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Mathematical modelling of induction motor with a saturated magnetic circuit during changes in moment of inertia

Abstract. *In the paper the equations of a circuit mathematical model of an induction motor are presented. The variable magnetization inductance and variable moment of inertia of rotating elements connected to the rotor are included in the model in order to improve its adequacy. A dynamical torque, being the component of an equation of rotor motion, is determined. The rightness of the trend, consisting in taking into account variable parameters of mathematical models of electrical machines, in order to increase the accuracy of the achieved results, is also shown.*

Streszczenie. *W pracy przedstawiono równania obwodowego modelu matematycznego silnika indukcyjnego z uwzględnieniem zmiennej indukcyjności magnesowania oraz zmiennego momentu bezwładności elementów wirujących dołączonych do wirnika w celu uściślenia ww. modelu. Określono moment dynamiczny występujący w równaniu ruchu wirnika oraz wykazano słuszność kierunku, polegającego na uzmiennianiu parametrów modeli matematycznych maszyn elektrycznych w celu zwiększenia dokładności otrzymywanych wyników. (**Modelowanie matematyczne silnika indukcyjnego z nasyconym obwodem magnetycznym przy zmianach momentu bezwładności**).*

Keywords: mathematical modelling, induction motor, magnetic circuit, moment of inertia.
Słowa kluczowe: modelowanie matematyczne, silnik indukcyjny, obwód magnetyczny, moment bezwładności.

Introduction

An analysis of static and dynamic phenomena during an operation of an electrical motor is necessary at a stage of designing efforts as well as during usage of drive systems. Mathematical models of the tested motors should be developed in order to carry out this analysis [1]. Depending on requirements, the circuit models, field models or field-circuit models are applied for mathematical description of motors and expanded electromechanical systems, as well. The circuit models (models based on lumped parameters) are the basis for mathematical description of drive systems, including automated drive systems with converter-fed motors, used in numerous technological processes. The circuit models of drive systems, that contain motor, converter and control system, are commonly used despite the intensive development of the field models of electrical machines and methods of their simulation [2].

In the case of circuit modelling, the problem is to solve two fundamental tasks [1]: (1) formulating a system of differential equations, (2) determining values of coefficients (parameters) of the abovementioned equations. A well carried out estimation of model parameters of the chosen structure is the key to its diagnosis and simulation [3]. If the model and estimation methods were properly selected for the given structure, then this solution may be copied, providing satisfactory results of simulation and diagnosis of the considered structure [3].

Induction motors are used in majority drive systems. The equations based on space vectors are commonly used in order to describe mathematically induction motors. The system of these equations is the vector mathematical model of induction motor. The abovementioned model is obtained as a result of application of space vector definition to equations of balanced voltages across stator and rotor circuits of three-phase induction motor [2]. As a consequence of this transformation, the voltage equations are obtained, that contain vector variables, represented on complex plane. The vector variables may also be represented in Cartesian coordinates if the vector equations transformed to matrix equations containing components of the vector variables.

The type and performance of windings and parameters of magnetic circuit determine the electromechanical properties of induction motors. Magnetic circuits of electrical machines are made of ferromagnetic materials, that have nonlinear properties, including saturation phenomenon and hysteresis phenomenon. Magnetization curves i.e. dependencies between magnetic induction and magnetic field intensity $B(H)$ should be known in order to conclude whether the given material has linear or nonlinear properties. The core assembly and stator windings determine magnetic flux, that determines starting torque and pull-out torque of induction motor.

Mathematical model of induction motor

Induction machine, consisting of stator winding and rotor winding, is the system of six circuits magnetically coupled: three immovable circuits of stator and three circuits of rotor (factual or equivalent one for squirrel-cage motor).

The following assumptions are usually adopted in order to simplify the mathematical model of induction machine [2,4]:

1. Phase windings of machine are balanced.
2. The influence of winding capacity are omitted.
3. Lumped windings of machine are taken into account.
4. The regularity of air-gap is assumed.
5. Harmonics of spatial distribution of magnetic field in air-gap are omitted.
6. The influence of anisotropy is omitted.
7. Hysteresis loss, eddy-current loss and anomalous loss are omitted.
8. Magnetic circuits are linear and the equivalent inductances are independent of currents.

Taking into account the subject matter of the paper, the assumption no. 8 shall not be respected in the further considerations.

The equations (1) and (2) of electromagnetic transient processes, occurring in magnetically coupled circuits of balanced two-pole induction machine, contain space vectors of fluxes, currents and voltages, and they are referenced to the rotating coordinate system d-q connected with the space vector of main flux $\underline{\psi}_m = \psi_m + \mathrm{j}0$ (flux in main magnetic circuit). These equations are given as follows:

$$\underline{u}_s = R_s \underline{i}_s + \frac{d\underline{\psi}_s}{dt} + \mathrm{j}\omega\underline{\psi}_s \tag{1}$$

$$\underline{u}_r = R_r \underline{i}_r + \frac{d\underline{\psi}_r}{dt} + \mathrm{j}(\omega - \omega_m)\underline{\psi}_r \tag{2}$$

where: $\underline{u}_s$, $\underline{u}_r$, $\underline{i}_s$, $\underline{i}_r$, $\underline{\psi}_s$, $\underline{\psi}_r$ are space vectors of voltages, currents and fluxes of stator and rotor, ω_m is angular velocity of rotor, j is imaginary unit. The quantity ω in equations (1) and (2) is determined as $\omega = \mathrm{d}\gamma/\mathrm{dt}$, where ω, γ are angular velocity and angle of rotation of main flux space vector $\underline{\psi}_m$. In the case of squirrel-cage induction motor the following relationship should be taken into account: $\underline{u}_r = 0 + j0$. The dependencies between space vectors of both, fluxes and currents, should be included in order to solve the equations (1) and (2). They may be expressed as follows:

(3) $$\underline{\psi}_s = L_{\sigma s}\underline{i}_s + \underline{\psi}_m \ , \qquad \underline{\psi}_r = L_{\sigma r}\underline{i}_r + \underline{\psi}_m$$

where: $L_{\sigma s}$, $L_{\sigma r}$ are leakage inductances of stator winding and rotor winding. The equation of current balance, according to adopted assumptions, is given below:

(4) $$\underline{i}_s + \underline{i}_r = \underline{i}_m$$

where: $\underline{i}_m$ is space vector of magnetizing current, whereas the dependency between flux and magnetizing current:

(5) $$\underline{\psi}_m = L_m \underline{i}_m$$

where: L_m is magnetization inductance of main magnetic circuit. In the case of linear magnetic circuits, the quantity L_m is constant parameter. For nonlinear magnetic circuits, L_m may be expressed as a function of magnetizing current or flux. For purposes of mathematical modelling of induction motors the dependency between L_m and main flux: $L_m(\psi_m)$ seems to be more useful.

The equation of torque balance should be added in order to solve the system of equations (1) – (5):

(6) $$m_e - m_o = m_d$$

where: m_e, m_o, m_d are electromagnetic torque of a motor, load torque applied to motor shaft and dynamical torque, respectively, while losses (frictions, clearances), that occur during transmission of power between motor and mechanism, are usually included in values of m_e or m_o. Electromagnetic torque is the result of magnetic fields interaction in electrical machine:

(7) $$m_e = \mathrm{Im}\left(\underline{\psi}_s^* \underline{i}_s\right)$$

Approximation of magnetization curve of induction motor core

A nonlinear curve $I_0 = f(U)$ of non-loaded induction motor results from nonlinear magnetization curve of main magnetic circuit, where U, I_0 are feeding voltage and current of non-loaded motor. In author's publications, i.a. [4], the approximation of nonlinear magnetization inductance L_m of induction motor, using dependency describing the attenuation diagram of the Butterworth's low-pass filter, was proposed. In this dependency the relative voltage replaced frequency as argument. The no-load curve of motor was obtained as a result of transformation. Introducing some modifications into Butterworth's polynomial the expected gradient of the linear segment of the motor no-load curve can be achieved. The abovementioned segment corresponds to unsaturated magnetic circuit. The proposed dependency is given below:

(8) $$I_0 = I_n \frac{U}{U_n}\frac{1}{a}\sqrt{b\left(\frac{U}{U_n}\right)^{2q} + 1}$$

where: U_n, I_n are rated voltage and rated current of motor, a, b, q are parameters determined experimentally or on the basis of the characteristic of motor prototype. The exemplary no-load curve of induction motor, drawn on the basis of the dependency (8), and points being the result of measurements [4] are shown in Fig. 1. According to Fig. 1, the dependency (8) allows reproducing precisely the no-load curve of induction motor, even if the magnetizing current exceeds the rated magnetizing current ($I_{mn} \approx I_{0n} < I_n$) several times.

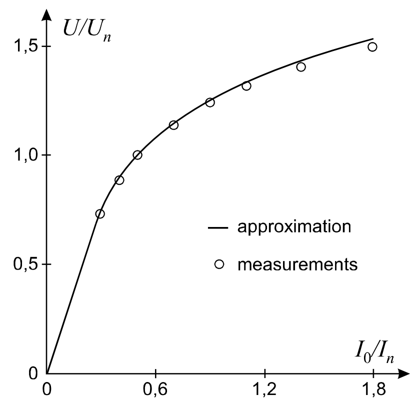

Fig. 1. No-load curve of induction motor [4]

Adopting the following assumptions: $U/U_n \approx \Psi_m/\Psi_{mn}$ and $I_0 \approx I_m$, the dependency (8) may be replaced by the dependency (9), that allows for approximation of magnetizing current as the function of main flux Ψ_m:

(9) $$I_m \approx I_n \frac{\Psi_m}{\Psi_{mn}}\frac{1}{a}\sqrt{b\left(\frac{\Psi_m}{\Psi_{mn}}\right)^{2q} + 1}$$

or for instantaneous values:

(10) $$i_m \approx I_n \frac{\psi_m}{\Psi_{mn}}\frac{1}{a}\sqrt{b\left(\frac{\psi_m}{\Psi_{mn}}\right)^{2q} + 1}$$

where: Ψ_{mn} is rated flux in main magnetic circuit.

Variable parameters of circuit mathematical models of electrical motors are controversial for some scientists due to the fact that the variable parameters are inconsistent with the assumptions concerning the method of formulating mathematical model based on Lagrange's formalism. According to this formalism the parameters of the modelled system should be constant. This controversy was mentioned in the former paper of the author [5].

Determination of dynamical torque. Electromagnetic and mechanical analogies

In teaching studies the equation of motion together with formula for calculation of dynamical torque are derived on the basis of energy conservation low:

(11) $$w_e = w_u + w_k$$

where: w_e is motor energy transmitted to mechanical system, w_u is energy output, w_k is kinetic energy stored in rotating masses. The kinetic energy is expressed as:

(12) $$w_k = \tfrac{1}{2} J\omega^2$$

where moment of inertia J generally is a function of angle of rotation γ or angular velocity ω. The equations of power balance is obtained as a result of differentiation of equation (11) taking into account (12):

(13) $$p_e = p_u + J\omega\frac{\mathrm{d}\omega}{\mathrm{d}t} + \frac{\omega^2}{2}\frac{\partial J}{\partial \gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t}$$

hence, dividing (13) by ω and taking into account $\omega = \mathrm{d}\gamma/\mathrm{d}t$, the equation of rotor motion is derived:

(14) $$m_e - m_o = J\frac{\mathrm{d}\omega}{\mathrm{d}t} + \frac{\omega^2}{2}\frac{\partial J}{\partial \gamma} = m_d$$

The formula for calculation of dynamical torque, resulting from the above considerations, may be given as follows:

(15) $$m_d = \frac{1}{\omega}\frac{\mathrm{d}w_k}{\mathrm{d}t}$$

In classical mechanics, a force of inertia is defined as time derivative of momentum, according to the following formula (ignoring vector properties of force and velocity):

(16) $$F = \tfrac{\mathrm{d}}{\mathrm{d}t}(mv)$$

The dynamical torque in rotary motion may be defined in a similar way:

(17) $$m_d = \frac{\mathrm{d}}{\mathrm{d}t}(J\omega) = J\frac{\mathrm{d}\omega}{\mathrm{d}t} + \omega^2\frac{\partial J}{\partial \gamma}$$

The first components on the right sides of the equations (14) and (17) are the same, whereas the second components in both equations differ in factor ½. Thus, the following question may be asked: which equation – (14) or (17) – determines correctly the dynamical torque for a mechanical system containing rotating elements with variable $J = f_1(\gamma)$?

In the well-known publication of White and Woodson “Electromechanical energy conversion” flux linkages are classified as generalized momentums, whereas voltages – as generalized forces. The following dependencies may be defined by analogy to (16) for flux linkages of self-induction or mutual induction, respectively:

(18) $$e_{jj} = \tfrac{\mathrm{d}}{\mathrm{d}t}(L_j i_j), \qquad e_{jk} = \tfrac{\mathrm{d}}{\mathrm{d}t}(M_{jk} i_k)$$

Self-inductances L and mutual inductances M may be constant or dependent on magnetizing current (or flux) – in the case of saturated magnetic circuits. They may also be functions of angle of rotation γ or displacement x – in the case of electromechanical transducers.

Limiting the considerations to self-inductances, the following equation may be written for saturated magnetic circuit (e.g. magnetic circuit of choke coil):

(19) $$e = \frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L(i)i\right] = L(i)\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{\partial L(i)}{\partial i}\frac{\mathrm{d}i}{\mathrm{d}t}i = \left[L(i) + \frac{\partial L(i)}{\partial i}i\right]\frac{\mathrm{d}i}{\mathrm{d}t}$$

On the other hand, for the case $L = f_2(x)$, the following equation may be written:

(20) $$e = \frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\left[L(x)i\right] = L(x)\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{\partial L(x)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t}i = L(x)\frac{\mathrm{d}i}{\mathrm{d}t} + vi\frac{\partial L(x)}{\partial x}$$

The term within square brackets on the right side of the equation (19) is a dynamical inductance of winding and derivation of its is known from literature sources.

Adopting analogical to (15) formula for calculation of inductive voltage (e.g. in winding of electromagnet) and including analogical to (12) formula for calculation of energy stored in magnetic field, it may be written for $L = f_2(x)$:

(21) $$e = \frac{1}{i}\frac{\mathrm{d}}{\mathrm{d}t}\left[L(x)\frac{i^2}{2}\right] = L(x)\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{i}{2}\frac{\partial L(x)}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} = L(x)\frac{\mathrm{d}i}{\mathrm{d}t} + \frac{vi}{2}\frac{\partial L(x)}{\partial x}$$

Using the variational calculus method, the inductive voltage of a coil may be determined on the basis of Lagrange’s function of the first kind and Euler-Lagrange’s condition in the following way:

(22) $$e = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}}\left[L(x)\frac{\dot{q}^2}{2}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[L(x)\dot{q}\right] = L(x)\ddot{q} + \frac{\partial L(x)}{\partial x}\dot{x}\dot{q} = L(x)\frac{\mathrm{d}i}{\mathrm{d}t} + vi\frac{\partial L(x)}{\partial x}$$

whereas for the dynamical torque:

(23) $$m_d = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{\gamma}}\left[J(\gamma)\frac{\dot{\gamma}^2}{2}\right] = \frac{\mathrm{d}}{\mathrm{d}t}\left[J(\gamma)\dot{\gamma}\right] = J(\gamma)\ddot{\gamma} + \frac{\partial J(\gamma)}{\partial \gamma}\dot{\gamma}^2 = J(\gamma)\frac{\mathrm{d}\omega}{\mathrm{d}t} + \omega^2\frac{\partial J(\gamma)}{\partial \gamma}$$

The abovegiven results are in accordance with (20) and (17), respectively, i.e. they are contrary to (21) and (14). In connection with the above, it may be concluded that the formula (17) allows determining properly the dynamical torque in a mechanical system containing rotating elements with variable J being a function of angle of rotation γ.

In the previously mentioned publication of White and Woodson the term of magnetic co-energy was defined as:

(24) $$w'_m = \int_{0,\dots,0}^{i_1,\dots,i_n} \sum_{k=1}^{n} \psi'_k \mathrm{d}i'_k$$

and for the single winding:

(25) $$w'_m = \int_0^i \psi' \mathrm{d}i'$$

The kinetic co-energy for a mechanical system may be defined in a similar way:

(26) $$w'_k = \int_0^v p'\mathrm{d}v' = \int_0^v m'v'\mathrm{d}v' \qquad \text{or} \qquad w'_k = \int_0^\omega J'\omega'\mathrm{d}\omega'$$

Taking into account the second dependency of (26), the moment of inertia can be determined correctly by using the formula (15):

(27) $$m_d = \frac{1}{\omega}\frac{\mathrm{d}w_k}{\mathrm{d}t} = \frac{1}{\omega}\frac{\mathrm{d}}{\mathrm{d}t}\left(J\omega^2 - w'_k\right) = \frac{1}{\omega}\frac{\partial J}{\partial \gamma}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\omega^2 + \frac{1}{\omega}\left(J\frac{\partial \omega^2}{\partial \omega} - \frac{\partial w'_k}{\partial \omega}\right)\frac{\mathrm{d}\omega}{\mathrm{d}t} = J\frac{\mathrm{d}\omega}{\mathrm{d}t} + \omega^2\frac{\partial J}{\partial \gamma}$$

In the cases, where mass or moment of inertia are constant, the known formulas are in common use:

(28) $$w_k = w'_k = \tfrac{1}{2}mv^2 \quad \Leftrightarrow \quad m = \mathrm{const}$$

(29) $$w_k = w'_k = \tfrac{1}{2}J\omega^2 \quad \Leftrightarrow \quad J = \mathrm{const}$$

It should be noted, that the relation $m = f_3(v)$ is useful only in theory of relativity and does not have application in analysis of electrical machines. On the other hand, the analogous relation for rotary motion $J = g_3(\omega)$ is useful not only in relativistic aspect. A classical example, however one of many, is Watt's centrifugal governor. A mechanism used in clocks in order to stabilize rotational speed may also be mentioned among examples (Fig. 2).

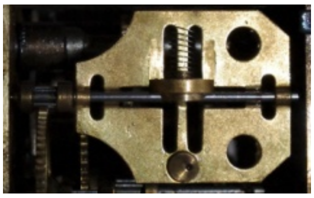

Fig. 2. Mechanism used in clocks in order to stabilize rotational speed – the example of variable moment of inertia J, dependent on angular velocity ω

The examples of $J = f_1(\gamma)$ and $m = g_1(x)$ are also known. They are connected e.g. with increasing a rotating mass of reel or mixer in chemical reactor, caused by reeling the line or sticking a mixed substance to the mixer [4], respectively.

The considered formulas for calculation of the dynamical torque can be negligible if the system of equations (1) – (5) will be expanded upon the following equation of motion of rotor and rotating elements with variable J connected to rotor:

(30) $$\frac{\mathrm{d}}{\mathrm{d}t}\left(J\omega\right) = m_e - m_o$$

In order to calculate the rotor angular velocity, the difference in torques on the right side of the equation (30) should be integrated and then the integration result should be divided by J, i.e.:

(31) $$\omega = \frac{1}{J}\left[\int_0^t (m_e' - m_o')\mathrm{d}t' + b(0)\right]$$

where: $b(0)$ is initial angular momentum determined as product of initial values of moment of inertia $J(0)$ and angular velocity $\omega(0)$. If a value of J is constant, i.e. independent of angular velocity and rotor position, the angular velocity may be determined on the basis of the equation (32):

(32) $$\omega = \int_0^t \left(\frac{m_e' - m_o'}{J}\right)\mathrm{d}t' + \omega(0) \quad \Leftrightarrow \quad J = \mathrm{const}$$

On the basis of the above considerations it may be concluded, that questioning the variability of magnetization inductance in the dependencies such as $i = L^{-1}(\psi)\psi$ or $\psi = L(i)i$ is equivalent to questioning the variability of mass or moment of inertia in the dependencies $p = m(v)v$ or $b = J(\omega)\omega$, e.g. $p = mv[1 - (v/c)^2]^{-1/2}$. The problem with variable parameters concerns the Lagrange's formalism based on dependencies such as (33), that range of application is limited to linear systems (coefficients A, B should be independent of generalized coordinates x and derivatives of theirs).

(33) $$T = \tfrac{1}{2}A\dot{x}^2, \qquad U = \tfrac{1}{2}Bx^2$$

Conclusions

In the paper the equations of circuit mathematical model of induction motor are presented. The variable magnetization inductance and variable moment of inertia of rotating elements connected to rotor is taken into account in order to improve the adequacy of the model. The dependency, that allows approximating precisely a nonlinear magnetization curve of ferromagnetic-based cores of electrical machines, is mentioned. On the basis of the selected electromagnetic and mechanical analogies, the rightness of the trend, consisting in taking into account the variable parameters of mathematical models of electrical machines in order to increase the accuracy of the achieved results, is shown.

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