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Analysis of numerical efficiency of TD-FEM computation with BDF

Abstract. *A study is presented on the topic of numerical efficiency of TD-FEM with the use of BDF. The BDF coefficients are calculated with the use of Lagrange polynomials. The paper also presents two strategies of time-step adaptation. The obtained numerical solutions have been compared with analytical solutions of the selected problems in order to check their accuracy.*

Streszczenie. *Przedstawiono badania na temat efektywności numerycznej metody TD-FEM przy wykorzystaniu metody różnic wstecznych. Współczynniki metody różnic wstecznych otrzymano za pomocą wielomianów Lagrange'a. W artykule przedstawiono również dwie strategie automatycznego doboru kroku czasowego. W celu sprawdzenia dokładności rozwiązań numerycznych porównano je z rozwiązaniami analitycznymi rozpatrzonych zagadnień. (**Analiza efektywności numerycznej metody TD-FEM z wykorzystaniem metody różnic wstecznych**).*

Keywords: backward differentiation formula, time domain finite element method, time step adaptation
Słowa kluczowe: metoda różnic wstecznych, metoda elementów skończonych w dziedzinie czasu, automatyczne dostosowanie kroku całkowania

doi:10.12915/pe.2014.03.22

Introduction

The paper presents studies that are a part of a larger project concerning electromagnetic field computation and the analysis of efficiency of the finite element method. Many complicated problems require analyses that are computationally expensive (e.g. coupled field, nonlinear problems [1] or ones complicated geometry). This is why many studies are performed to find possible ways of reducing computation time of the finite element method (e.g. by parallel computing [2] or by choice of optimal methods [3]).

Whether the problem concerns a simplified or complicated model – the computation of the electromagnetic field distribution involves the solution of a system of linear equations:

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F}, \tag{1}$$

or nonlinear equations if the environment parameters are dependent on the electromagnetic field. In equation (1) $\boldsymbol{K}$ is the stiffness matrix, while $\boldsymbol{F}$ is the load vector and $\boldsymbol{u}$ is the vector of the computed values (degrees of freedom).

There exist many problems e.g. like in [1, 4] that require a transient analysis (even in time-periodic problems when nonlinearity is involved). This requires a more advanced variant of the finite element method to be used. For some problems it may be computationally efficient to apply special methods like HBFEM [5] (for time periodic problems) but a more general way is the application of TD-FEM methods [6, 7] (**T**ime-**D**omain **F**inite **E**lement **M**ethod). This approach is used in most well-known commercial software created for the purposes of solving electromagnetic field problems with the finite element method.

The application of the TD-FEM requires an additional discretization in time, where the previously mentioned system of equations is solved for consecutive time-steps. For a large number of degrees of freedom the computation of the solution often requires a considerable amount of time, hence in the TD-FEM method, among many other factors generally influencing the accuracy of the finite element method, it is important how the approximation of the time derivative is performed.

This paper discusses an application of the BDF (**B**ackward **D**ifferentiation **F**ormulae) [8] for the time derivative approximation.

The issue of time-step adaptation has also been raised as it determines not only the amount of steps taken but also has an influence on the solution accuracy.

Backward Differentiation Formulae

The time derivative of a component u (e.g. a component of the magnetic vector potential A, which is used later on) by means of a BDF (with various time-steps between values) is given by:

$$\left.\frac{\partial u}{\partial t}\right|_{t=t_m} \approx \sum_{i=0}^{n} p_i\, u_{m-i}, \tag{2}$$

where:

$$p_i = \left.\frac{\partial \ell_{m-i}}{\partial t}\right|_{t=t_m}, \tag{3}$$

n denotes the order of the BDF ($n = 1$ means the backward Euler scheme), t_m is the time value for the current step, u_m is the component value for the current time-step. ℓ_j is the Lagrange polynomial which is 1 at t_j and 0 at every other included time-step.

The choice of BDF for the time derivative approximation is motivated by the fact that in this case:

- the number of unknowns does not change with BDF order in the solved system of equations and it is the same as when solving a quasi-stationary problem (however in the case of TD-FEM the system is solved for each time-step),
- the multipliers of consecutive values of u in time are common for all degrees of freedom.

Finite element formulation

The diffusion equation for the electromagnetic field, for a single component of the magnetic vector potential, is given by:

$$\nabla^2 A = -\mu J_{\text{ext}} + \mu\gamma\frac{\partial A}{\partial t}, \tag{4}$$

where J_{ext} is current density component produced by external voltage.

By applying the Galerkin formulation, (4) yields a system of equations, which for linear environment parameters can preliminarily be expressed by:

$$\boldsymbol{h}\boldsymbol{A} = \boldsymbol{f} + \boldsymbol{g}\frac{\partial \boldsymbol{A}}{\partial t}, \tag{5}$$

where e.g. for a 2D problem the values of the matrix $\boldsymbol{h}$ are given by [3, 9]:

(6) $$\boldsymbol{h}_{ij} = -\iint_{\Omega}\left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}\right)\mathrm{d}x\mathrm{d}y,$$

while for the vector $\boldsymbol{f}$:

(7) $$\boldsymbol{f}_i = -\iint_{\Omega}\mu J_{\mathrm{ext}\,i}N_i\mathrm{d}x\mathrm{d}y - \oint_{\Gamma}N_i\frac{\partial A}{\partial n}\mathrm{d}\Gamma,$$

and the matrix $\boldsymbol{g}$:

(8) $$\boldsymbol{g}_{ij} = \iint_{\Omega}\mu\gamma N_i N_j\mathrm{d}x\mathrm{d}y.$$

i and j are node numbers and N with an appropriate index denotes the respective finite element shape function. Γ is the contour describing the boundary of the region Ω, while n is the direction normal to the boundary at a given point. Taking into account the approximation (2) one can write the rightmost expression of (5) as:

(9) $$\boldsymbol{g}\left.\frac{\partial \boldsymbol{A}}{\partial t}\right|_{t=t_m} = \boldsymbol{g}\sum_{k=0}^{n}p_k\left(\left.\boldsymbol{A}\right|_{t=t_{m-k}}\right).$$

At $t = t_m$ the only unknown value for each degree of freedom is u_m. In consequence, for a system (1), the multiplier of u_m contributes to $\boldsymbol{K}$ while others are added to the load vector:

(10) $$\boldsymbol{K} = \boldsymbol{h} - p_0\boldsymbol{g},$$

(11) $$\boldsymbol{F} = \boldsymbol{f} + \boldsymbol{g}\sum_{k=1}^{n}p_k\left(\left.\boldsymbol{A}\right|_{t=t_{m-k}}\right).$$

Time-step adaptation

A time-step adaptation technique can significantly increase the accuracy of the solution and reduce computation time [3, 10] by obtaining the electromagnetic field distribution at time values distant by optimally chosen Δt. In the discussed study the choice of the time-step value Δt is made by means of a predictor-corrector scheme. Two different strategies are applied and compared:

A) a BDF of order n is chosen; for every time-step the predictor step uses the solution for a BDF of order $n-1$ while the corrector step then applies a solution with the time derivative approximated by the BDF of order n,
B) two solutions are compared for different BDF of order n [10].

Strategy B requires further clarification. For the predictor step instead of a time-derivative of $t = t_m$ a formula is applied for the value between the current and previous t. In this case the time-derivative approximation is:

(12) $$\left.\frac{\partial u}{\partial t}\right|_{t=\frac{(t_m+t_{m-1})}{2}} \approx \sum_{i=0}^{n}q_i\,u_{m-i},$$

q_i are coefficients computed from the same Lagrange polynomials as in (3) but at a different time t:

(13) $$q_i = \left.\frac{\partial \ell_{m-i}}{\partial t}\right|_{t=\frac{(t_m+t_{m-1})}{2}}.$$

The left-hand side of (12) is replaced by a linear interpolation between time derivatives at t_m and t_{m-1}. Equation (12) then takes the form:

(14) $$\left.\frac{\partial u}{\partial t}\right|_{t=t_m} = 2\sum_{i=0}^{n}q_i\,u_{m-i} - \left.\frac{\partial u}{\partial t}\right|_{t=t_{m-1}}.$$

The time derivative at $t = t_{m-1}$ is obtained from a previous time-step hence the only unknown on the right-hand side is again the value u_m. In this case the rightmost component of (5) yields:

(15) $$\boldsymbol{g}\left.\frac{\partial \boldsymbol{A}}{\partial t}\right|_{t=t_m} = 2\boldsymbol{g}\sum_{k=0}^{n}q_k\left(\left.\boldsymbol{A}\right|_{t=t_{m-k}}\right) - \boldsymbol{g}\left.\frac{\partial \boldsymbol{A}}{\partial t}\right|_{t=t_{m-1}}.$$

A simple error estimation is applied in this paper. The local error at node j is assumed as:

(16) $$\boldsymbol{e}_j = \frac{\boldsymbol{A}''_j - \boldsymbol{A}'_j}{\max(\boldsymbol{A}'', \boldsymbol{A}')} = \boldsymbol{c}_j\left(\Delta t\right)^{n+1},$$

where the $\boldsymbol{A}''$ vector denotes the values of the corrector solution and $\boldsymbol{A}''$ are the predictor solution values. The error value for a supposed optimal time-step is:

(17) $$\boldsymbol{e}_{j\,\mathrm{ctrl}} = \boldsymbol{c}_j\left(\Delta t'\right)^{n+1},$$

from the maximum difference between the predictor and corrector solutions is taken into account:

(18) $$\frac{\left(\Delta t\right)^{n+1}}{\left(\Delta t'\right)^{n+1}} = \frac{\max|\boldsymbol{e}|}{e_{\mathrm{ctrl}}} = \frac{\max|\boldsymbol{A}''-\boldsymbol{A}'|}{e_{\mathrm{ctrl}}\max(\boldsymbol{A}'', \boldsymbol{A}')},$$

and thus the time-step change is applied according to:

(19) $$\eta = \frac{\Delta t'}{\Delta t} = \sqrt[n+1]{\frac{e_{\mathrm{ctrl}}\max(\boldsymbol{A}'', \boldsymbol{A}')}{\max|\boldsymbol{A}''-\boldsymbol{A}'|}},$$

where $\Delta t'$ is the newly chosen time-step; e_{ctrl} is the acceptable error value. Furthermore, a value $e_{\max}$ is determined, such that if $\max|\boldsymbol{A}''-\boldsymbol{A}'| > e_{\max}$ then the time-step is changed according to (19) and computations are repeated for a new t.

Exemplary problems

In order to perform the analysis of numerical efficiency two simple problems are solved – one of linear symmetry, where the weak formulation is:

(20) $$\left.\frac{\partial A}{\partial x}\Phi\right|_{\partial\Omega} - \int_{\Omega}\frac{\partial A}{\partial x}\frac{\partial \Phi}{\partial x} - \int_{\Omega}\mu\gamma\frac{\partial A}{\partial t}\Phi = 0,$$

and the other of cylindrical symmetry, where for a cylindrical coordinate system one obtains:

(21) $$\left.\frac{\partial A}{\partial r}\Phi r\right|_{\partial\Omega} - \int_{\Omega}\frac{\partial A}{\partial r}\frac{\partial \Phi}{\partial r}r - \int_{\Omega}\mu\gamma\frac{\partial A}{\partial t}\Phi r = 0.$$

Φ is the test function, Ω is the considered region and $\partial\Omega$ denotes its boundary.

Neumann boundary conditions of the form:

(22) $$\left.\frac{\partial A}{\partial n}\right|_{\partial\Omega} = a\sin(\omega_1 t) + 0.1a\sin(9\omega_1 t),$$

are applied, where n is the direction normal to the boundary $\partial\Omega$ at a given point and $\omega_1 = 2\pi f_1$. The two different frequencies are imposed in order to add an additional difficulty to the time adaptive process – the task is to ascertain if the strategies described in the previous paragraph can capture the faster and less significant changes (because of the ten times smaller amplitude in the boundary condition).

The choice of such simple problems allows the numerical result to be compared with analytical solutions obtained from literature [11] (and applied separately for each imposed frequency) thus making the error analysis of the selected methods simple and reliable.

Analysis of numerical efficiency

The analysis is performed in terms of the computation time and the solution accuracy. The latter is ascertained in a comparison to the analytical solution. An average relative error e_r has been computed (denoted later by e_{r1} and e_{r9} for the respective frequencies f_1 and $9f_1$) by the formula:

$$e_r = \left(\left| \overline{\underline{\boldsymbol{A}}_{num} - \underline{\boldsymbol{A}}_{an}} \right| / \left| \overline{\underline{\boldsymbol{A}}_{an}} \right| \right) \cdot 100\%, \tag{23}$$

where the overlined values indicate averages for the degrees of freedom. $\underline{\boldsymbol{A}}_{num}$ are the complex values obtained by applying the Vaníček method [12] on the numerical solution. $\underline{\boldsymbol{A}}_{an}$ are the values of the analytical solution.

Numerous trials have been performed, where e_{ctrl} and e_{max} are set respectively to:

$$e_{ctrl} = 10^{-m-1}\%, \quad e_{max} = 10^{-m}\%, \tag{24}$$

where m is changed at every trial. This has been done for both strategies (denoted by A and B) for BDF orders n from 2 to 6 ($n > 6$ is not analyzed since it is known for its stability issues [13]).

Once the results have been gathered the total computation time values, for the two strategies and various BDF orders, have been compared for results with close values of e_{r1} and e_{r9}. Typical results from such comparisons are displayed in Figure 1 for the problem with linear symmetry and in Figure 2 for the cylindrically symmetric problem.

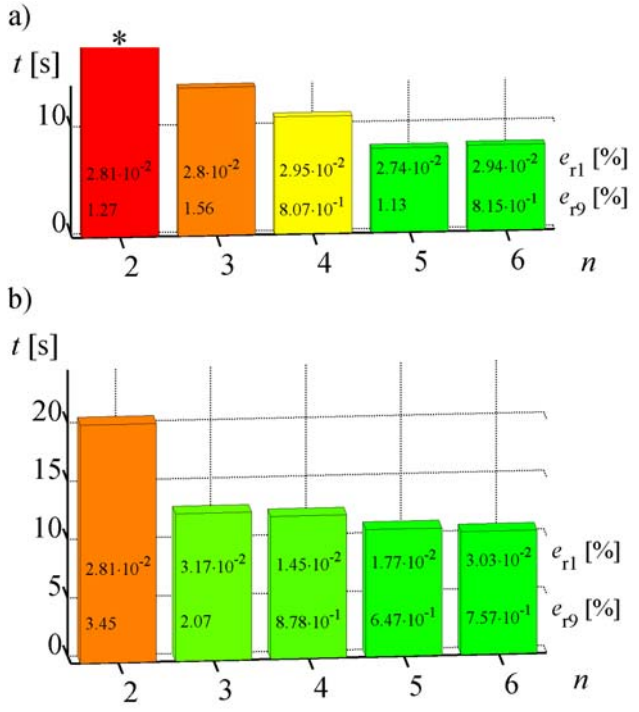

Fig.1. Comparison of selected trials of computations (performed with a) strategy A, b) strategy B) for a considered problem of linear symmetry. * – for strategy A and $n = 2$ the computation time was 94.0 seconds

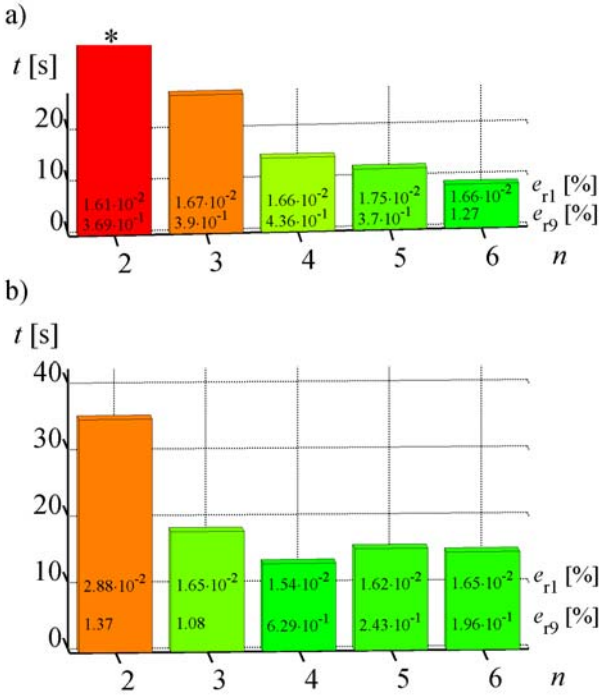

Fig.2. Comparison of selected trials of computations (performed with a) strategy A, b) strategy B) for a considered problem of cylindrical symmetry. * – for strategy A and $n = 2$ the computation time was 116.7 seconds

From the results presented in the figures and others that have been obtained it has been noticed that strategy B handles the problems much more efficiently in the case of $n = 2$ (about 4 to 5 times shorter computation time) and for $n = 3$. An apparent increase in efficiency is noticed for the increase of n up to 4. At this BDF order both strategies exhibit similar efficiencies. However, for strategy B, the efficiency doesn't change much for a greater n. In the case of strategy A the efficiency improvement is noticeable up to $n = 5$.

In the presented study the comparison of computation times is similar to comparing the numbers of performed iterations. This is so because the most time consuming process in each time-step was the solution of the system of equations built as a consequence of (20) and (21). The significance of the computation time of the system-solving process is even bigger e.g. for nonlinear problems with large numbers of degrees of freedom. A comparison of the numbers of iterations for selected trials (same ones as in Figures 1 and 2) is presented in Figure 3 (the results for $n = 2$ have been omitted since they are significantly worse than the rest).

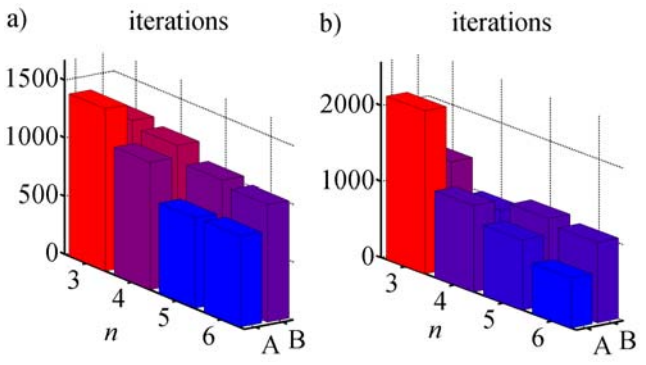

Fig.3. Comparison of numbers of performed iterations for various time adaptation strategies and BDF orders

Conclusions

The study has shown that satisfactory accuracies can be obtained when using a predictor-corrector scheme that applies the BDF. The accuracy is dependent on the chosen values e_{ctrl} and $e_{\max}$. In most cases an increase in the order of the BDF has allowed for the computations to be completed in a shorter period of time while keeping a close error value.

Two strategies of the time-step adaptation process have been compared (denoted respectively as strategy A – where BDF of order $n-1$ and n are used and strategy B – that applies two different backward differentiation formulae of order n). Several trials have been performed for both strategies and various BDF orders. For solutions of similar accuracies (expressed by obtained error values) the results have been compared in terms of computation time and the number of performed iterations. It has been noticed that for strategy A the efficiency has improved significantly up to $n = 5$ while only up to $n = 4$ when applying strategy B. The results are clearly better for strategy B in case of $n = 2$ and $n = 3$, while the smallest computation times in general have been achieved for $n = 5$ and $n = 6$ when applying strategy A.

One can of course argue about whether in all cases the rise of n will contribute to the decrease of computation time, which is why problems with changes in 2 and 3 spatial coordinates will be studied in the future. It is however evident that the increase in the algorithm from $n = 2$ to $n = 3$ causes the computations to be performed even two times faster (while keeping relatively similar error values or improving them). Also, the BDF of higher order can be applied without the risk of a requirement of additional computation time, since the obtained coefficients (denoted as p and q in the paper) are common to all degrees of freedom in a single time-step.

Two additional analyses are planned for the future as far as the application of a time-step adaptation process is concerned:

– the analysis of the time-step choice basing on a more complicated scheme, e.g. one where a local discretization error is calculated by means of more composite formulae [10] (which could improve the efficiency of the presented strategies even more),
– an attempt to check the efficiency of other schemes in contrast to BDF which is commonly used in TD-FEM.

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