Zdzisław TRZASKA

Warsaw University of Ecology and Management

doi:10.15199/48.2015.03.19

Impact and chaotic phenomena in nonlinear nonsmooth electrical dynamical systems

Abstract. The article presents results of the analysis of nonlinear oscillators with nonsmooth elements and nonlinear systems with nonsmooth excitations. Main attention has been focused on highlighting specific properties of nonsmooth systems compared to their smooth counterparts. Non-smooth transformation of the time variable and the replacement of initial issues by boundary problems have been taken as the base for the analytical method. Results of numerical simulations and computing in the form of graphs of voltage and current waveforms and attractors are presented.

*Streszczenie. W artykule przedstawiono wyniki analizy nieliniowych oscylatorów z niegładkimi elementami oraz układów nieliniowych z niegładkimi wymuszeniami. Zasadnicza uwaga została skupiona na uwypukleniu specyficznych właściwości układów niegładkich w porównaniu do ich odpowiedników gładkich. Niegładką transformację zmiennej czasowej i zastąpienie zagadnień początkowych zagadnieniami brzegowymi przyjęto jako podstawę metody analitycznej. Przedstawiono wyniki symulacji numerycznych i obliczeń komputerowych w postaci wykresów przebiegów napięć i prądów oraz atraktorów (**Uderzeniowe i chaotyczne zjawiska w nieliniowych niegładkich elektrycznych układach dynamicznych**).*

Keywords: nonlinear oscillators, impact phenomena, nonsmooth characteristics, chaotic states
Słowa kluczowe: nieliniowe oscylatory, zjawisko uderzenia, niegładkie charakterystyki, chaotyczne przebiegi.

Introduction

The rapid development of the present technology and the ever-increasing requirements for installed devices imply stimulation of research centers not only to design new practical systems, but also to search for new components with improved operational characteristics compared to the previously used [1, 2]. Modern methods of design and optimization techniques coexist with numerical modeling and computer simulations. Using such tools allows getting relevant information concerning, for example, critical values, dynamic changes, and the expected or projected usage of the planned or actually tested objects and so on. Numerical simulation of measurement and control appliances widely used in power systems is essential in order to solve problems at an early design stage and accelerate their development and implementation to the practice, as well as to explore and analyze the new topologies and/or control devices [3]. The difficulties associated with numerical simulations of oscillations of deterministic systems can be caused by various disadvantages such as, e.g., discontinuity of the elements characteristics, the manifested ambiguity of dynamic and damping parameters, caused by hysteresis, saturation and dead zones, the phenomena of attenuation and impact, timing delays, the occurrence of switching mono-and bistable states [4, 5].

The behaviour of such a complex system, which is the power system, is determined increasingly by computerized control and automatic analogue or digital decision-making devices, when the treatment and detection of events representing discontinuities is notably indispensable. To fully understand the system's behaviour and meet high performance specifications a recourse to model all dynamics together with their interactions is needed, and this is most important when there are strong interactions among the parts of the system [6]. This important class of non-smooth dynamical systems shows not only the typical features of smooth nonlinear systems, such as generic bifurcations, multiple solutions and chaos, but also displays a new phenomenon such as, for instance, the phenomenon of the impact [7]. It means a sudden change of the system state, when a periodic orbit reaches the barrier at zero speed, but non-zero acceleration: small nonzero perturbations of such orbits may or may not have to disclose. So far, extensive research was put on smooth dynamic systems; however, non-smooth systems are not as well understood.

Studies of non-smooth dynamical systems can be based on a general formulation that enables a simulation and control of the system with all its possible modes together. Helpful in this regard are mathematical methods of identifying parameters to establish and optimize transient response as well as methods of approximation dedicated to solving these problems [8 -10].

Suppression of surges caused by switching and/or by lightning is crucial for energy quality of any three-phase power system, and for its protection. Transient processes cause damages to sensitive electronic equipment in the home, schools, shopping centers, and industrial as well as medical equipments, sewage treatment plants, factories, etc. In order to protect the power system against overvoltage transients, peaks or surges caused by lightning surge arresters and transient voltage surge suppressors are installed in practice [11 – 15].

To represent a power system periodically driven in a specific way, an impact oscillator is a term used herein which exhibits an intermittent or continuous oscillation sequence of commuting states with the restriction of limits. Another important aspect of the dynamics of the impact oscillator, which attracts especially interested-plied to preserve low speed closes to the shock, characterized by the phenomenon of flutter (with chatters) [16 -18].

The plan of the paper is the following. Section 2 provides basic formulations and introductory results to be used in this article. Section 3 derives the main problem concerning solutions of impact oscillators with periodic discontinuous excitations. In Section 4, we give results of applications of the NSTT method to determination of *T*-periodic solution of equations describing strongly nonlinear impact oscillator with periodic pulsed excitations. Section 5 is devoted to studies of chaotic phenomena caused by surge arresters in the overvoltage protection of power cables. Finally, in Section 6 conclusions are presented.

Basic properties of nonsmooth dynamical systems

Over the past decade, dynamical systems, in which there are non-smooth signals, have gained increasing recognition in electromechanical engineering and other applied sciences, as the non-smooth effects can already be really taken into account and there is no need of smoothing. This is due to the fact that have been developed new analytical and numerical tools suited for testing of non-smooth systems. This problem has steadily gaining in importance as the applications for the construction of electrical and electronic components with increasingly merge, as well as less and less resistance to current and voltage surge pulses are growing constantly [19, 20].

A key feature of the shock dynamics of nonlinear systems is the phenomenon of sudden change, when the periodic orbit reaches the barrier at zero speed, but non-zero acceleration. Then a small nonzero perturbation of such orbits may disclose slight irregular oscillations that bring different forms compared to the smooth nonlinear dynamics. Particularly vulnerable are the controllers (PLC) acquiring signals (data) from sensors spread over large areas and long lines combined with other controllers, control apparatus in the control room, etc..

In the group of dynamic interactions we introduce the basic division:

♣ oscillations, characterized by a long duration of action, involving many cycles and the limited amplitude of characteristic signals (charge, flux, current, voltage),

♣ single shock excitations of short duration pulse operation and high amplitude (e.g. the impact of an aircraft at the power line, the impact of electromagnetic waves during *E*-bomb explosions),

♣ shock incident signals of repeated action, in which the strong pulse interactions occur periodically (e.g. violent gusts of wind at the over head power line, dynamic overvoltage emerging during emergency relief system, ground fault arising during persistent short-circuit system with the ground, resonance resulting at favourable conditions for the formation of resonance and ferroresonance, defibrillation, or stop of harmful ventricular flickers by using current pulses delivered by the defibrillator electrodes) [21-23].

Non-smooth dynamical systems exhibit more complex and enriched dynamics, when compared with their smooth counterparts. However, the qualitative analysis and design is still the subject of intensive research. Recently, it has been found that non-smooth dynamical systems reveal significant wealth of nonlinear phenomena, including a chaotic, that are unique to this potentially important class of nonlinear systems. Furthermore, in the case of highly non-linear systems a significant unpredictability appears in the course of their dynamics. This is the main feature of such systems. For instance, a sudden transition from a stable periodic oscillation to the full range of chaotic oscillations may often occur in non-smooth system at small change of parameters, while such phenomenon is not observed in smooth configurations, if they are not in series with period-doubling bifurcation [4, 8]. The presence of shocks in electromagnetic processes is a classic source of non-linear and complex behaviours of power systems. Therefore, a better understanding of the effects of such processes can be used to monitor system instantaneous states.

The dynamics of nonsmooth oscillations with shock forcing is analyzed in the sequel by using a relatively new mathematical tool, which appears to be hyperbolic algebra [7, 18]. The key idea of this tool is steeped in of non-smooth time transformations (NSTT) proposed originally in [7] for strongly nonlinear, but still smooth models. The NSTT is based on the algebra of hyperbolic numbers, an approach corresponding to the algebra of complex numbers and functions in the case of smooth excitations. The solution efficiency of NSTT results from explicit links between impact dynamics and hyperbolic algebras analogously to the link between harmonic oscillations and conventional complex analyses. Presently, this is one of the principal challenges at the crossroad between non-smooth dynamical systems, mathematics, and computer science [8, 24]. Basic details in this direction are presented in the next Section.

Impact oscillator and its fundamental properties

In the domain of nonlinear, non-smooth dynamical systems, impact oscillator is a fundamental issue, for it can be used to describe a much broader area of application than it is possible with the classical approach. Therefore, it is of fundamental importance to study the impact oscillator model of the harmonic oscillator and to discuss the specific properties of its solutions.

Impact oscillator is a term used herein to represent a system which is periodically driven in a different way, which also is an intermittent or continuous time-varying sequence of switchings with the limit restrictions. This important structure of non-smooth dynamical systems shows not only the typical features of smooth nonlinear systems, such as generic bifurcations, multiple solutions and chaos. Moreover, it also displays new phenomena appearing as, for instance, the sudden change of the system state, where a periodic orbit reaches the barrier at zero speed, but non-zero acceleration. Small nonzero perturbations of such orbits may or may not have to disclose the impact.

An impact oscillator is represented by

$$\ddot{x}(t) + x^{2n-1}(t) = 0, \qquad n \to \infty \tag{1}$$

where the upper dot stands for the time derivative. This mathematical model describes two dual nonlinear circuits shown in Table 1, right panel, first position. It has to be emphasized here that the impact oscillator is composed of a capacitor and an inductor but one of these elements is linear and the other must be nonlinear displaying characteristic with saturation. Such relatively simple circuits with one nonlinear element may yield quite complex periodic response diagrams with appropriate initial conditions impacting those diagrams.

Despite the strong nonlinearity leading to impacts, the limit oscillator is notwithstanding described by quite simple elementary functions such as triangular sine and rectangular cosine, say $p(t)$ and $\dot{p}(t) = e(t)$ which are presented in Table 1, right panel, second position.

Alongside the above mathematical challenges, this case admits interpretation by means of the total energy

$$\frac{\dot{x}^2}{2} + \frac{x^{2n}}{2n} = \frac{1}{2} \tag{2}$$

where the number 1/2 on the right-hand side corresponds to the initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$.

Taking into account that the state variable of the oscillator reaches its amplitude value at zero kinetic energy, gives the estimate $-n^{-1/(2n)} \le x(t) \le n^{1/(2n)}$ for any time t. Since $n^{1/(2n)} \to 1$ as $n \to \infty$ then the limit oscillation is restricted by the interval $-1 \le x(t) \le 1$. Inside of this interval, the second term on the left-hand side of expression (2) vanishes and hence, $\dot{x} = \pm 1$or $x = \pm t + \alpha_{\pm}$,where $\alpha_{\pm}$ are constants. By manipulating with the signs and constants one can construct the sawtooth sine $p(t)$ - triangular wave – since there is no other way to providing the periodicity condition. So, the family of oscillators (1) includes the two quite simple cases associated with the boundaries of the interval $1 \le n < \infty$. Respectively, one has the two couples of periodic functions

$$\{x, \dot{x}\} = \{\sin(t), \cos(t)\} \quad \text{if } n = 1 \tag{3}$$

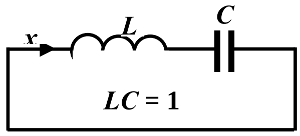
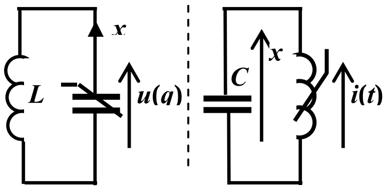
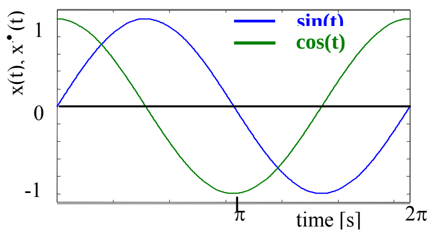
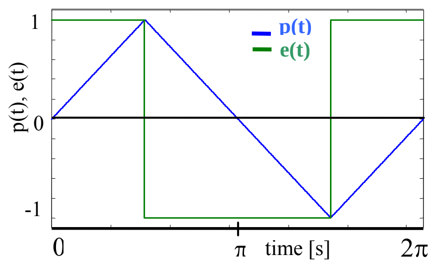
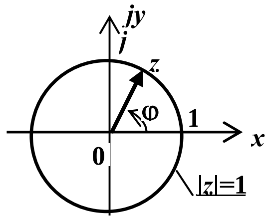
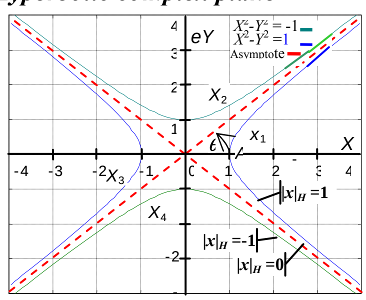
and

$$\{x, \dot{x}\} = \{p(t), \dot{p}(t) = e(t)\} \quad \text{if } n \to \infty \tag{4}$$

where $\dot{p}(t) = e(t)$ is a generalized derivative of the sawtooth sine and will be named as a rectangular cosine. The analogy of respective relations in the hyperbolic and the conventional complex planes is shown in Table 1.

Hitherto, the power-form characteristics with integer exponents were employed for phenomenological modelling of amplitude limiters in oscillating capacitive networks [25] and for illustrations of impact asymptotic [26]. It should be noted that such phenomenological approaches to impact modelling are designed to capture the integral effect of interaction with physical constraints bypassing local details of the dynamics near constraints. Such details, first of all, depend upon both the oscillating system and amplitude limiter of physical properties. In many cases, Hertz model of interaction may be adequate to describe the local dynamics near constraint surfaces [7]. Note that direct replacement of the characteristic x^{2n-1} in (1) by inertial term $kx^{3/2}$ gives no oscillator. It is also worth to mention that for internal values of n in the range $1 < n < \infty$ all oscillator waveforms are described by special features. Any presence of functions $p(t)$ and $e(t)$ in further developed analytical

Table 1. Representations of smooth linear and nonsmooth nonlinear dynamical systems

	Harmonic State	**Impact State**
1	***Harmonic osillator***  $\ddot{x} + x = 0$	***Impact oscillators***  $\ddot{x} + x^{2n-1} = 0,\ n \to \infty$
2	***Signals: sin(t) and cos(t)*** 	***Signals: p = sin$_1$(t) and e = cos$_p$ (t)*** 
3	***Conventional complex numbers*** $z = x + jy$	***Complex hyperbolic numbers*** $x(t) = X_p(t) + Y_p(t)e(t)$ $e(t\_ = \dot{p}(t), e^2(t) = 1$
4	***Conventional complex plane***  $\varphi = \arctan(\frac{y}{x})$ $z = \exp(j\varphi)$	***Hyperbolic complex plane***  $-\infty < \theta < \infty$ $x_{1,3} = \pm\exp(e\theta),\ \theta = \operatorname{arctanh}(\frac{Y}{X});$ $x_{2,4} = \pm e \cdot \exp(e\theta),\ \theta = \operatorname{arctanh}(\frac{X}{Y});$
5	***Fourier series*** $\sum_{k=0}^{\infty}[A_k\cos(kt) + B_k\sin(kt)]$	***Sawtooth power series*** $\sum_{k=0}^{\infty}[\frac{1}{k!}X^{(k)}(0)p^k + \frac{1}{k!}Y^{(k)}(0)p^k\dot{e}]$

an algorithm is not a simple match of different pieces of solutions but it has its real physical basis and invokes specific mathematical tools. Their effectiveness is determined by the following statement [18]:

Any periodic process x(t) of the period normalized to T=4a can be expressed through the dynamic state of the impact oscillator, {p(t), e(t)}, in the form of ' hyperbolic complex number'

(5) $$x(t) = X(p) + Y(p)e(t)$$

where the functions X(p) and Y(p) on the right-hand side are easily expressed through the original function $x(t)$*, if this function is known.*

In a case when $x(t)$ is an unknown periodic oscillation of some dynamical system, equations for $X(p)$ and $Y(p)$ components are obtained by substituting (5) into the corresponding differential equation of oscillation. Then either analytical or numerical procedures can be applied. Therefore expression (5) can be qualified as non-smooth time transformation, $t \rightarrow p$, on the manifold of periodic oscillations.

Nowadays it is a well recognized conviction that hyperbolic calculus leads to better results than classical one [27]. In some cases, it is possible to find out closed form solutions of nonlinear differential equations [8]. Therefore, a description of strongly nonlinear dynamical systems with using hyperbolic algebra may lead to results of major importance. The occurrence of such algebraic structures seems to be essentials feature of the approach since it justifies and simplifies analytical manipulations with non-invertible temporal substitutions such as NSTT. In particular, analytical solutions can be obtained under lossless conditions. These solutions show a good matching with the corresponding numerical solutions at any energy level even within the first order asymptotic approximation. To illustrate the above statement let us consider an oscillator with constant inductance *L*=1H connected with a multilayer ceramic capacitor (MLCC) with dielectric made of X7R material ($BT\text{-}Nb_2O_5\text{-}Co_3O_4$) [28]. The hysteretical effects in MCCLs with X7R characteristics are attenuated by the dopants and small granulation of ceramic (0.5μm). The voltage-charge characteristic $v_C(q_C)$ of such a capacitor takes the form presented in Figure 1. Such an oscillator is described by the following equation

(6) $$\ddot{x} + \frac{\tan(x)}{\cos^2(x)} = 0$$

where x denotes the non-dimensional state variable

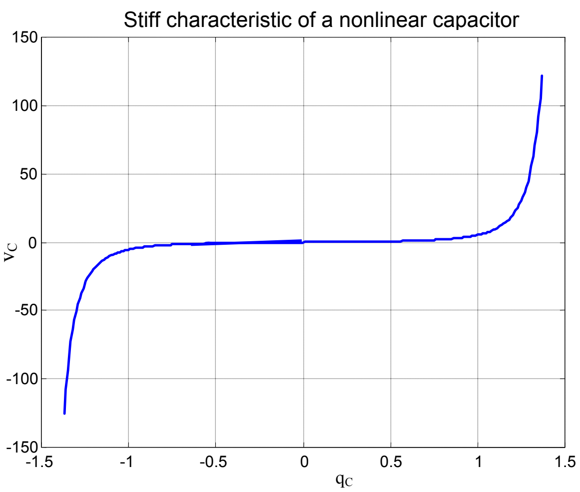


Fig.1. Characteristic of a nonlinear multilayer ceramic capacitor

expressing charge of nonlinear capacitor and dots mean derivatives with respect to time.

Observe that the second term in left-hand side of (6) denoted by

(7) $$f(x) = \frac{\tan(x)}{\cos^2(x)}$$

is an odd function in respect to state variable $x(t)$.

Following the rules introduced in [7, 18, 27] and making the substitutions

(8) $$t \rightarrow p(t/a) \quad \text{and} \quad x(t) = X(p) + Y(p)e(t)$$

in (6), gives the boundary value problem

(9) $$X'' + a^2 f(X) = 0 \qquad X'_{|p=\pm 1} = 0 \qquad Y \equiv 0$$

where primes mean derivatives with respect to $p(t/a)$ with the period $T = 4a$.

Solving the boundary value problem (9) and substituting the result into (8) yields

(10) $$x(t) = \arcsin(\alpha \sin(p(t)/\sqrt{1-\alpha^2}))$$

where α is a constant dependent on the amplitude *A* of oscillations, namely

(11) $$\alpha = \sin(A)$$

Figure 2 illustrates the evolution of the oscillation shapes in the normalized coordinates.

It has to be noted that the stiff characteristic (Fig.1) is close to linear for relatively small amplitudes but becomes infinitely growing as the amplitude approaches certain limits. As a result, the corresponding temporal mode of oscillation (Fig.2) changes its shape from smooth quasi harmonic to nonsmooth triangular sine. When the amplitude *A* is close to zero, the oscillator linearizes whereas solution gives the corresponding sine-wave temporal shape. On the other hand, the total energy

(12) $$E = \frac{1}{2}\tan^2(A)$$

becomes infinitely large as the parameter *A* approaches the upper limit π/2. In this case, the period vanishes while the oscillation takes the triangular wave shape, as it is seen from expression (10).

It is now evident that the studied oscillator belongs to a class of strongly nonlinear systems admitting unusual

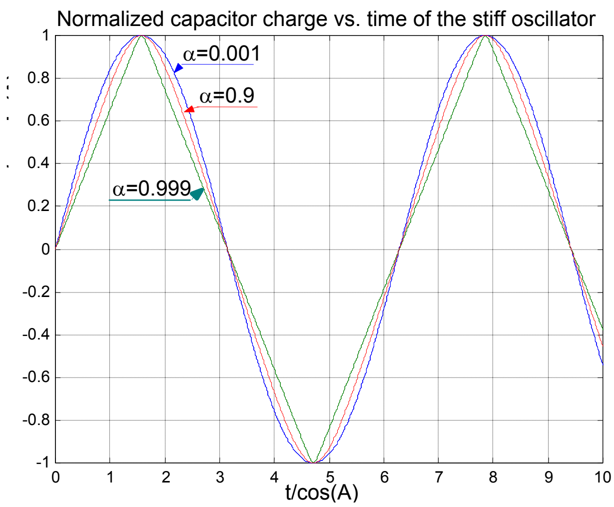


Fig.2.Changes in time of capacitor relative charge of the stiff oscillator for α=sin(A)

and surprisingly simple exact general solutions at any level of the total energy. It is worth to underline that the characteristic of the nonlinear MCCL captures succinctly general physical situations with stiff behaviour of the restoration variable. In recent years, the use of BT ceramics for manufacturing MLCCs of smaller size and higher capacitance is increasing sharply.

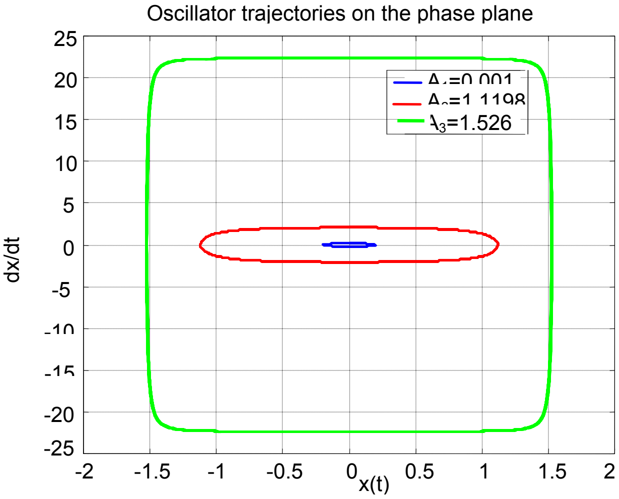


Fig.3. Steady state trajectories of stiff oscillator for different amplitudes (coordinates corresponding A_1 are rescaled by 10^3)

Taking into account expression (10) we can calculate the normalized current in the stiff oscillator and obtain

(13) $$\dot{x}(t) = \frac{\tan(A)\cos(t/\cos(A))}{\sqrt{1-[\sin(A)\sin(t/\cos(A)]^2}}$$

The above state variable can be considered as the linear momentum of the oscillator Hamiltonian $\Im$.

Taking advantages of (10) and (13) we may define the so-called action coordinate *S* as the area bounded by the oscillator trajectory on the phase plane $(x(t), \dot{x}(t))$ and divided by 2π. Thus we have

(14) $$S = \frac{1}{2\pi}\int_0^T \dot{x}(t)dx(t) = \frac{I_a}{2\pi} = \frac{1}{\cos(A)} - 1$$

where *T* denotes the oscillation period.

In the case of stiff oscillator (6), for three amplitude values we obtain from (14) with A_1 =0.001, A_2 =1.1198 and A_3 =0.999 the following areas I_{a1} = 3.141e-006, I_{a2} =8.1314, and I_{a3} = 134.0912., while the actual values estimated from the numerical solution in Matlab are I_{a1n} =3.1684e-006, I_{a2n} =8.1788, and I_{a3n} =134.7864. This indicates a good agreement of the results obtained from analytical and numerical calculations. In all numerical calculations presented in this paper we used Matlab's procedure with *ruler* = *baser* = 10^{-10}.

It has been shown that a more complicated nonsmooth (or a quasi nonsmooth) dynamical regime can be represented by this nonsmooth oscillation and its different deformations, according to the power forms, p^3, p^5, .., or in another simple way. Hence, the introduction of the pair {*p*(*t*); *e(t*)} is not a purely mathematical manipulation, but it has a valuable physical meaning

Since the initial conditions, corresponding to the periodic regimes are known, one can integrate numerically the differential equations of oscillations (6) in order to check the analytical solutions. Applying procedure *ode15s* from the program package Matlab [29] gives the diagrams shown in Figure 4.

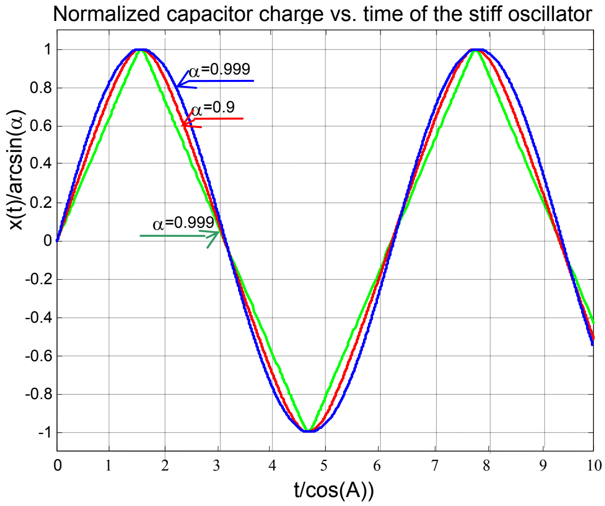


Fig.4. Numerical results for the stiff oscillator with α = sin(*A*)

Notice that in the above case of stiff oscillators one would obtain that both, analytical and numerical, results are perfectly matching.

Strongly nonlinear oscillator under periodic pulsed excitation

Instantaneous impulses acting on a nonsmooth oscillator can be modelled either by imposing specific matching conditions on the oscillator state variables at pulse times or by introducing Dirac's impulses into the differential equations of oscillation. The analytical tool, which is described in [7, 18, 27]], on the one hand, eliminates the singular terms from the equations and, on the other hand, brings solutions to the unit-form of a single analytic expression for the whole time interval. Moreover, a special feature of this technique is that the smooth solutions to be defined are expressed through a pair of nonsmooth functions such as a saw-tooth sine and rectangular cosine.

Introducing the saw-tooth temporal argument may significantly simplify solutions whenever exciting functions are combined of the triangular wave and its derivatives. Accordingly to these points, the introduction of a non-smooth independent variable into differential equations of oscillation can be purposeful.

To illustrate the effectiveness in applications of the nonsmooth temporal transformation with no additional complications which may be brought about by technical details related to a concrete practical problem we will seek periodic solutions of equations describing stiff nonlinear oscillators. For this purpose let us consider the case of strongly nonlinear exactly solvable oscillator described in the precedent Section by equation (6) with added periodic impulsive excitation on the right-hand side as follows

(15) $$\ddot{x} + \frac{\tan(x)}{\cos^2(x)} = 2H\sum_{-\infty}^{\infty}(-1)^k\delta[t-(2k-1)a]$$

where *H* > 0 and *a* = *T*/4 > 0 characterize the amplitude and period of the impulsive excitation, respectively.

Taking into account properties of the rectangular cosine e(*t* /*a*) and applying them to the right hand side of equation (13) we get

(16) $$2H\sum_{-\infty}^{\infty}(-1)^k\delta[t-(2k-1)a] = \frac{H}{a}\frac{d}{d(t/a)}e(t/a)$$

Now, representing periodic solutions in the form of (8), $x(t) = X(p) + Y(p)e(t)$, and substituting this into equation (13), gives

$$\begin{aligned} & a^{-2}X''(p) + \tan(X(p)) + \tan^3(X(p)) \\ (17) \qquad & \quad = a^{-2}(aH - X')\cdot \dot{e}(t/a), \\ & X'_{|p=\pm 1} = aH, \\ & Y(p) \equiv 0. \end{aligned}$$

Boundary value problem (17) describes the class of steady state periodic oscillations of the period *T*=4*a* of impulsively excited oscillations.

With reference to equations (9) and (1), we can present the solution of equation (17) exactly in the form

$$(18) \qquad X(p) = \arcsin(\sin(A)\cdot\sin(p/\cos(A)))$$

where *A* is a constant, which can be computed by taking into account condition for *X'*(*p*) in (17) and the symmetry of solution (18).

Substituting the result for $p = \pm 1$ from (5) in (18) and performing analytical manipulations with elementary functions, yields

$$(19a) \quad T = 4q\cdot\arccos\left(\frac{a^2 H}{\sqrt{(1-q^2)(1-q^2H^2)}}\right), \quad k = 0$$

and

(19b)

$$T = 4q\cdot\left(k\pi \pm \arccos\left(\frac{a^2 H}{\sqrt{(1-q^2)(1-q^2H^2)}}\right)\right), \; k = 1, 2, \ldots,$$

where $q = \cos(A)$, and $T = 4a$ is the period of impulsively excited oscillation.

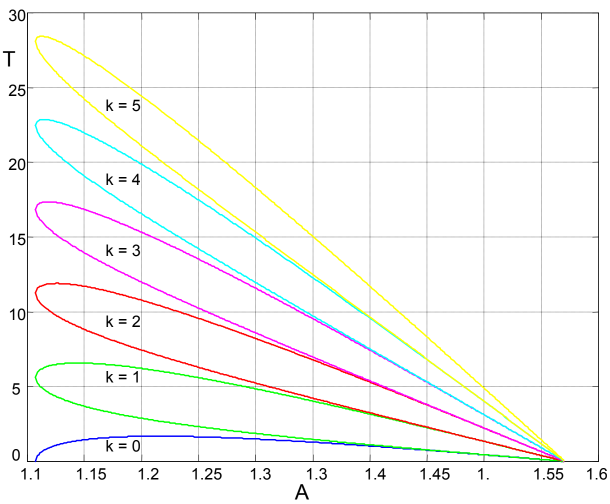


Fig.5. Branches of the period T vs. oscillation amplitudes for the pulse amplitude H = 2

The sequence of branches of solutions (19) at different numbers *k*, and the parameter *H* = 2 are presented in Figure 5. The diagram gives such combinations of the period and amplitude of the oscillation at which oscillator (15) can exhibit periodic oscillations with the period of external impulses, *T* = 4*a*. The upper and lower branches of each loop correspond to plus and minus signs in expression (19b), respectively. Solution (19a) which corresponds to the number *k* = 0 has the only upper branch.

It can be shown that solutions (19a) and (19b) exist in the interval

$$(20) \qquad \arccos\left(\frac{1}{\sqrt{1+H^2}}\right) = A_{\min} < A < A_{\max} = \frac{\pi}{2}$$

For the selected magnitude of *H*, the minimal amplitude is found to be *A* =1.1071 [rad], which corresponds to the left edges of the amplitude-period loops in Figure 5. As a result, further slight increase of the amplitude is accompanied by bifurcation of the solutions as shown in Figures 6 and 7. The diagram includes only first three couples of new solutions (*k* =1, 2, 3) from the infinite set of solutions.

As follows from these diagrams, the influence of excitation pulses on the temporal shapes is decreasing as the amplitude grows. This is the result of dominating the restoring variable over the external pulses. When the amplitude becomes close to its maximum A_{min} = π/2, the oscillator itself generates high-frequency impacts.

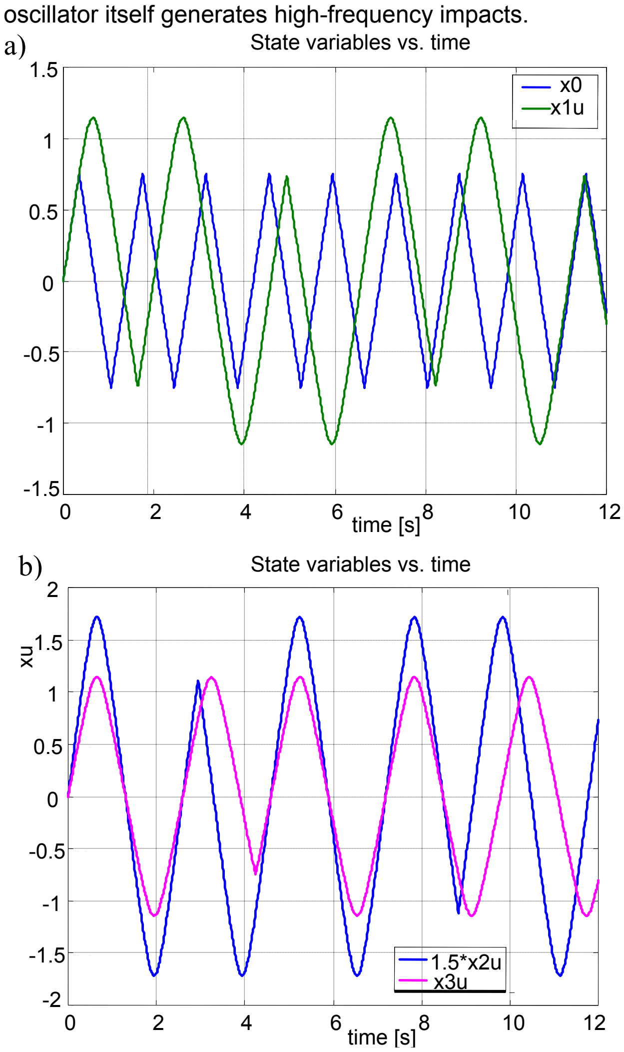


Fig.6. Periodic oscillations at A=1.145 for upper branches of closed loops of the corresponding periods shown in Figure 4: a) *k*=0; 1, b) *k*=2; 3 with x_{2u} rescaled by 1.5

It is worth mentioning that a slight increase of the amplitude is accompanied by bifurcation of oscillations. From the presented diagrams it follows that the temporal shapes of oscillations are decreasing as the amplitude of external pulses grows and this influence is strong.

The fact of exact solvability of the studied oscillators attracts much attention possibly due to the specific form of the oscillator characteristics with uncertain physical interpretations.

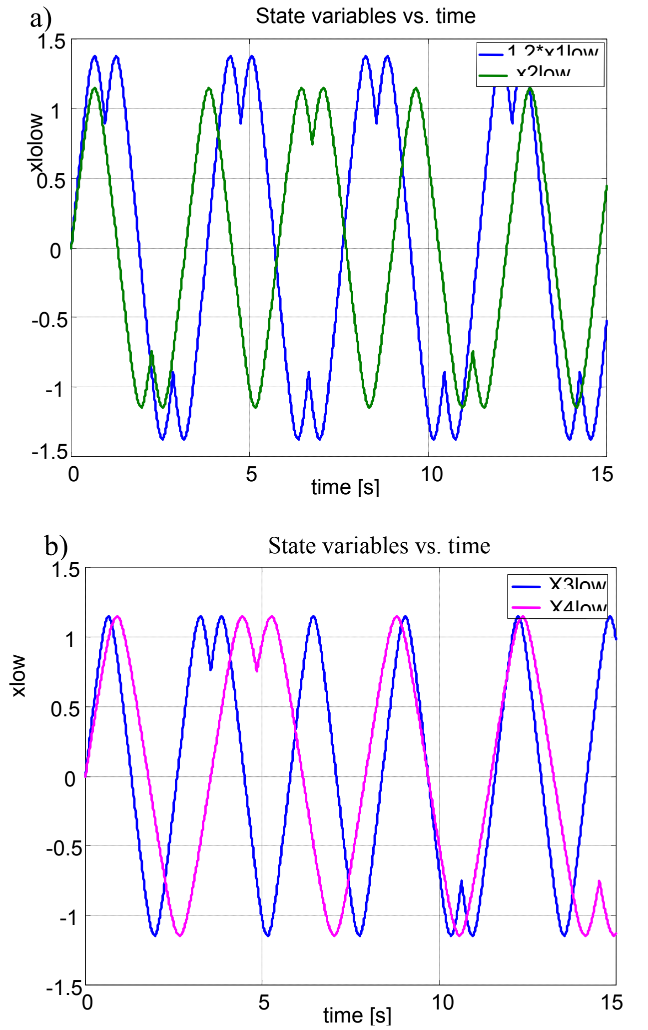


Fig. 7. Periodic oscillations at A=1.145 for lower branches of closed loops of the corresponding periods shown in Figure 4: a) k=1; 2, with x_{2low} rescaled by 1.2 b) k=3; 4

Chaos caused by surge arresters in the overvoltage protection

The overvoltages in power supply systems are caused mainly by lightning and switching operations. They are inevitable in practice [12]. To protect electrical equipment against overvoltages, surge limiters are installed possibly in close proximity to these devices [14]. Currently, surge protection is implemented almost exclusively by no spark surge arresters formed by the zinc oxide varistors. They are made of ceramic exhibiting nonlinear voltage-current characteristic, which can be determined on the basis of their dimensions.

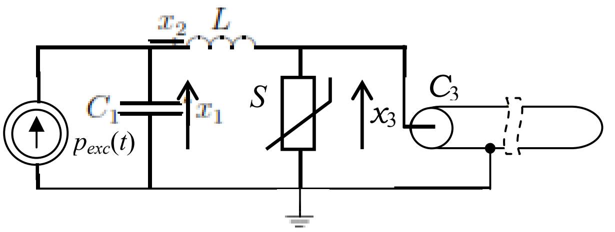


Fig.8. Overvoltage protection of power cable by surge arrester S

In the sequel we take into account a model of the surge arrester in the form which does not need of knowing the arresters' physical characteristics, but there is only a need for the knowledge of the electrical data, given by the manufacturer.

The voltage U given by the surge protective device, while diverting the surge current to the ground, must not exceed the voltage withstand value of the equipment connected downstream. In this paper, the adopted surge arrester model is described by a third degree power polynomial

$$i(t) = av(t) + bv^3(t) = f(v(t)) \quad (21)$$

whose coefficients are fitted to match the nonlinear characteristics of the given device. The voltage across and current through the surge arrester are denoted by $v(t)$ and $i(t)$, respectively. The transient state of the system (Fig.8) is described by the simultaneous differential equations

$$\begin{aligned} \dot{x}_1(t) &= p_{exc}(t) - x_2(t), \\ \dot{x}_2(t) &= x_1(t) - x_3(t), \\ \dot{x}_3(t) &= x_2 - f(x_3(t)). \end{aligned} \quad (22)$$

where $x_3(t) = v(t)$ denotes the voltage at the surge arrester and the dimensionless state variables are taken into considerations. The shapes of repeated excitation waveforms are shown in Figures 9a), b) for rectangular and triangular waveforms of excitations, respectively. Results of computer simulations are presented in Figures 10 and 11 for respective excitation waveforms.

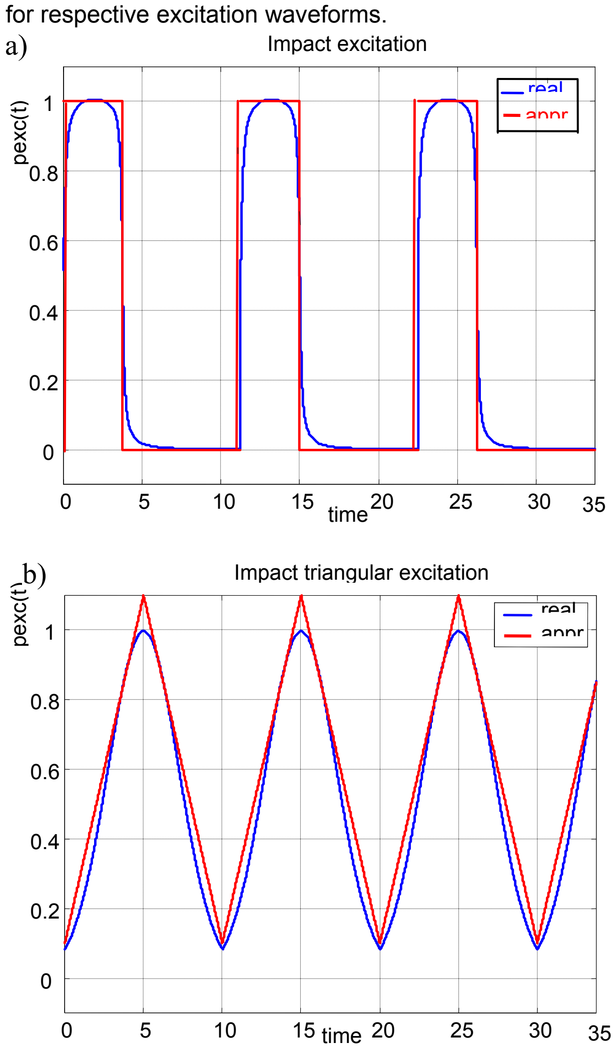


Fig.9. Two impact excitations: a) rectangular, b) triangular

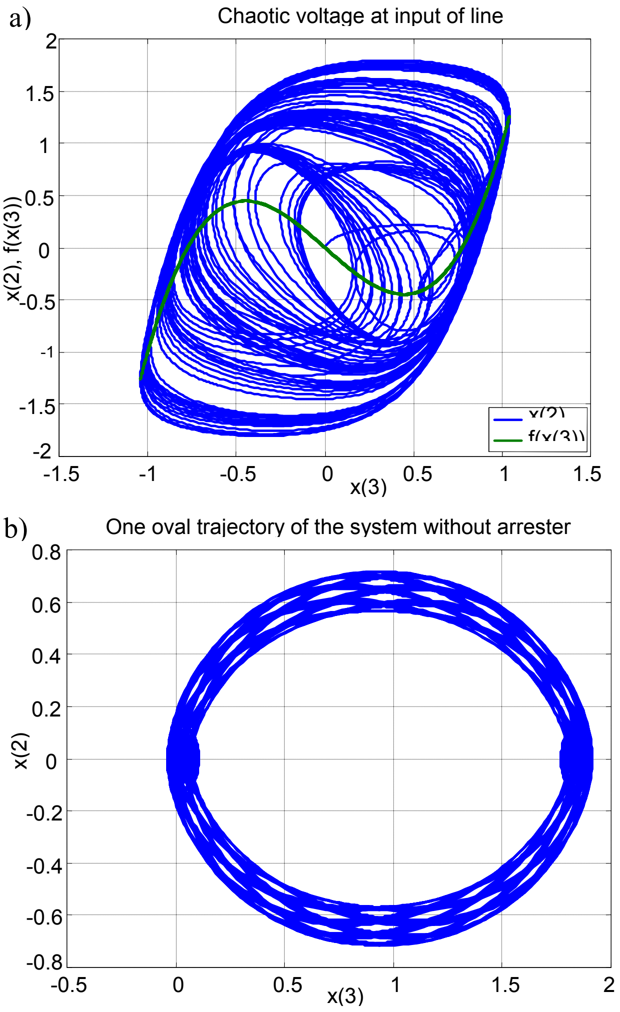


Fig.10. Phase trajectory of the system: a) with surge arrester *S*, b) without surge arrester

Looking at the above results in more details we can find that the application of the surge arrester deceases importantly the level of the voltage at the input of the protected power cable. On the other hand the current in the overhead line increases approximately in the same rate. It is also worth noticing that impact excitations in the form of triangular and rectangular waveforms significantly deform the system outputs what is evident from the shapes of corresponding phase trajectories. It is easily seen in particular (Fig.11b) in the case of the excitation current exhibiting the rectangular waveform.

The system with a surge arrester shows a greater tendency for chaos when the excitation takes the triangular waveform in comparing with the incident rectangular current waveform (Figs.10a and 11a). On the other hand particular incident current waves operating in the same structure but without surge arresters force more complex phase portrait in the case of rectangular waveform than the triangular one (Figs.10b and 11b). Thus, connection of nonlinear surge arrester at the input of the power cable importantly improves its protection against overvoltages. The investigation shows that when using a surge arrester the cable phase to ground overvoltage is limited to the protection level of the surge arrester. It has to be noted that the applied model is appropriate for studies where the incident current impulses do not reach very high values, such as surges produced by a reignition in the circuit breaker [16, 21]. Then, there are generally several feedback arcs (subsequent arcs) which may occur one after the other.

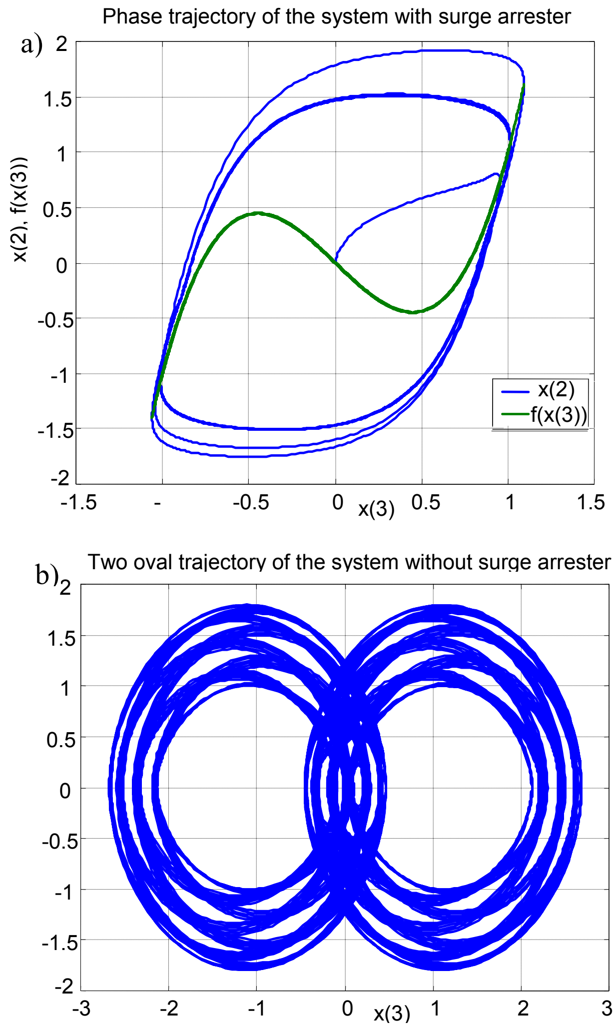


Fig.11. The phase trajectory of the system: a) with surge arrester, b) without surge arrester

Conclusions

In this paper, a version of nonsmooth substitution, specifically the nonsmooth time variable transformation is applied to analyze waveforms of currents and voltages in nonlinear nonsmooth dynamical systems. The basic rules for algebraic and differential manipulations are presented to apply the non-smooth argument substitutions in differential equation on the entire time interval. The two main features of the presented approach are to generate a particular algebraic structure and switching the initial value formulation to a boundary value problem. Because of the idempotent basis the boundary value problems with real and hyperbolic parts of a solution still remain coupled. The nonsmooth time transformation shows an explicit link between the underlying dynamics and hyperbolic algebra, analogously to the link between the harmonic approach and complex analysis. Notice that the transformation itself implies no constraints on dynamical systems and is applicable to both smooth and non-smooth systems [18, 27].

Basic rules for algebraic and differential calculations are described. In particular, in the second section we have showed how to implement nonsmooth argument substitutions in the differential equations. These impose two principal features on the dynamical systems by generating specific algebraic structures and switching formulations to boundary-value problems. Moreover, nonlinear coordinate transformations can simplify significantly the nonlinear nonsmooth dynamic problems. This means that the studies of dynamical systems with discontinuities can be simplified by means of appropriate non-smooth transformations of variables. However it has to be emphasized that the transformation itself is a preliminary stage of analysis finalized by specific boundary value problems on standard intervals. To solve boundary value problems appropriate methods must be applied in accord to the related physical content.

Main features induced by nonsmooth temporal substitutions can be briefly presented as follows:

• Applying non-smooth temporal variables, in particular triangular sine wave transforms manipulations of the system coordinates into the algebra of hyperbolic numbers;

• Fulfilling appropriate conditions keeps the result of differentiation or integration of the coordinates within the same algebra and therefore eases the corresponding manipulations with the dynamic system variables;

• In order to describe amplitude and/or frequency modulated processes the explicit time argument can be used together with the nonsmooth one.

The effectiveness of the presented method has been illustrated by several examples of analysis of basic nonlinear nonsmooth circuits constituting starting points for more complex systems.

Note that the illustrating models are strongly nonlinear, nevertheless the triangular wave temporal argument adequately captures specifics of the impulsive excitation and, as a result, provides asymptotic solutions in closed forms. The case of rectangular cosine waveform of incident currents has been also considered and illustrated.

Voltage surge protectors are essential elements in the protection of modern installations, but choosing the right one and ensuring compliance with their installation rules are essential conditions for their effectiveness. They should be tailored to the likely surge waveforms expected at the point of installation.

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***Author**: prof. dr hab. inż. Zdzisław Trzaska, Wyższa Szkoła Ekologii i Zarządzania, , ul. Olszewska 12, 00-792 Warszawa, E-mail: zdzislaw.trzaska@netlandia.pl*