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The method of analysis of the stationary thermal field in insulation of a lead with the variable heat transfer coefficient

***Abstract**. In the paper the stationary thermal field was analyzed in a ring of insulation of the lead with a variable coefficient of the heat transfer on the external perimeter. Different functions were considered modelling the total heat transfer coefficient from the surface of insulation. The analytical computer aided method was developed for solution of the two-dimensional elliptical problem. Eigenfunctions of the problem were determined by the separation of variables. The unknown coefficients of eigenfunctions and the constants were computed numerically solving the respective system of algebraic equations. The obtained results were verified by means of the finite element method.*

Streszczenie.** W pracy analizowano stacjonarne pole termiczne w pierścieniu izolacji przewodu ze zmiennym współczynnikiem przejmowania ciepła na zewnętrznym obwodzie. Uwzględniono różne funkcje modelujące całkowity współczynnik przejmowania ciepła z powierzchni izolacji. Do rozwiązania dwuwymiarowego zagadnienia eliptycznego opracowano metodę analityczną wspomaganą komputerową. Funkcje własne zagadnienia określono za pomocą separacji zmiennych. Nieznane współczynniki funkcji własnych oraz stałe obliczono rozwiązując numerycznie odpowiedni układ równań algebraicznych. Otrzymane wyniki zweryfikowano za pomocą metody elementów skończonych. (Metoda analizy stacjonarnego pola termicznego w izolacji przewodu ze zmiennym współczynnikiem przejmowania ciepła***).*

Keywords: insulation of the lead, stationary thermal field, analytical methods, computer aid.
Słowa kluczowe: izolacja przewodu, stacjonarne pole termiczne, metody analityczne, wspomaganie komputerowe.

Introduction

In recent works [1],[2] of the authors the transient thermal field in insulation of a DC lead was analyzed in terms of the convective heat transfer. A constant value of the heat transfer coefficient on the insulation perimeter was assumed with that. However, results of numerous investigations (e.g. [4], [13]) show, that the mentioned coefficient depends on location on the perimeter of cylindrical solids. That is because above the top point of the system (Fig. 1) some kind of the heat stream is formed, which results in the worse heat transfer in the above and the better one down below.

The authors took the mentioned phenomenon into account in papers [3], [5]. In [3] the transient thermal field of a DC cable was computed by the numerical way with the variable heat transfer coefficient on the cable perimeter. In [5], in turn, the analytical method of determination of the stationary thermal field of a bare lead was presented, with the variable heat transfer coefficient, as well. In the present paper the subject-matter of the analytical method was extended taking into considerations a ring shaped configuration instead of the circular one [5]. It enabled the stationary thermal field analysis in a layer of insulation of the lead. The variable heat transfer coefficient was assumed on the perimeter, similar to [3], [5]. It should be mentioned about many advantages of the analytical methods, which results are described by formulas. They provide a lot of information making more easy a discussion on the influence of particular parameters and the physical interpretation of the obtained results, as well. In the proposed method eigenfunctions of the solution were determined by the analytical way. Unknown coefficients of those functions and constants were obtained solving the respective system of algebraic equations.

Boundary problem of the modeled thermal field

The subject of investigations is a layer of insulation of the lead (Fig. 1). It was assumed, that the whole system is located in air of the temperature T_0 and it is shielded from direct solar radiation. Besides, constant and averaged material parameters were assumed in analysis of the considered problem.

A two-dimensional equation of the heat conduction was obtained [6], [7] assuming that a length of the system (layer of insulation) is considerably greater than its diameter and assuming variable cooling conditions on the perimeter

$$\frac{\partial^2 T(r,\varphi)}{\partial r^2}+\frac{1}{r}\frac{\partial T(r,\varphi)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T(r,\varphi)}{\partial \varphi^2}=0 \tag{1}$$

for $R_1 \le r \le R_2$, $0 \le \varphi \le 2\pi$, where:
$T(r,\varphi)$ - stationary temperature field, R_1 - internal radius of insulation, R_2 - external radius of insulation, r - radial coordinate, φ - angular coordinate.

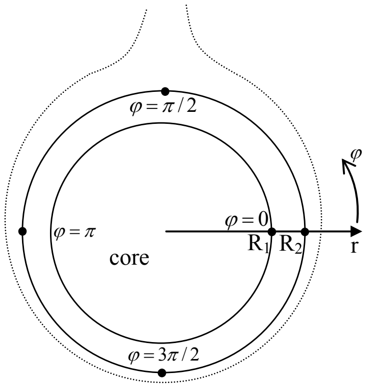

Fig. 1 Model of insulation with the boundary layer

The external surface of insulation ($r=R_2$) gives up the heat by means of a natural convection and radiation. The mentioned heat transfer is described by Hankel's boundary condition [4]

$$-\lambda \left.\frac{\partial T(r,\varphi)}{\partial r}\right|_{r=R_2} = \alpha(\varphi)\left[T(R_2,\varphi)-T_o\right] \quad \text{for} \quad 0 \le \varphi \le 2\pi, \tag{2}$$

where: λ - thermal conductivity.

The total heat transfer coefficient $\alpha(\varphi)$ occurred in equation (2) depends on location of the considered point on the perimeter of insulation. Insulation of the lead is strictly adherent to a core. The thermal conductivity of the last one is more than 2000 times larger than the one of insulation. Therefore a uniform distribution of the thermal field can be assumed in the whole region $r \le R_1$ of a core (copper or aluminium). From the above premises it follows, that constant value T_c of the temperature on the internal surface of insulation ($r=R_1$) can be assumed. It is described by Dirichlet's boundary condition

(3) $T(R_1,\varphi)=T_c$ for $0\le\varphi\le 2\pi$.

Value T_c in (3) obviously depends on the current intensity in a core of the lead and it cannot exceed the maximum sustained temperature for insulation.

Relations (1-3) determine the elliptical boundary problem of the thermal field in insulation.

Solution of the boundary problem

The homogeneous two-dimensional partial equation (1) of the heat conduction was solved by the separation of variables method [8],[9]. After elimination of singular solutions and non-physical ones (non-periodical with respect to the angular coordinate) it was obtained:

(4)

$$T(r,\varphi)=A\ln(r)+B+\sum_{n=1}^{\infty}(C_n r^n+D_n r^{-n})\left[E_n\cos(n\varphi)+F_n\sin(n\varphi)\right]$$

for $R_1\le r\le R_2$, $0\le\varphi\le 2\pi$ where:

A,B - constants, C_n,D_n,E_n,F_n - coefficients of eigenfunctions. Then the number of constants and coefficients of solution (4) was reduced taking advantage of Dirichlet's boundary condition (3). For example constant B was eliminated in the result of substitution (4) to (3) and integration of the obtained relation with respect to the angular coordinate φ within interval $<0,2\pi>$. In turn the number of unknown coefficients in (4) was reduced this way, that (4) was substituted to (3) once again. Then the obtained relation was multiplied by $cos(m\varphi)$ and integrated both sides with respect to the angular coordinate φ within interval $<0,2\pi>$. Advantage of the functions *{cos(mφ), sin(mφ)}* ortogonality was taken into account with that. In the result it was obtained:

(5)

$$T(r,\varphi)=T_c+A\ln(\frac{r}{R_1})+\sum_{n=1}^{\infty}r^n\left(1-\left(\frac{R_1}{r}\right)^{2n}\right)\left[G_n\cos(n\varphi)+H_n\sin(n\varphi)\right].$$

In order to determine G_n,H_n and constant A the summation of series (5) was limited to a finite number of L terms and (5) was substituted to Henkel's boundary condition (2). In the result it was obtained:

(6)

$$\frac{A}{R_2}+\sum_{n=1}^{L}nR_2^{n-1}\left(1+\left(\frac{R_1}{R_2}\right)^{2n}\right)\left(G_n\cos(n\varphi)+H_n\sin(n\varphi)\right)=$$
$$=-\frac{\alpha(\varphi)}{\lambda}\left[T_c-T_o+A\ln\frac{R_2}{R_1}+\sum_{n=1}^{L}R_2^n\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)\left(G_n\cos(n\varphi)+H_n\sin(n\varphi)\right)\right].$$

Such obtained relation (6) was multiplied by $cos(m\varphi)$ and integrated both sides with respect to the angular coordinate φ within interval $<0,2\pi>$. This way equation (7a) was obtained for $m=1,2...L$. A successive equation was obtained multiplying (6) by $sin(m\varphi)$ and integrating both sides with respect to the same coordinate and within the same interval as in the above. In the result one comes to equation (7b) for $m=1,2...L$. Last equation (7c) was obtained in the result of integration of relation (6) both sides with respect to the angular coordinate φ within interval $<0,2\pi>$. In computation of some integrals present in (7a,b) advantage of the functions $\{cos(m\varphi), sin(m\varphi)\}$ ortogonality was taken within interval $<0,2\pi>$. In the result relations (7a-c) determine the system of $2L+1$ equations with respect to G_n,H_n,A

$$\begin{cases}\sum_{n=1}^{L}G_nI_1(m,n)+\sum_{n=1}^{L}H_nI_2(m,n)+A\frac{\ln\frac{R_2}{R_1}}{\lambda}I_3(m)=-\frac{T_c-T_o}{\lambda}I_3(m) & m=1,2...L,\\ \sum_{n=1}^{L}G_nI_4(m,n)+\sum_{n=1}^{L}H_nI_5(m,n)+A\frac{\ln\frac{R_2}{R_1}}{\lambda}I_6(m)=-\frac{T_c-T_o}{\lambda}I_6(m) & m=1,2...L,\\ \sum_{n=1}^{L}G_nI_7(n)+\sum_{n=1}^{L}H_nI_8(n)+A\left(\frac{2\pi}{R_2}+\frac{\ln\frac{R_2}{R_1}}{\lambda}I_9\right)=-\frac{T_c-T_o}{\lambda}I_9 \end{cases}$$

(7a,b,c), where:

(8a)

$$I_1(m,n)=\begin{cases}\frac{R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)\int_0^{2\pi}\alpha(\varphi)\cos(n\varphi)\cos(m\varphi)d\varphi & for\ m\neq n\\ \frac{R_2^m}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2m}\right)\int_0^{2\pi}\alpha(\varphi)\cos^2(m\varphi)d\varphi+m\pi R_2^{m-1}\left(1+\left(\frac{R_1}{R_2}\right)^{2m}\right) & for\ m=n,\end{cases}$$

(8b) $$I_2(m,n)=\frac{R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)\int_0^{2\pi}\alpha(\varphi)\sin(n\varphi)\cos(m\varphi)d\varphi ,$$

(8c) $$I_3(m)=\int_0^{2\pi}\alpha(\varphi)\cos(m\varphi)d\varphi ,$$

(9a) $$I_4(m,n)=\frac{R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)\int_0^{2\pi}\alpha(\varphi)\cos(n\varphi)\sin(m\varphi)d\varphi ,$$

$$I_5(m,n)=\begin{cases}\frac{R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)\int_0^{2\pi}\alpha(\varphi)\sin(n\varphi)\sin(m\varphi)d\varphi & for\ m\neq n\\ \frac{R_2^m}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2m}\right)\int_0^{2\pi}\alpha(\varphi)\sin^2(m\varphi)d\varphi+m\pi R_2^{m-1}\left(1+\left(\frac{R_1}{R_2}\right)^{2m}\right) & for\ m=n,\end{cases}$$

(9b)

(9c) $$I_6(m)=\int_0^{2\pi}\alpha(\varphi)\sin(m\varphi)d\varphi ,$$

(10a) $$I_7(n)=\frac{R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)\int_0^{2\pi}\alpha(\varphi)\cos(n\varphi)d\varphi ,$$

(10b) $$I_8(n)=\frac{R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)\int_0^{2\pi}\alpha(\varphi)\sin(n\varphi)d\varphi ,$$

(10c) $$I_9=\int_0^{2\pi}\alpha(\varphi)d\varphi .$$

The computation of integrals (8-10) for given $\alpha(\varphi)$ leads to determination of unknown coefficients G_n,H_n and constant A from the system of equations (7). This way the lacking elements of solution (5) are determined. The results of computations of integrals (8-10) for three different functions $\alpha(\varphi)$ were placed in the appendix.

Computational examples

The computer programme was developed based on Mathematica 6.0 package [10]. The program computes the temperature field distribution by means of the presented method. The layer of PVC insulation heated up to maximum sustained temperature on the critical surface $r=R_1$ by a core of the lead (with cross section *300 mm*^{2}) was considered as an example. The following data were assumed: $R_1=0{,}0117m$, $R_2=0{,}0141m$, $\lambda=0{,}017W/(mK)$, $T_0=25^oC$, $T_c=70^oC$, $L=100$. Besides three different functions $\alpha(\varphi)$ were assumed modeling the total heat transfer coefficient from insulation. The first approximation $\alpha_1(\varphi)$ [3] relatively accurate models influence of the boundary layer,

(11)

$$\alpha_1(\varphi) = (\alpha_{max} - \alpha_{min})(1 - e^{\frac{\varphi - \pi/2}{B}}) + \alpha_{min} \text{ for } \varphi \in \langle 0, \pi/2 \rangle ,$$

$$\alpha_1(\varphi) = (\alpha_{max} - \alpha_{min})(1 - e^{\frac{-\varphi + \pi/2}{B}}) + \alpha_{min} \text{ for } \varphi \in \langle \pi/2, 3\pi/2 \rangle ,$$

$$\alpha_1(\varphi) = (\alpha_{max} - \alpha_{min})(1 - e^{\frac{\varphi - 5\pi/2}{B}}) + \alpha_{min} \text{ for } \varphi \in \langle 3\pi/2, 2\pi \rangle ,$$

where:

$$\alpha_{max} = 14{,}64\, W/(m^2K),\ \alpha_{min} = 10{,}52\, W/(m^2K),\ B = 0{,}4 .$$

The above formulas consider the diameter of a lead and proper criterial numbers, as well.

In the second approximation $\alpha_2(\varphi)$ constant values of the coefficient on the top and on the bottom half of insulation were assumed

(12) $$\alpha_2(\varphi) = \begin{cases} \alpha'_{min} = 13\, W/(m^2K) & for \quad 0 < \varphi < \pi \\ \alpha'_{max} = 15\, W/(m^2K) & for \quad \pi < \varphi < 2\pi. \end{cases}$$

The above relation models better give up of the heat by the lower surface of insulation than by the upper one. The third case of the same heat exchange on the surface $r=R_2$ was considered assuming $\alpha_3(\varphi)=const=14W/(m^2K)$.

The field distributions for $\alpha_1(\varphi), \alpha_2(\varphi), \alpha_3(\varphi)=const$ were presented in diagrams. The temperature distribution on the external perimeter of insulation ($r=R_2$) in the function of angular coordinate was shown in Fig. 2. Temperature distributions on the circles $r=const.$ for $r \in \langle R_1, R_2 \rangle$ have a very similar shape and they are mutually shifted along the temperature axis. Fig. 3 and Fig. 4 illustrate the temperature distributions in the function of radial coordinate with constant values of angular coordinates for $\alpha_1(\varphi)$ and $\alpha_2(\varphi)$, respectively. On the mentioned diagrams (Fig. 3 and Fig. 4) the field distributions for $\alpha_3(\varphi)=const$ were plotted for a comparison.

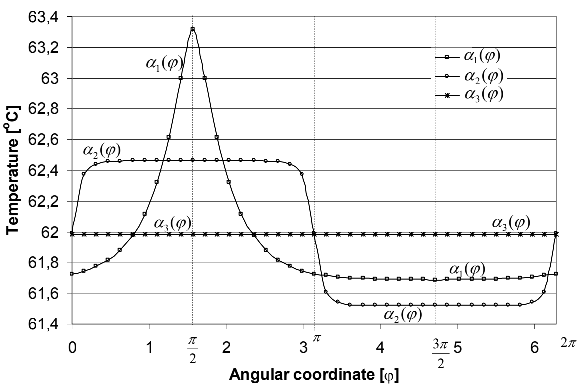

Fig. 2 Temperature distributions on the perimeter of insulation ($r=R_2$) in the function of angular coordinate for $\alpha_1(\varphi), \alpha_2(\varphi), \alpha_3(\varphi)$

The developed method was verified, as well. For this purpose the obtained results were compared with the computations made by means of the finite element method FE [11]. It is a base of the professional program NISA v.16 [12]. A two-dimensional model of insulation was approximated by the mesh consisting of 800 quadrangular elements of 2727 nodes placed in the vertices and in the half of sides of the mentioned figures. Relative differences of the temperature distributions

(13) $$100\% \frac{T_{FE}(R_2,\varphi) - T_A(R_2,\varphi)}{T_{FE}(R_2,\varphi)}$$

were illustrated in Fig. 5, where $T_{FE}(R_2,\varphi)$ - temperature distribution obtained by the finite element method, $T_A(R_2,\varphi)$ - temperature distribution obtained by the developed method.

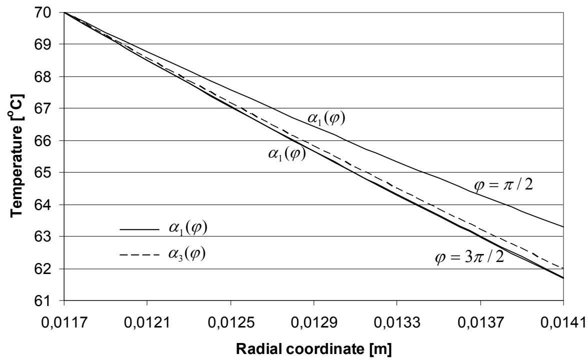

Fig.3 Temperature distributions in insulation in the function of radial co-ordinate for $\alpha_1(\varphi)$ and $\alpha_3(\varphi)=const$ for the selected values of angular co-ordinate

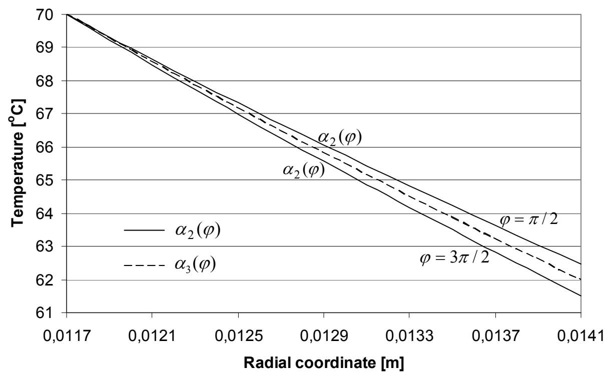

Fig.4 Temperature distributions in insulation in the function of radial co-ordinate for $\alpha_2(\varphi)$ and $\alpha_3(\varphi)=const$ for the selected values of angular co-ordinate

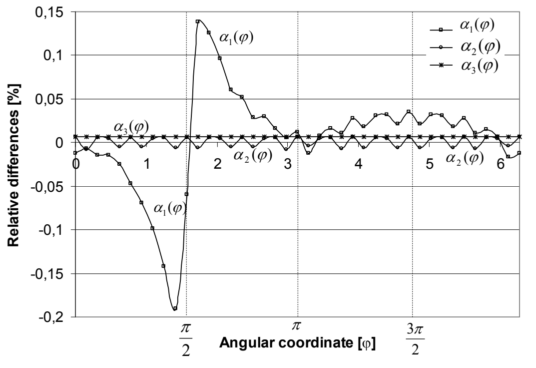

Fig.5 Relative differences of the temperature distributions on the perimeter of insulation ($r=R_2$) in the function of angular co-ordinate obtained by the finite element method and by the analytical one

Conclusions

A) Analyzing distributions in the function of angular co-ordinate evident differences between the temperature values at the top ($\varphi=\pi/2$) and in the bottom point ($\varphi=3\pi/2$) are observed for $\alpha_1(\varphi)$ and $\alpha_2(\varphi)$ (Fig. 2). It results from the influence of a boundary layer of air created around insulation. As it is seen in Fig. 2, with more accurate modeling i.e. for $\alpha_1(\varphi)$, the differences mentioned in the above are larger than ones for $\alpha_2(\varphi)$. For a constant value of the heat transfer coefficient $\alpha_3(\varphi)$, the temperature of surface $r=R_2$ in insulation is uniform (Fig. 2 for $\alpha_3(\varphi)=const.$).

Then analyzing the field distributions in the function of radial co-ordinate (for constant values of angular co-ordinates) it is seen, that the temperature of insulation decreases for all coefficients of the heat transfer (Fig. 3,4) with the radial co-ordinate increase. For $\alpha_1(\varphi)$ and $\alpha_2(\varphi)$ inequality $T(r,\varphi=\pi/2)\geq T(r,\varphi=3\pi/2)$ is satisfied for $R_1\leq r\leq R_2$, what results from better give up of the heat by the bottom part of the system.

B) Relative differences (13) of the temperature distributions computed by the finite element method (FE) and by the analytical one (A) are the biggest for the heat transfer coefficient $\alpha_1(\varphi)$ (Fig. 5 for $r=R_2$). The absolute maximal value is about 0.19%. For the remained coefficients i.e. for $\alpha_2(\varphi)$ and $\alpha_3(\varphi)=const.$ the above discussed differences are smaller. At other points of insulation ($R_1<r<R_2$) the considered differences are almost the same or less than the one shown in Fig. 5. Then the developed method should be considered as numerically verified.

Appendix

A) Results of the computation of integrals (8-10) for the heat transfer coefficient $\alpha_1(\varphi)$:

(A1)

$$I_1(m,n)=\frac{B(\alpha_{\max}-\alpha_{\min})R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)\left[\frac{e^{-\frac{\pi}{B}}\cos\frac{3\pi(m-n)}{2}-\cos\frac{(m-n)\pi}{2}}{B^2(m-n)^2+1}+\frac{e^{-\frac{\pi}{B}}\cos\frac{3\pi(m+n)}{2}-\cos\frac{(m+n)\pi}{2}}{B^2(m+n)^2+1}\right]$$

for $m\neq n$,

(A2)

$$I_1(m,n)=\frac{R_2^m}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2m}\right)\frac{e^{-\frac{\pi}{B}}}{4B^2m^2+1}\Big[B(\alpha_{\max}-\alpha_{\min})(1+4B^2m^2+\cos(3m\pi))+ e^{\frac{\pi}{B}}((4B^2m^2+1)(\pi\alpha_{\max}-B(\alpha_{\max}-\alpha_{\min}))-B(-1)^m(\alpha_{\max}-\alpha_{\min}))\Big]+\pi mR_2^{m-1}\left(1+\left(\frac{R_1}{R_2}\right)^{2m}\right)$$

for $m=n$ **,**

(A3)

$$I_2(m,n)=\frac{B(\alpha_{\max}-\alpha_{\min})R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)e^{-\frac{\pi}{B}}\left[\frac{\sin\frac{\pi(m-n)}{2}(e^{\frac{\pi}{B}}-1-2\cos((m-n)\pi)}{B^2(m-n)^2+1}+\frac{\sin\frac{3\pi(m+n)}{2}-e^{\frac{\pi}{B}}\sin\frac{(m+n)\pi}{2})}{B^2(m+n)^2+1}\right],$$

(A4) $$I_3(m)=2Be^{-\frac{\pi}{B}}(\alpha_{\max}-\alpha_{\min})\frac{\left[2(-1)^m-1-e^{\frac{\pi}{B}}\right]\cos(\frac{m\pi}{2})}{B^2m^2+1},$$

(A5)

$$I_4(m,n)=\frac{B(\alpha_{\max}-\alpha_{\min})R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)\left[\frac{e^{-\frac{\pi}{B}}\sin\frac{3\pi(m-n)}{2}-\sin\frac{(m-n)\pi}{2}}{B^2(m-n)^2+1}+\frac{e^{-\frac{\pi}{B}}\sin\frac{3\pi(m+n)}{2}-\sin\frac{(m+n)\pi}{2}}{B^2(m+n)^2+1}\right],$$

(A6)

$$I_5(m,n)=\frac{B(\alpha_{\max}-\alpha_{\min})R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)\left[\frac{e^{-\frac{\pi}{B}}\cos\frac{3\pi(m-n)}{2}-\cos\frac{(m-n)\pi}{2}}{B^2(m-n)^2+1}+\frac{\cos\frac{\pi(m+n)}{2}-e^{-\frac{\pi}{B}}\cos\frac{3(m+n)\pi}{2}}{B^2(m+n)^2+1}\right]$$

for $m\neq n$,

(A7)

$$I_5(m,n)=\frac{R_2^m}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2m}\right)\frac{e^{-\frac{\pi}{B}}}{4B^2m^2+1}\Big[B(\alpha_{\max}-\alpha_{\min})(1+4B^2m^2-(-1)^m)+ e^{\frac{\pi}{B}}(B(-1)^m(\alpha_{\max}-\alpha_{\min})+(4B^2m^2+1)(\pi\alpha_{\max}-B(\alpha_{\max}-\alpha_{\min})))\Big]+\pi mR_2^{m-1}\left(1+\left(\frac{R_1}{R_2}\right)^{2m}\right)$$

for $m=n$,

(A8)

$$I_6(m)=2Be^{-\frac{\pi}{B}}(\alpha_{\max}-\alpha_{\min})\frac{\left[1+2(-1)^m-e^{\frac{\pi}{B}}\right]\sin(\frac{m\pi}{2})}{B^2m^2+1},$$

(A9)

$$I_7(n)=\frac{2B(\alpha_{\max}-\alpha_{\min})R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)e^{-\frac{\pi}{B}}\frac{\left[2(-1)^n-1-e^{\frac{\pi}{B}}\right]\cos(\frac{n\pi}{2})}{B^2n^2+1},$$

(A10)

$$I_8(n)=\frac{2B(\alpha_{\max}-\alpha_{\min})R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)e^{-\frac{\pi}{B}}\frac{\left[1+2(-1)^n-e^{\frac{\pi}{B}}\right]\sin(\frac{n\pi}{2})}{B^2n^2+1},$$

(A11) $$I_9=2B(\alpha_{\max}-\alpha_{\min})(e^{-\frac{\pi}{B}}-1)+2\pi\alpha_{\max}.$$

B) Results of the computation of integrals (8-10) for the heat transfer coefficient $\alpha_2(\varphi)$:

(A12)

$$I_1(m,n)=\begin{cases}0 & \text{for } m\neq n\\ \frac{R_2^m}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2m}\right)\frac{\pi(\alpha'_{\max}+\alpha'_{\min})}{2}+\pi mR_2^{m-1}\left(1+\left(\frac{R_1}{R_2}\right)^{2m}\right) & \text{for } m=n,\end{cases}$$

(A13)

$$I_2(m,n)=\begin{cases}\frac{R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)\frac{n(\alpha'_{\max}-\alpha'_{\min})}{(n^2-m^2)}((-1)^{m+n}-1) & \text{for } m\neq n\\ 0 & \text{for } m=n,\end{cases}$$

(A14) $$I_3(m)=0,$$

(A15)

$$I_4(m,n)=\begin{cases}\frac{R_2^n}{\lambda}\left(1-\left(\frac{R_1}{R_2}\right)^{2n}\right)\frac{m(\alpha'_{\max}-\alpha'_{\min})}{(m^2-n^2)}((-1)^{m+n}-1) & \text{for } m\neq n\\ 0 & \text{for } m=n,\end{cases}$$

(A16) $$I_5(m,n) = I_1(m,n)\ ,$$

(A17) $$I_6(m) = \frac{2\left[\alpha'_{\max} + \alpha'_{\min} + 2(-1)^m \alpha'_{\max}\right]\sin^2(\frac{m\pi}{2})}{m}\ ,$$

(A18), $$I_7(n) = 0\ ,$$

(A19)

$$I_8(n) = \frac{R_2^n}{\lambda}\left(1 - \left(\frac{R_1}{R_2}\right)^{2n}\right)\frac{2\left[\alpha'_{\max} + \alpha'_{\min} + 2(-1)^n \alpha'_{\max}\right]\sin^2(\frac{n\pi}{2})}{n}\ ,$$

(A20) $$I_9 = \pi(\alpha'_{\max} + \alpha'_{\min})\ .$$

C) Results of the computation of integrals (8-10) for the heat transfer coefficient $\alpha_3(\varphi)=const$:

(A21)

$$I_1(m,n) = \begin{cases} 0 & for\ m \neq n \\ \frac{R_2^m}{\lambda}\left(1 - \left(\frac{R_1}{R_2}\right)^{2m}\right)\alpha\pi + \pi m R_2^{m-1}\left(1 + \left(\frac{R_1}{R_2}\right)^{2m}\right) & for\ m = n, \end{cases}$$

(A22) $$I_2(m,n) = 0\ ,$$

(A23) $$I_3(m) = 0\ ,$$

(A24) $$I_4(m,n) = 0\ ,$$

(A25) $$I_5(m,n) = I_1(m,n)\ ,$$

(A26) $$I_6(m) = 0\ ,$$

(A27) $$I_7(n) = I_8(n) = 0\ ,$$

(A28) $$I_9 = 2\pi\alpha\ .$$

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