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Simple mathematical models of transmission shafts and gear trains. Electrical and mechanical circuits

Abstract. *In the paper simple kinematic models of mechanical connections between the driving motor and the working mechanism are analyzed. As a result, the mathematical models of connections considered have been formulated. The scope of carried out analysis has included: rigid transmission shaft with zero mass, rigid transmission shaft with non-zero mass, elastic transmission shaft with zero mass, two-mass system, single-stage rigid gear train without clearance (non-slack gear train) and multipath rigid mechanical power transmission without clearances.*

Streszczenie. *W pracy poddano analizie proste modele kinematyczne połączeń mechanicznych silnika z mechanizmem roboczym. Sformułowano modele matematyczne rozważanych połączeń. W zakresie przeprowadzonej analizy znalazły się: sztywny wał napędowy o zerowej masie, sztywny wał napędowy o niezerowej masie, sprężysty wał napędowy o zerowej masie, układ dwumasowy, jednostopniowa sztywna przekładnia mechaniczna bez luzu oraz wielodrożna sztywna transmisja mocy mechanicznej bez luzów. (**Proste modele matematyczne wałów napędowych i przekładni mechanicznych. Obwody elektryczne i mechaniczne**).*

Keywords: mechanical power transmissions in drive systems, electrical and mechanical analogies, kinematic and mathematical models.
Słowa kluczowe: transmisje mocy mechanicznej w napędach, analogie elektryczno-mechaniczne, modele kinematyczne i matematyczne.

Introduction

Electric motors are connected with working mechanisms via transmission shafts that are elements of mechanical power transmissions. The shafts have various lengths and cross-sections. Mechanical power transmissions include also driving machineries, i.e. gear trains and clutches. Depending on length and cross-section, transmission shafts can demonstrate different susceptibilities to the impact of torsional moment, as measured by a value of torsional angle. In the case of short mechanical connections, values of torsional angles are insignificant and they may be omitted by assumption of rigid mechanical connections. In the case of longer mechanical connections the values of torsional angles cannot be ignored and such connections should be considered as the elastic ones. In turn, toothed gears and clutches are the reason of clearances in mechanical system, in which the mechanical power may be transmitted between motor and working mechanism after so-called "taking in the clearance", i.e. when the one part of mechanism has been turned in relation to the another part by a certain angle $2\Delta\gamma_0$.

In the paper simple kinematic models of mechanical connections between the driving motor and the working mechanism are analyzed. The equivalent circuits, typical for electrical systems, are defined for the mechanical systems concerned.

The issues based on the electrical and mechanical similarities were already considered in the previous papers of the author [1,2,3,4]. Identifying these similarities is very helpful for electricians in finding a relevant interpretation of mechanical systems, which is particularly important in the case of professionals dealing with electromechanical energy converters or drive systems. The comprehensive studies regarding mechanical connections used in drive systems can also be found in other papers, e.g. [5,6,7,8,9].

Rigid transmission shaft with zero mass

The rigid (stiff) and weightless shaft is the example of idealized shaft that transmits a torque. In other words, this is the rigid shaft with assumed zero mass m (zero moment of inertia J). The kinematic model of considered transmission shaft is shown in Fig. 1, whereas, the corresponding equivalent circuit is shown in Fig. 2, where M_1, M_2 are external torques applied to the both sides of the shaft, in particular, drive torque and anti-torque, ω_1, ω_2, γ_1, γ_2 are angular velocities and angles of rotation at the points of application of external torques to the shaft, C_s is torsional elasticity coefficient, D_1, D_2 are mechanical friction coefficients (resistances) defined for bearings.

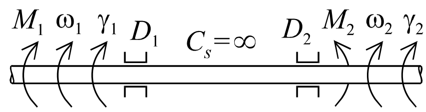

Fig. 1. Kinematic model of the rigid transmission shaft with zero mass

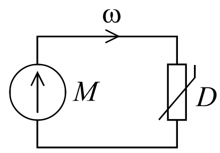

Fig. 2. Equivalent circuit for the considered transmission shaft (Fig.1)

The following dependencies may be adopted for the considered mechanical connection:

$$\omega = \omega_1 = \omega_2, \qquad \gamma = \gamma_1 = \gamma_2$$
$$M = M_1 - M_2, \qquad D = D_1 + D_2 \tag{1}$$

as well as the equation of torques:

$$M - D\omega = 0 \tag{2}$$

In most cases, mechanical friction coefficients (D) do not have constant values and they depend on angular velocity or torque. Under certain conditions they can have constant or almost constant values. Mechanical friction coefficients determine loss torques associated with a type of friction or air resistance. In the case of kinetic dry friction, the corresponding loss torque is practically constant: $\Delta M = D\omega = b_1 = \text{const}$ i.e. mechanical friction coefficient D is inversely proportional to the angular velocity: $D = b_1|\omega^{-1}|$ and it (D) is always positive independently of direction of rotation. The constant friction coefficient corresponds with lubricated friction for which the loss torque is proportional to the angular velocity, whereas, the parabolic dependency between loss torque and angular velocity, i.e. $D = b_2\omega$, where $b_2 = \text{const}$, has been adopted in order to represent the air resistance.

Rigid transmission shaft with non-zero mass

The mass of real connecting element (transmission shaft) is distributed continuously along the longitudinal axis of the element. In simple terms, the mass may be considered as a lumped parameter. As a result, the mechanical structure and analysis become simpler. The rigid transmission shaft with non-zero mass is depicted in Fig. 3, where the real connecting element with continuous mass distribution has been substituted by the connection including two weightless elements representing its longitudinal dimension and one element representing its mass. The corresponding equivalent circuit is shown in Fig. 4, where J is moment of inertia of shaft lumped mass.

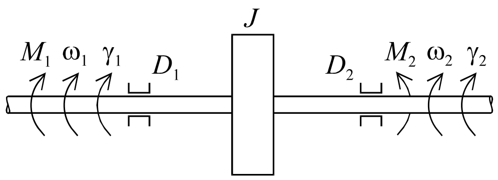

Fig. 3. Kinematic model of rigid transmission shaft with non-zero mass

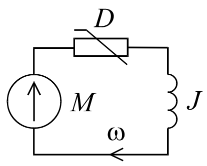

Fig. 4. Equivalent circuit for the considered transmission shaft (Fig.3)

On the basis of the equivalent circuit (Fig. 4) the following equation of torques may be written by analogy to the equation of voltages of non-branched electrical circuit including voltage source and two passive elements i.e. resistance and inductance:

$$M - D\omega - \tfrac{\mathrm{d}}{\mathrm{d}t}\left(J\omega\right) = 0 \qquad (3)$$

The last term on the left side of the Eq. 3 is the dynamic torque [1].

Elastic transmission shaft with zero mass

The weightless elastic transmission shaft (Fig. 5), i.e. the shaft described by the torsional elasticity coefficient C_s of finite value and zero mass (negligible moment of inertia), may be considered as an element of the simplified drive system consisting of two lumped masses coupled via this shaft (two-mass system).

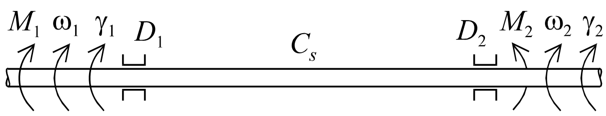

Fig. 5. Kinematic model of elastic transmission shaft (long transmission shaft) with negligible moment of inertia

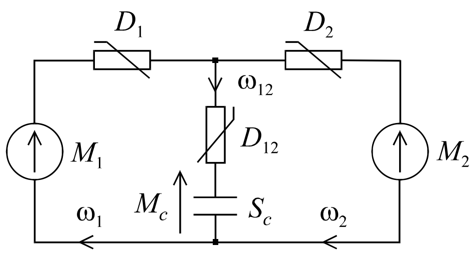

Fig. 6. Equivalent circuit for the considered shaft (Fig. 5)

The corresponding equivalent circuit is shown in Fig. 6, where M_c is torsional moment. The viscous friction inside the shaft has been taken under consideration (D_{12}).

The quantity S_c, analogous to the capacity of capacitor in an electrical circuit, is the torsional susceptibility coefficient being equal to inversed torsional elasticity coefficient C_s:

$$S_c = 1/C_s \qquad (4)$$

On the basis of the equivalent circuit (Fig. 6) the following equations of angular velocities and torques, corresponding with the Kirchhoff's circuit laws applied in order to analyze electrical circuits, may be written:

$$\begin{gathered} \omega_1 - \omega_2 - \omega_{12} = 0 \\ M_1 - D_1\omega_1 - D_{12}\omega_{12} - M_c = 0 \\ M_c + D_{12}\omega_{12} - D_2\omega_2 - M_2 = 0 \end{gathered} \qquad (5)$$

In addition, by analogy to the dependency between current and voltage of capacitor, the following dependency between angular velocity ω_{12} and torsional moment M_c may be written:

$$\omega_{12} = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\gamma_1 - \gamma_2\right) = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(S_c M_c\right) = \tfrac{\mathrm{d}}{\mathrm{d}t}\left(C_s^{-1} M_c\right) \qquad (6)$$

where $(\gamma_1 - \gamma_2)$ is angle of shaft torsion.

Two-mass system

Representing the real mechanical systems with continuous mass distribution by kinematic models based on lumped parameters causes discrepancies in results of analysis in relation to accurate models [5], but considerably simplifies this analysis. In addition, these discrepancies decrease with the number of points of concentration in the model. Representing a drive system, containing elastic elements, by the model with two points of concentration (two-mass system – Fig. 7) allows for simplification of the model as much as possible but it may not in all cases be applied. Such mathematical description is best suited to mechanical systems containing the connection between electric motor and moving unit of working machine via long shaft with negligible moment of inertia in contrast to the significant moments of inertia of the abovementioned elements of mechanical system.

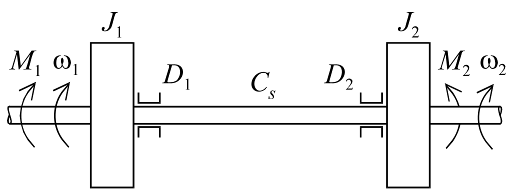

Fig. 7. Kinematic model of two-mass system (two lumped masses connected via long shaft with negligible moment of inertia)

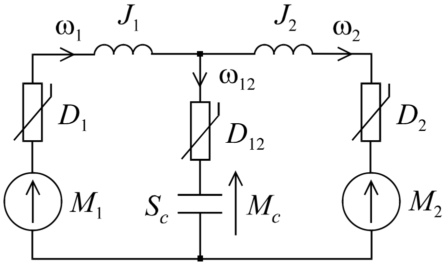

Fig. 8. Equivalent circuit for the considered drive system (Fig. 7)

Kirchhoff's lows based equations for the considered system are as follows:

$$\omega_1 - \omega_2 - \omega_{12} = 0$$

(7) $$M_1 - D_1\omega_1 - \tfrac{d}{dt}\left(J_1\omega_1\right) - D_{12}\omega_{12} - M_c = 0$$

$$M_c + D_{12}\omega_{12} - \tfrac{d}{dt}\left(J_2\omega_2\right) - D_2\omega_2 - M_2 = 0$$

where the dependency between angular velocity ω_{12} and torsional moment M_c is given as Eq. 6.

Single-stage rigid gear train without clearance

The kinematic models of single-stage gear trains, bevel and helical, respectively, are depicted in Figs. 9 and 10, whereas the corresponding circuit diagram, analogous to the equivalent circuit of transformer, is shown in Fig. 11, where N is gear ratio defined as follows:

(8) $$N = \omega_1 / \omega_2$$

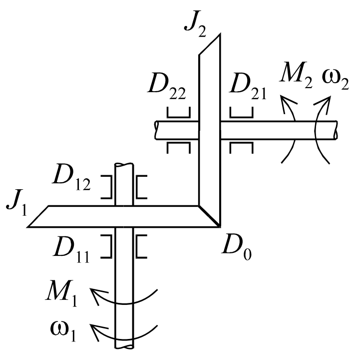

Fig.9. Kinematic model of single-stage bevel gear train

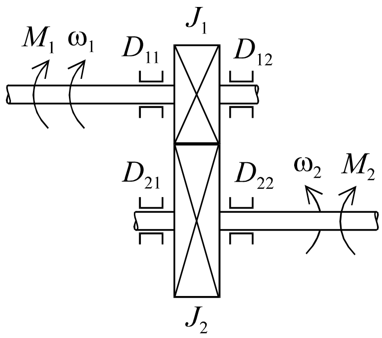

Fig. 10. Kinematic model of single-stage helical gear train

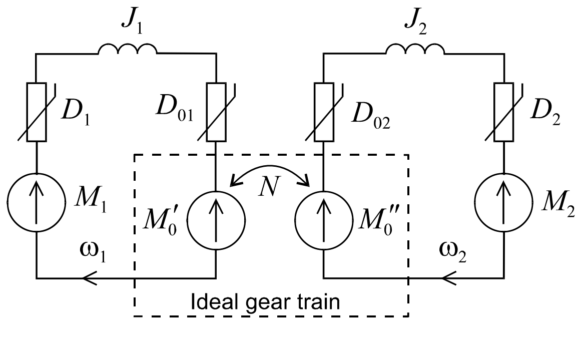

Fig. 11 Circuit diagram of single-stage non-slack gear train

The equations of torques in accordance with the circuit diagram (Fig. 11) are given as follows:

(9) $$M_1 - D_1\omega_1 - \tfrac{d}{dt}\left(J_1\omega_1\right) - D_{01}\omega_1 - M_0' = 0$$

(10) $$M_0'' - D_{02}\omega_2 - \tfrac{d}{dt}\left(J_2\omega_2\right) - D_2\omega_2 - M_2 = 0$$

The following dependency is true in the case of ideal gear train (mechanical power balance):

(11) $$M_0'\omega_1 = M_0''\omega_2$$

Thus, taking Eq. 8 into account, it may be derived:

(12) $$M_0' N = M_0''$$

After defining the new calculation quantities:

(13) $$M_2' = M_2 N^{-1}, \qquad D_2' = D_2 N^{-2}$$
$$D_{02}' = D_{02} N^{-2}, \qquad J_2' = J_2 N^{-2}$$

and dividing the Eq. 10 by N, the equation 14 of torques for the gear train secondary side (gear train output shaft), expressed in terms of gear train primary side (gear train input shaft), has been obtained. This equation together with the Eq. 9 corresponds with the equivalent circuit of gear train (Fig. 12).

(14) $$M_0' - D_{02}'\omega_1 - \tfrac{d}{dt}\left(J_2'\omega_1\right) - D_2'\omega_1 - M_2' = 0$$

The equivalent circuit considered may also be described by a single equation:

(15) $$M - D\omega_1 - \tfrac{d}{dt}\left(J\omega_1\right) = 0$$

where: M is the resultant external torque, J is the equivalent moment of inertia, D is the equivalent mechanical friction coefficient:

(16) $$M = M_1 - M_2', \qquad J = J_1 + J_2'$$
$$D = D_0 + D_1 + D_2', \qquad D_0 = D_{01} + D_{02}'$$

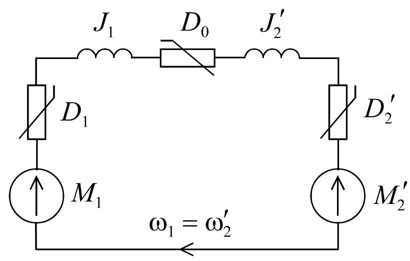

Fig. 12 Equivalent circuit of single-stage non-slack gear train

In the general case the mechanical friction coefficient D_0 defined for gear train (the coefficient determining a part of mechanical power converted into heat dissipation) depends on the mutual torque $M_0 = M_0'$, i.e. $D_0 = f(M_0)$. Thus, if $M_0 = 0$ then also $D_0 = 0$. In the simplest case the following dependency may be adopted:

(17) $$D_{01} = D_{02}' = \frac{1}{2} D_0 = \frac{M_0}{\omega_n} \frac{1-\eta}{1+\eta}$$

where: η is the efficiency of gear train.

In more advanced terms, the respective components of friction coefficient may be determined as follows:

$$D_{01} = \begin{cases} D_0 & \text{for} \quad M_0\omega_1 > 0 \\ 0 & \text{for} \quad M_0\omega_1 \le 0 \end{cases}$$

(18)

$$D'_{02} = \begin{cases} D_0 & \text{for} \quad M_0\omega_2 < 0 \\ 0 & \text{for} \quad M_0\omega_2 \ge 0 \end{cases}$$

where positive values of torque and angular velocities correspond with directions of the arrows marked in Figs. 11 and 12.

Multipath rigid mechanical power transmission without clearances

The kinematic model of multipath mechanical power transmission is depicted in Fig. 13, whereas, the equivalent circuit, corresponding with the multipath mechanical power transmission without clearances, is shown in Fig. 14.

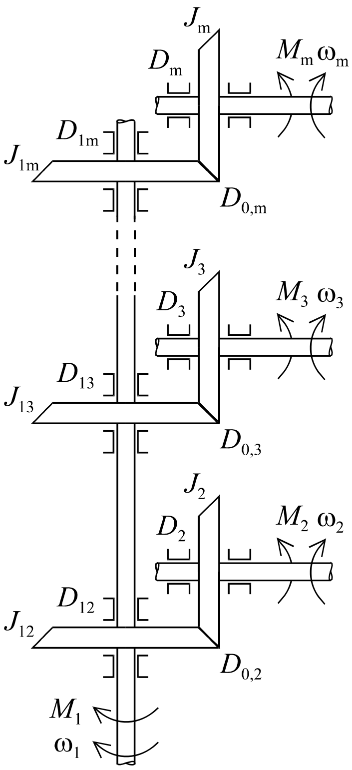

Fig. 13. Kinematic model of multipath mechanical power transmission

The equation of torques in accordance with the equivalent circuit (Fig. 14) is given as follows:

$$M_1 - D_1\omega_1 - \tfrac{d}{dt}\left(J_1\omega_1\right) -$$

(19)

$$-\sum_{k=2}^{m}\left(\left(D'_{0,k} + D'_k\right)\omega_1 + \tfrac{d}{dt}\left(J'_k\omega_1\right) + M'_k\right) = 0$$

where: $D_1 = D_{11} + D_{12} + D_{13} + \ldots + D_{1m}$ and $J_1 = J_{11} + J_{12} + J_{13} + \ldots + J_{1m}$, whereas D_{11} is equivalent mechanical friction coefficient defined for driving motor, J_{11} is equivalent moment of inertia of rotating masses concentrated at the rotor of driving motor as lumped mass (parameters omitted in Fig. 13).

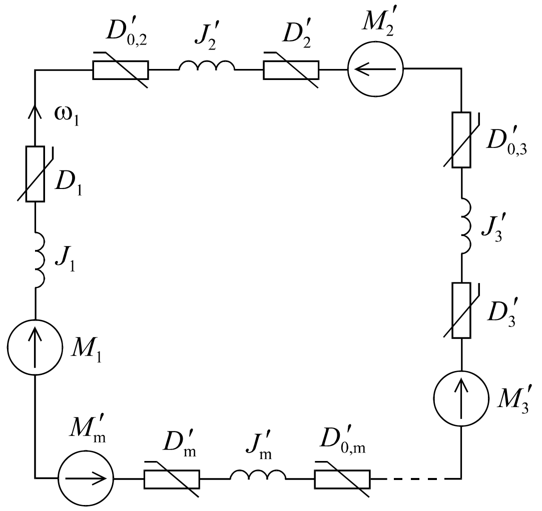

Fig. 14. Equivalent circuit of multipath mechanical power transmission without clearances

Conclusions

In the paper simple models of mechanical connections between the driving motor and the working mechanism are analyzed. The equivalent circuits, typical for electrical systems, are defined for the mechanical systems concerned. Particular attention is paid to the similarities between electrical and mechanical systems. Identifying these similarities is very helpful for electricians in finding a relevant interpretation of mechanical systems, which is particularly important in the case of professionals dealing with electromechanical energy converters or drive systems.

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