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A multiple-point estimator of sinusoidal signal power and its errors

Abstract. *The subject of research is a point estimator of the sinusoidal signal power calculated on the basis of signal samples. The errors of the estimator have been determined. It has been shown that appropriate setting of the signal parameters may lead to bringing the errors of an estimator to zero. A model of an estimator error that takes into account the influence of signal parameters and its processing parameters and the influence of Gaussian noise, has been developed.*

Streszczenie. *Przedmiotem badań jest punktowy estymator mocy sygnału sinusoidalnego obliczany na podstawie próbek sygnału. Wyznaczono błędy estymatora. Pokazano, że odpowiednie ustawienie parametrów sygnału może skutkować sprowadzeniem błędów estymatora do zera. Opracowano model błędu estymatora uwzględniający wpływ parametrów sygnału i parametrów jego przetwarzania oraz wpływ szumu Gaussa. (**Wielopunktowy estymator mocy sygnału sinusoidalnego i jego błędy**).*

Keywords: sinusoidal signal, power, point estimator, Gaussian noise, quantisation.
Słowa kluczowe: sygnał sinusoidalny, moc, estymator punktowy, szum Gaussa, kwantowanie.

Introduction

In the technique, the power is one of the most important energy parameters, which gives an idea of the intensity and stationarity of signal [1]. The article presents the results of a study of a sinusoidal signal power estimator properties. Three measurement cases were considered, when the estimator is determined on the basis of signal samples, quantised signal samples and quantised samples of a signal disturbed by Gaussian noise. It is shown that an appropriate choice of signal parameters may lead to bringing the errors of the estimator to zero. The studies led to the development of a model of an estimator error, which takes into account the influence of signal parameters and its processing parameters, the influence of a finite number of samples and the disturbance in the form of Gaussian noise.

Sinusoidal signal and its samples

Let $x(t)$, $t\in\mathbf{R}$ be a sinusoidal signal with the amplitude $A\in\mathbf{R}_+\backslash\{0\}$, the DC component $A_0\in\mathbf{R}$, the period $T\in\mathbf{R}_+\backslash\{0\}$ and the initial phase $\varphi\in\mathbf{R}$. Then

$$x(t) = A_0 + A\sin\left(\frac{2\pi}{T}t + \varphi\right). \tag{1}$$

Let us assume that the signal $x(t)$ is uniformly sampled with the number of samples $M\in\mathbf{N}\backslash\{0, 1\}$. Under the conditions of synchronous sampling, the signal samples assume the form

$$\mathrm{x}[i] = A_0 + A\sin\left(\frac{2\pi}{M}i + \varphi\right),\ i = 0,1,...,M-1. \tag{2}$$

If signal $x(t)$ is disturbed by the Gaussian noise $n(t)$ with the standard deviation $\sigma_n\in\mathbf{R}_+$, then

$$\mathrm{y}[i] = \mathrm{x}[i] + \mathrm{n}[i], \tag{3}$$

will be the signal samples of the signal

$$y(t) = x(t) + n(t). \tag{4}$$

Let us denote by

$$\mathrm{SNR} = 10\log\left(\frac{A^2}{2\sigma_n^2}\right), \tag{5}$$

an expressed in decibels the signal-to-noise ratio. Based on this, we obtain that

$$\sigma_n = \frac{A}{\sqrt{2\cdot 10^{\frac{\mathrm{SNR}}{10}}}}. \tag{6}$$

Adding noise to a signal aims at simulating situations in which the result of signal power estimation is affected by disturbances occurring in a real measurement channel.

Let us assume that samples $\mathrm{x}[i]$ and $\mathrm{y}[i]$ are quantised in an ideal round-off A/D converter with the step

$$q = \frac{2A}{2^B}, \tag{7}$$

where $B\in\mathbf{N}\backslash\{0, 1\}$ is the converter resolution, while

$$\mathrm{Q}(\pm z) = q\cdot\mathrm{round}\left(\pm\frac{z}{q}\pm\frac{1}{2}\right), \tag{8}$$

where $\mathrm{round}(w)$ is a round-off function rounding off the number $w\in\mathbf{R}$ to the nearest integer [2]. Then

$$\mathrm{x}_q[i] = \mathrm{Q}(\mathrm{x}[i]),\ \mathrm{y}_q[i] = \mathrm{Q}(\mathrm{x}[i] + \mathrm{n}[i]), \tag{9}$$

will be the quantised signal samples of signals $x(t)$ and $y(t)$.

Sinusoidal signal power estimation

The power of the signal $x(t)$ is defined as follows [1]

$$P_{x(t)} = \frac{1}{T}\int_0^T x^2(t)\mathrm{d}t = A_0^2 + \frac{A^2}{2}. \tag{10}$$

Let $2\le m\le M$ be the number of samples of signal $x(t)$ such that mod(M, m)=0. Fixing the value of m means that a power estimator will be calculated on the basis of samples with indices kM/m, $0\le k<m$. Therefore, let us consider the three measuring situations.

(a) Signal power estimation based on signal samples.

Let

$$P_{\mathrm{x}[i]}^{(mp)} = \frac{1}{m}\sum_{k=0}^{m-1}\mathrm{x}^2\left[k\frac{M}{m}\right], \tag{11}$$

be a m-point estimator of power $P_{x(t)}$ calculated on the basis of samples $\mathrm{x}[i]$ of signal $x(t)$. This means that the relative error of estimator (11) expressed in percent can be determined on the basis of the formula

$$\delta_{P_{x(t)}}^{(mp)} = \frac{\left|P_{\mathrm{x}[i]}^{(mp)} - P_{x(t)}\right|}{P_{x(t)}}100. \tag{12}$$

If m=2, then

$$P^{(2p)}_{\mathrm{x}[i]} = \frac{1}{2}\sum_{k=0}^{1} \mathrm{x}^2\left[k\frac{M}{2}\right] = A_0^2 + A^2\sin^2(\varphi), \tag{13}$$

and

$$\begin{aligned}\delta^{(2p)}_{P_{x(t)}} &= \left|\frac{2}{2A_0^2 + A^2}\left(A_0^2 + A^2\sin^2(\varphi)\right) - 1\right|100 \\ &= \frac{A^2}{2A_0^2 + A^2}\left|\cos(2\varphi)\right|100.\end{aligned} \tag{14}$$

Assuming $\varphi=\pi(r+1)/4$, $r\in\mathbf{Z}$, we obtain

$$\begin{aligned}\delta^{(2p)}_{P_{x(t)}} &= \frac{A^2}{2A_0^2 + A^2}\left|\cos\left(\frac{\pi}{2}(r+1)\right)\right|100 \\ &= \frac{A^2}{2A_0^2 + A^2}\left|\sin(\pi r)\right|100 = 0.\end{aligned} \tag{15}$$

If m>2, then

$$\begin{aligned}P^{(mp)}_{\mathrm{x}[i]} &= \frac{1}{m}\left(mA_0^2 + A^2\sum_{k=0}^{m-1}\sin^2\left(\frac{2\pi}{m}k + \varphi\right)\right. \\ &\left. + 2AA_0\sum_{k=0}^{m-1}\sin\left(\frac{2\pi}{m}k + \varphi\right)\right).\end{aligned} \tag{16}$$

Since for m>2 the components

$$\sum_{k=0}^{m-1}\sin^2\left(\frac{2\pi}{m}k + \varphi\right) = \frac{m}{2},\ \sum_{k=0}^{m-1}\sin\left(\frac{2\pi}{m}k + \varphi\right) = 0, \tag{17}$$

then

$$\delta^{(mp)}_{P_{x(t)}} = \left|\frac{2}{m\left(2A_0^2 + A^2\right)}\left(mA_0^2 + A^2\frac{m}{2}\right) - 1\right|100 = 0. \tag{18}$$

Thus, estimator (11) can be determined without error on the basis of at least three samples $\mathrm{x}[i]$ of signal $x(t)$.

(b) Signal power estimation based on quantised signal samples.

Let

$$P^{(mp)}_{\mathrm{x}_q[i]} = \frac{1}{m}\sum_{k=0}^{m-1}\mathrm{x}_q^2\left[k\frac{M}{m}\right], \tag{19}$$

be a m-point estimator of power $P_{x(t)}$ calculated on the basis of quantised samples $\mathrm{x}_q[i]$ of signal $x(t)$. This means that the relative error of estimator (19) expressed in percent can be determined on the basis of the formula

$$\delta^{(mp)}_{P_{x(t)}} = \frac{\left|P^{(mp)}_{\mathrm{x}_q[i]} - P_{x(t)}\right|}{P_{x(t)}}100. \tag{20}$$

Taking into account the results from point (a), let it be assumed that m=3. Then

$$\begin{aligned}P^{(3p)}_{\mathrm{x}_q[i]} &= \frac{1}{3}\sum_{k=0}^{2}\mathrm{x}_q^2\left[k\frac{M}{3}\right] \\ &= \frac{1}{3}\left(\mathrm{Q}^2\left(A_0 + A\sin(\varphi)\right) + \mathrm{Q}^2\left(A_0 + A\sin\left(\frac{2\pi}{3} + \varphi\right)\right)\right. \\ &\left. + \mathrm{Q}^2\left(A_0 + A\sin\left(\frac{4\pi}{3} + \varphi\right)\right)\right),\end{aligned} \tag{21}$$

If A_0=0 and $\varphi=\pi/2+2\pi r$, then

$$\begin{aligned}P^{(3p)}_{\mathrm{x}_q[i]} &= \frac{1}{3}\left(\mathrm{Q}^2\left(A\sin\left(\frac{\pi}{2} + 2\pi r\right)\right)\right. \\ &+ \mathrm{Q}^2\left(A\sin\left(\frac{7\pi}{6} + 2\pi r\right)\right) \\ &\left. + \mathrm{Q}^2\left(A\sin\left(\frac{11\pi}{6} + 2\pi r\right)\right)\right) \\ &= \frac{1}{3}\left(\mathrm{Q}^2(A) + 2\mathrm{Q}^2\left(-\frac{A}{2}\right)\right).\end{aligned} \tag{22}$$

Since

$$\begin{aligned}\mathrm{Q}(A) &= \frac{A}{2^{B-1}}\left(\left\lfloor 2^{B-1} + \frac{1}{2}\right\rfloor\right) = A, \\ \mathrm{Q}\left(-\frac{A}{2}\right) &= -\frac{A}{2^{B-1}}\left(\left\lceil -2^{B-2} - \frac{1}{2}\right\rceil\right) = -\frac{A}{2},\end{aligned} \tag{23}$$

then

$$\delta^{(3p)}_{P_{x(t)}} = \left|\frac{2}{3A^2}\left(A^2 + 2\left(\frac{A^2}{4}\right)\right) - 1\right|100 = 0. \tag{24}$$

If m=4, then

$$\begin{aligned}P^{(4p)}_{\mathrm{x}_q[i]} &= \frac{1}{4}\sum_{k=0}^{3}\mathrm{x}_q^2\left[k\frac{M}{4}\right] = \frac{1}{4}\left(\mathrm{Q}^2\left(A_0 + A\sin(\varphi)\right)\right. \\ &+ \mathrm{Q}^2\left(A_0 + A\sin\left(\frac{\pi}{2} + \varphi\right)\right) + \mathrm{Q}^2\left(A_0 + A\sin(\pi + \varphi)\right) \\ &\left. + \mathrm{Q}^2\left(A_0 + A\sin\left(\frac{3\pi}{2} + \varphi\right)\right)\right).\end{aligned} \tag{25}$$

Assuming A_0=0 and $\varphi=2\pi r$ or $\varphi=\pi/2+2\pi r$, we obtain

$$P^{(4p)}_{\mathrm{x}_q[i]} = \frac{1}{4}\left(\mathrm{Q}^2(A) + \mathrm{Q}^2(-A) + 2\mathrm{Q}^2(0)\right). \tag{26}$$

Since $\mathrm{Q}(-A)=-\mathrm{Q}(A)$ and $\mathrm{Q}(0)=0$, then

$$\delta^{(4p)}_{P_{x(t)}} = \left|\frac{1}{2A^2}\left(A^2 + (-A)^2\right) - 1\right|100 = 0. \tag{27}$$

Attention should also be drawn to the case where m=6. Then

$$\begin{aligned}P^{(6p)}_{\mathrm{x}_q[i]} &= \frac{1}{6}\sum_{k=0}^{5}\mathrm{x}_q^2\left[k\frac{M}{6}\right] = \frac{1}{6}\left(\mathrm{Q}^2\left(A_0 + A\sin(\varphi)\right)\right. \\ &+ \mathrm{Q}^2\left(A_0 + A\sin\left(\frac{\pi}{3} + \varphi\right)\right) + \mathrm{Q}^2\left(A_0 + A\sin\left(\frac{2\pi}{3} + \varphi\right)\right) \\ &+ \mathrm{Q}^2\left(A_0 + A\sin(\pi + \varphi)\right) + \mathrm{Q}^2\left(A_0 + A\sin\left(\frac{4\pi}{3} + \varphi\right)\right) \\ &\left. + \mathrm{Q}^2\left(A_0 + A\sin\left(\frac{5\pi}{3} + \varphi\right)\right)\right).\end{aligned} \tag{28}$$

Assuming A_0=0 and $\varphi=\pi/2+2\pi r$, we obtain

$$P^{(6p)}_{\mathrm{x}_q[i]} = \frac{1}{6}\left(\mathrm{Q}^2(A) + \mathrm{Q}^2(-A) + 2\mathrm{Q}^2\left(\frac{A}{2}\right) + 2\mathrm{Q}^2\left(-\frac{A}{2}\right)\right). \tag{29}$$

Since Q(-A/2)=-Q(A/2), then

$$\delta^{(6p)}_{P_{x(t)}} = \left| \frac{1}{3A^2}\left(2A^2+4\left(\frac{A}{2}\right)^2\right)-1\right|100=0. \tag{30}$$

The case when m=6 is equivalent to a situation when m=3.

(c) Signal power estimation based on quantised samples of a signal disturbed by Gaussian noise.

Let

$$P^{(mp)}_{y_q[i]} = \frac{1}{m}\sum_{k=0}^{m-1} y_q^2\left[k\frac{M}{m}\right], \tag{31}$$

be a m-point estimator of power $P_{x(t)}$ calculated on the basis of quantised samples $y_q[i]$ of signal $y(t)$. Interference of the signal $x(t)$ with the noise $n(t)$ results in the necessity of K-fold averaging of the estimator (31). As a result of this operation we obtain the following estimator

$$\begin{aligned}\overline{P}^{(mp)}_{y_q[i]} &= \frac{1}{K}\sum_{j=1}^{K}\left(P^{(mp)}_{y_q[i]}\right)_j \\ &= \frac{1}{mK}\sum_{j=1}^{K}\sum_{k=0}^{m-1} Q^2\left(x\left[k\frac{M}{m}\right]+n_j\left[k\frac{M}{m}\right]\right).\end{aligned} \tag{32}$$

This means that the relative error of estimator (32) expressed in percent can be determined on the basis of the formula

$$\delta^{(mp)}_{P_{x(t)}} = \frac{\left|\overline{P}^{(mp)}_{y_q[i]} - P_{x(t)}\right|}{P_{x(t)}}100. \tag{33}$$

If m=3, then

$$\begin{aligned}\overline{P}^{(3p)}_{y_q[i]} &= \frac{1}{3K}\sum_{j=1}^{K}\sum_{k=0}^{2} Q^2\left(x\left[k\frac{M}{3}\right]+n_j\left[k\frac{M}{3}\right]\right) \\ &= \frac{1}{3K}\sum_{j=1}^{K}\Big(Q^2\left(A_0+A\sin(\varphi)+n_j[0]\right) \\ &\quad + Q^2\left(A_0+A\sin\left(\frac{2\pi}{3}+\varphi\right)+n_j\left[\frac{M}{3}\right]\right) \\ &\quad + Q^2\left(A_0+A\sin\left(\frac{4\pi}{3}+\varphi\right)+n_j\left[\frac{2M}{3}\right]\right)\Big).\end{aligned} \tag{34}$$

Assuming φ=2πr, we obtain

$$\begin{aligned}\overline{P}^{(3p)}_{y_q[i]} &= \frac{1}{3K}\sum_{j=1}^{K}\Big(Q^2\left(A_0+n_j[0]\right) \\ &\quad + Q^2\left(A_0+\frac{\sqrt{3}}{2}A+n_j\left[\frac{M}{3}\right]\right) \\ &\quad + Q^2\left(A_0-\frac{\sqrt{3}}{2}A+n_j\left[\frac{2M}{3}\right]\right)\Big).\end{aligned} \tag{35}$$

Then

$$\begin{aligned}\delta^{(3p)}_{P_{x(t)}} = \Bigg|&\frac{2}{3K\left(A^2+2A_0^2\right)}\Big(\sum_{j=1}^{K}\Big(Q^2\left(A_0+n_j[0]\right) \\ &+ Q^2\left(A_0+\frac{\sqrt{3}}{2}A+n_j\left[\frac{M}{3}\right]\right) \\ &+ Q^2\left(A_0-\frac{\sqrt{3}}{2}A+n_j\left[\frac{2M}{3}\right]\right)\Big)\Big)-1\Bigg|100.\end{aligned} \tag{36}$$

Quantisation of samples of signal $y(t)$ can be considered as a quantisation of Gaussian noise with standard deviation σ_n and the expected values

$$\mu_n = A_0,\ \mu_n = A_0+\frac{\sqrt{3}}{2}A,\ \mu_n = A_0-\frac{\sqrt{3}}{2}A. \tag{37}$$

If we apply the Widrow's statistical theory of quantization [2], then we can calculate the following error

$$\begin{aligned}\hat{\delta}^{(3p)}_{P_{x(t)}} = \Bigg|&\frac{2}{3\left(A^2+2A_0^2\right)}\Big(E\left[n_q^2\right]\Big|_{\mu_n=A_0} + E\left[n_q^2\right]\Big|_{\mu_n=A_0+\frac{\sqrt{3}}{2}A} \\ &+ E\left[n_q^2\right]\Big|_{\mu_n=A_0-\frac{\sqrt{3}}{2}A}\Big)-1\Bigg|100,\end{aligned} \tag{38}$$

where

$$E\left[n_q^2\right] = -\frac{d^2}{dv^2}\Phi_{n_q}(v)\Big|_{v=0}, \tag{39}$$

is an ordinary moment of the 2nd order of the random variable n_q assuming the values of quantised Gaussian noise $n(t)$, with

$$\Phi_{n_q}(v) = \sum_{i=-\infty}^{\infty}\Phi_n\left(v-\frac{2\pi}{q}i\right)\mathrm{sinc}\left[\frac{q}{2}\left(v-\frac{2\pi}{q}i\right)\right], \tag{40}$$

where

$$\Phi_n(v) = e^{j\mu_n v}e^{-0.5v^2\sigma_n^2},\ j=\sqrt{-1}, \tag{41}$$

is the characteristic function of the random variable n assuming the values of Gaussian noise $n(t)$. Between errors (36) and (38) an obvious relationship exists, i.e. if $K\to\infty$, then error (36) converges to error (38). Thus, error (38) is a mathematical model of error (36).

Let us now consider the case when φ=π/2+2πr. Based on (34), we obtain

$$\begin{aligned}\overline{P}^{(3p)}_{y_q[i]} &= \frac{1}{3K}\sum_{j=1}^{K}\Big(Q^2\left(A_0+A+n_j[0]\right) \\ &\quad + Q^2\left(A_0-\frac{A}{2}+n_j\left[\frac{M}{3}\right]\right) \\ &\quad + Q^2\left(A_0-\frac{A}{2}+n_j\left[\frac{2M}{3}\right]\right)\Big),\end{aligned} \tag{42}$$

and

$$\begin{aligned}\delta^{(3p)}_{P_{x(t)}} = \Bigg|&\frac{2}{3K\left(A^2+2A_0^2\right)}\Big(\sum_{j=1}^{K}\Big(Q^2\left(A_0+A+n_j[0]\right) \\ &+ Q^2\left(A_0-\frac{A}{2}+n_j\left[\frac{M}{3}\right]\right) \\ &+ Q^2\left(A_0-\frac{A}{2}+n_j\left[\frac{2M}{3}\right]\right)\Big)\Big)-1\Bigg|100.\end{aligned} \tag{43}$$

Then the mathematical model of error (43) assumes the form

$$\begin{aligned}\hat{\delta}^{(3p)}_{P_{x(t)}} = \Bigg|&\frac{2}{3\left(A^2+2A_0^2\right)}\Big(E\left[n_q^2\right]\Big|_{\mu_n=A_0+A} \\ &+ 2E\left[n_q^2\right]\Big|_{\mu_n=A_0-\frac{A}{2}}\Big)-1\Bigg|100.\end{aligned} \tag{44}$$

Determination of the mathematical model of error (33) can also be continued if m>3. Then we obtain the following error

$$\hat{\delta}^{(mp)}_{P_{x(t)}} = \left| \frac{2}{m\left(A^2+2A_0^2\right)} \cdot \sum_{k=0}^{m-1}\left(E\left[n_q^2\right]\Big|_{\mu_n=A_0+A\sin\left(\frac{2\pi}{M}k+\varphi\right)}\right) -1\right| 100. \tag{45}$$

It should be noted that error (45) takes into account not only the influence of signal parameters, the influence of quantization and the influence of noise but, above all, the influence of a finite number m of samples $y[i]$ of signal $y(t)$.

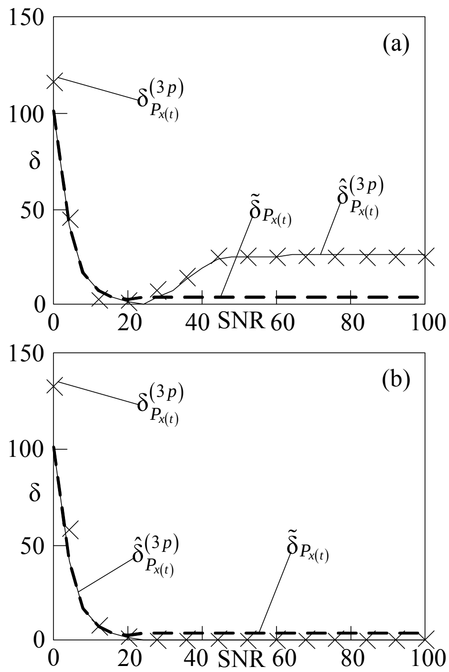

Fig.1. Errors (33), (45) and (46) as a function of SNR, where (a) φ=0 and (b) $\varphi=\pi/2$

On the basis of the literature [2], we can also define the following error of signal power estimator

$$\tilde{\delta}_{P_{x(t)}} = \left| \frac{2}{A^2+2A_0^2} E\left[y_q^2\right] -1\right| 100, \tag{46}$$

where

$$E\left[y_q^2\right] = -\frac{\mathrm{d}^2}{\mathrm{d}v^2}\Phi_{y_q}(v)\Big|_{v=0}, \tag{47}$$

while

$$\Phi_{y_q}(v) = \sum_{i=-\infty}^{\infty} \Phi_{x+n}\left(v-\frac{2\pi}{q}i\right)\mathrm{sinc}\left[\frac{q}{2}\left(v-\frac{2\pi}{q}i\right)\right], \tag{48}$$

and

$$\Phi_{x+n}(v) = e^{\mathrm{j}A_0 v} e^{-0.5v^2\sigma_n^2} J_0(Av), \tag{49}$$

where $J_0(\cdot)$ is a Bessel function of the 0th order. Unlike error (45), error (46) is calculated on the assumption that M is infinitely large.

Figure 1 presents the results of errors (33), (45) and (46) as a function of SNR, in a situation, where A=1 V, A_0=0, φ=0 and $\varphi=\pi/2$, M=12, m=3, B=3, K=100.

It should be noted that an appropriate selection of initial phase results in the errors of signal power estimator quickly assuming values close to zero (if A_0=0). We also noted that if $M\to\infty$ and $m\to M$, then error (45) converges to error (46).

Conclusion

In this article, it has been shown that an appropriate selection of the sinusoidal signal parameters and its processing parameters may result in a decrease of or bringing to zero the errors of signal power estimator.

The most important result of the research is a mathematical model of power estimator error, which takes into account the influence of signal parameters, the influence of quantization, the influence of noise, and the influence of a finite number of signal samples. In the author's opinion, the developed model is best suited to describe the situation of measurement, in which the power of signal is estimated on the basis of a small number of quantized samples of signal which is disturbed by noise.

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