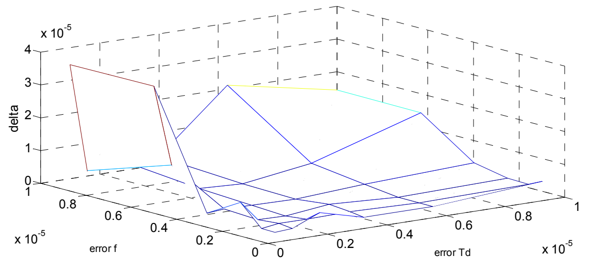

Fig. 5. The error δ as the function of δ_f and δ_{Td} for the sampling (N=256, P_n/P_s=0)

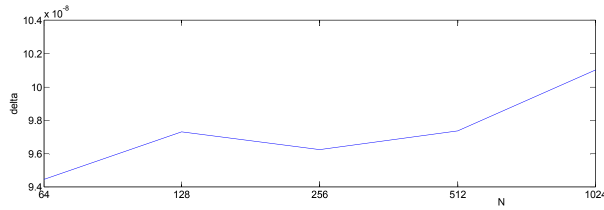

Fig. 6. The error δ as the function of N for the sampling (δ_f = δ_{Td} = 10^{-6}, P_n/P_s=0)

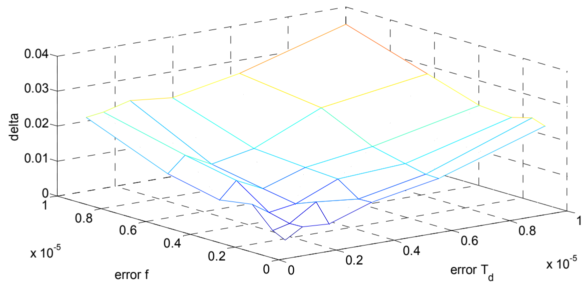

Fig. 7. The error δ as the function of δ_f and δ_{Td} for the undersampling (N=256, M=100, P_n/P_s=0)

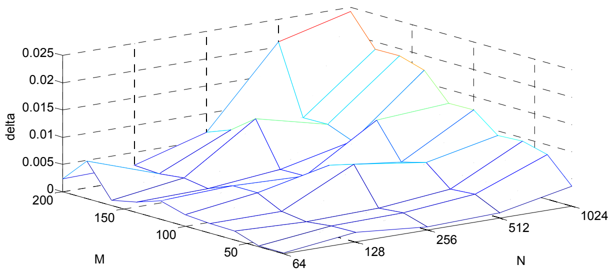

Fig. 8. The error δ as the function of M and N for the undersampling (δ_f=δ_{Td}=10^{-6}, P_n/P_s=0)

The results of the calculations, shown on Figures 5-8, concern the case, when the inaccuracy of the deceiver is neglected, whereas only fluctuations of both frequencies occur. In this case only the absolute values of δ are presented, because the relative errors are in the range of 10^3. It is caused by the summation of the phase errors, when the time of the measurement gets longer. However the absolute values of the errors obtained for the undersampling are less than 1% and absolutely acceptable.

Additionally, one should notice, that the errors in range of 10^{-5}, presented on Fig. 5, are practically unachievable.

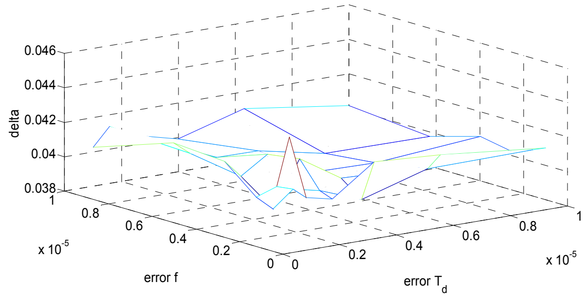

Fig. 9. The error δ for the sampling (N=256, P_n/P_s=0,25) as the function of δ_f and δ_{Td}

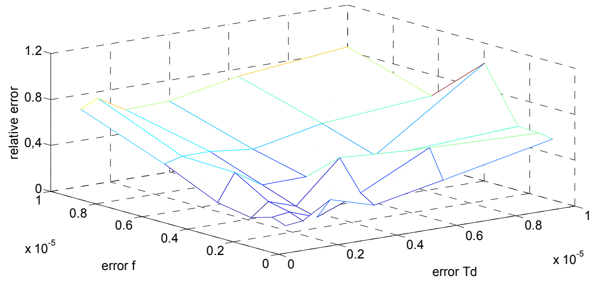

Fig. 10. The quotient of errors δ for the undersampling and the sampling (N=256, M=100,P_n/P_s=0,25) as the function of δ_f and δ_{Td}

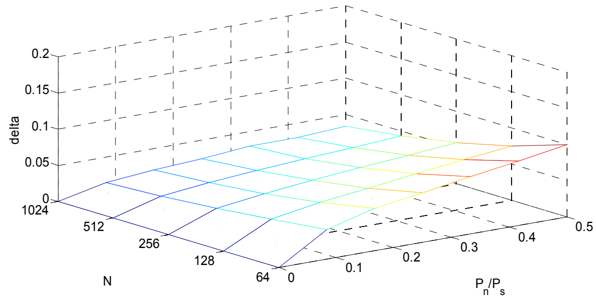

Fig. 11. The error δ as the function of δ_f and δ_{Td} for the sampling (N=256, P_n/P_s=0,25)

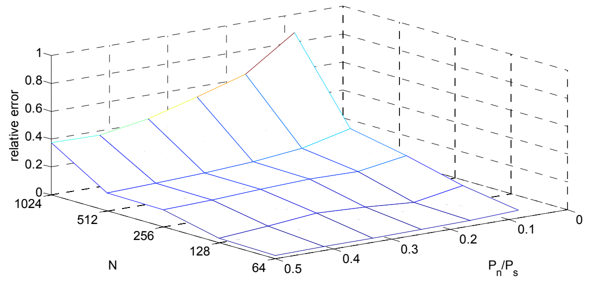

Fig. 12. The quotient of the errors δ for the sigma undersampling (M=100) and the sampling as the function of N and P_n /P_s, δ_f=δ_{Td}=10^{-6}

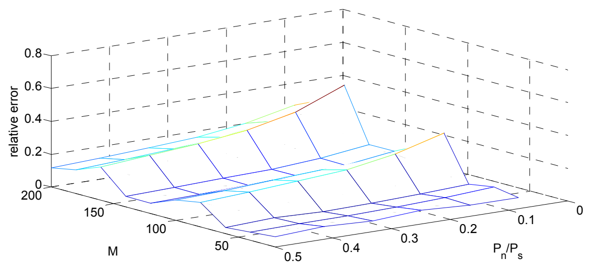

Fig. 13. The quotient of the errors δ for the sigma undersampling (N=256) and the sampling as the function of M and P_n /P_s, δ_f=δ_{Td}=10^{-6}

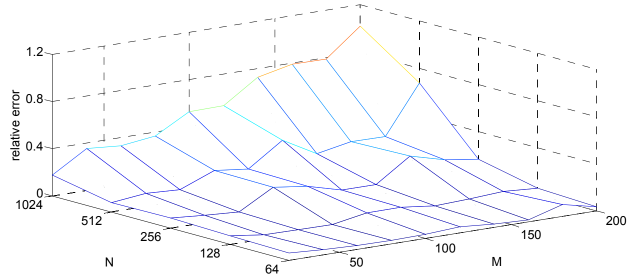

Fig. 14. The quotient of the errors δ for the sigma undersampling and the sampling as the function of M and N, P_n/P_s=0,25, $\delta_f=\delta_{Td}=10^{-6}$

The last set of results, presented on Figs. 9-14, concerns the case, when both kinds of disturbances have been taken into account. It proves a great efficiency of the Σ undersampling in comparison to the sampling, especially for greater values of the parameter P_n/P_s. The values of the relative errors are significantly less than 1, with few exceptions.

Conclusions

The results of the simulations prove, that Σ undersampling can be the effective method of the measurement of periodical signals in the presence of disturbances. It significantly decreases errors of the measurements in the comparison to the sampling. The Σ undersampling is especially effective in the cases, when the errors of the detector are comparable to the signal. The integrating of the signal decreases the influence of the noise on the results of the measurements. The increase of errors, caused by the summation of phase errors is less important.

The accuracy of the method is valid on the condition, that the algorithm, basing on FFT and IFFT is used to obtain the mean value of the signal.

Despite that the results of the simulations are presented for only a triangle signal, similar values have been obtained for other periodical signals.

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***Autor**: dr inż. Adam Nikołajew, Politechnika Białostocka, Wydział Elektryczny, ul. Wiejska 45 D, 15-351 Białystok, e-mail: a.nikolajew@pb.edu.pl*

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