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Inductance of a long two-rectangular busbar single-phase line

***Abstract.** In this paper, inductance of a long single-phase line with parallel straight rigid rectangular busbars is investigated. Using a definition of the self and mutual inductances for two conductors of any shape and finite lengths new exact closed formulae for these inductances of long rectangular busbars are proposed. In case of direct current (DC) or low frequency (LF) these inductances are given by analytical formula.*

***Streszczenie.** W artykule bada się indukcyjność linii jednofazowej z prostymi równoległymi sztywnymi szynoprzewodami prostokątnymi. Stosując definicję indukcyjności własnej i wzajemnej dla dwóch przewodów o dowolnym kształcie i skończonej długości zaproponowano dokładne wzory na obliczanie tych indukcyjności dla długich szynoprzewodów prostokątnych. W przypadku prądu stałego lub niskiej częstotliwości indukcyjności te wyrażono wzorami analitycznymi. (**Indukcyjność linii jednofazowej o długich szynoprzewodach prostokątnych**)*

Key words: rectangular busbar, self and mutual inductance, single-phase line
Słowa kluczowe: prostokątny przewód szynowy, indukcyjność własna i wzajemna, linia jednofazowa

Introduction

We consider a two-wire single-phase line with straight parallel rigid rectangular conductors A and B with the same dimensions $a \times b \times l$, separated by a distance d and conducting the currents $\underline{I}$ and $-\underline{I}$ respectively, as shown in Fig. 1.

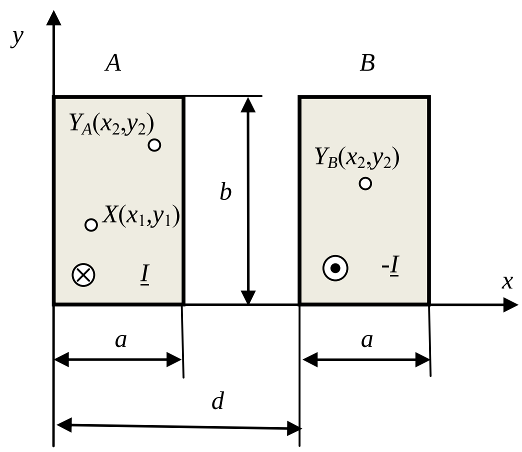

Fig. 1. A single-phase line with two conductors of rectangular cross section with width *a*, thickness *b*, length *l* and currents $\underline{I}$ and $-\underline{I}$

Total inductance of the single-phase line is given by well known formula [1-4]

(1) $$L_l = 2L - 2M$$

where L is the self inductance of rectangular busbar and M is the mutual inductance between two parallel busbars separated by a distance d.

The self and mutual inductances play an important role not only in power circuits [1, 2], but also in printed circuit board (PCB) lands [3, 4]. Formulae for the mutual inductances of set of conductors of rectangular cross-section are the subjects of many electrical papers and books. The most significant of them are: Grover's given in [3-6], Kalantarov and Tseitlin's presented in [7], Strunsky's shown in [8], Ruehli's presented in [6] and [9] as well as Hoer and Love's shown in [4], [6] and [10].

In this paper a new method for calculating self and mutual inductance of long rectangular conductors and consequently the total inductance of the single-phase line is presented. The method results in a system of two integral Fredholm's equations [11, 12] . We compare our analytical formulae with several well-known ones given in the literature for DC or low frequency.

Self inductance of rectangular busbar

If the length of busbar is much greater than the other dimensions, in case of DC or low frequency, its self inductance is given by following formula [13]

(2) $$L = \frac{\mu_0\, l}{2\,\pi}\left[\ln(2\,l) - 1 + G\right]$$

where

(3) $$G = -\frac{1}{2\,a^2 b^2}\int_0^b\int_0^b\int_0^a\int_0^a \ln\left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2$$

(x_1, y_1) are coordinates of point of observation $X = X(x_1, y_1) \in A$ and (x_2, y_2) are coordinates of source point $Y_A = Y_A(x_2, y_2) \in A$.

In general case the integral (3) is difficult to calculate. But if two variables, for example x_1 and x_2, can be replaced with only one variable $x = x_2 - x_1$ then a double definite integral can be calculated from following formula

(4) $$F(y,z) = \int_{s_3}^{s_4}\int_{s_1}^{s_2} f(x_2 - x_1, y, z)\,\mathrm{d}x_2\,\mathrm{d}x_1 = \\ = F(s_4 - s_1) - F(s_4 - s_2) + F(s_3 - s_2) - F(s_3 - s_1)$$

or in more general form as

(5) $$F(y) = \left[F(x,y)\right]_{s_4-s_2,\, s_3-s_1}^{s_4-s_1,\, s_3-s_2}(x) = \left[F(x,y)\right]_{p_2,\, p_4}^{p_1,\, p_3}(x) = \sum_{i=1}^{i=4}(-1)^{i+1}F(p_k, y)$$

where

(6) $$F(x,y) = \iint f(x,y)\,\mathrm{d}x\mathrm{d}x$$

is a double indefinite integral of $f(x,y)$. In (3) we can also omit terms proportional to one variable like $F(x,y) = x\,g(y)$. Now we can also put $y = y_2 - y_1$ and first calculate a quadruple indefinite integral

(7) $$G(x,y) = -\frac{1}{2\,a^2 b^2}\iiiint \ln\left[x^2 + y^2\right]\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}y = \frac{1}{288 a^2 b^2}\left\{150 x^2 y^2 - 6\left[8\,x\,y^3 \tan^{-1}\frac{x}{y} + 8 x^3 y \tan^{-1}\frac{y}{x} + \left(x^4 - 6\,x^2 y^2 + y^4\right)\ln\left(x^2 + y^2\right)\right]\right\}$$

After calculating this integral we determine the self inductance of the long busbar of rectangular cross section

$$L = \frac{\mu_0\, l}{2\,\pi}\left\{\ln(2\,l) - 1 + \left[\left[G(x,y)\right]\begin{matrix} a,-a \\ (x) \\ 0,0 \end{matrix}\right]\begin{matrix} b,-b \\ (y) \\ 0,0 \end{matrix}\right\} = \frac{\mu_0\, l}{2\,\pi}\left\{\ln(2\,l) - 1 + \sum_{i=1}^{i=4}\sum_{j=1}^{j=4}(-1)^{i+j} F(p_i, q_j)\right\} \tag{8}$$

On the basis of (8) we have the analytical formulae for the self inductance of the straight long busbar of rectangular cross section

$$L = \frac{\mu_0\, l}{2\,\pi}\left\{\ln\frac{2\,l}{a} + \frac{13}{12} - \frac{2}{3}\left[\frac{b}{a}\tan^{-1}\frac{a}{b} + \frac{a}{b}\tan^{-1}\frac{b}{a}\right] + \frac{1}{12}\left[\left(\frac{a}{b}\right)^2 - 6 + \left(\frac{b}{a}\right)^2\right]\ln\left[1+\left(\frac{a}{b}\right)^2\right] + \frac{1}{6}\left[6-\left(\frac{a}{b}\right)^2\right]\ln\frac{a}{b}\right\} \tag{9}$$

or

$$L = \frac{\mu_0\, l}{2\,\pi}\left\{\ln\frac{2\,l}{b} + \frac{13}{12} - \frac{2}{3}\left[\frac{b}{a}\tan^{-1}\frac{a}{b} + \frac{a}{b}\tan^{-1}\frac{b}{a}\right] + \frac{1}{12}\left[\left(\frac{a}{b}\right)^2 - 6 + \left(\frac{b}{a}\right)^2\right]\ln\left[1+\left(\frac{b}{a}\right)^2\right] + \frac{1}{6}\left[6-\left(\frac{b}{a}\right)^2\right]\ln\frac{b}{a}\right\} \tag{10}$$

as well as

$$L = \frac{\mu_0\, l}{2\,\pi}\left\{\ln\frac{2\,l}{a+b} + \frac{13}{12} - \frac{2}{3}\left[\frac{b}{a}\tan^{-1}\frac{a}{b} + \frac{a}{b}\tan^{-1}\frac{b}{a}\right] + \frac{1}{2}\ln\left[1 + \frac{a}{b}\frac{2}{1+\left(\frac{a}{b}\right)^2}\right] + \frac{1}{12}\left[\left(\frac{a}{b}\right)^2\ln\left[1+\left(\frac{b}{a}\right)^2\right] + \left(\frac{b}{a}\right)^2\ln\left[1+\left(\frac{a}{b}\right)^2\right]\right]\right\} \tag{11}$$

For the chosen traverse dimensions and different lengths of a busbar the calculations of its inductance have been made according to all previous, mentioned above, formulae – Table 1.

Table 1. Self-inductance of a busbar of rectangular cross section for DC or low frequency

l (m)	Grover *L* (nH)	Kalantarov *L* (nH)	Strunsky *L* (nH)	Bueno *L* (nH)	Ruehli *L* (nH)	Hoer *L* (nH)	Eq. (9) *L* (nH)
1.00 *a*	13.8593	17.7482	21.6067	17.7165	22.4732	22.4732	17.7165
10.0 *a*	542.007	545.896	549.469	545.579	550.911	550.911	545.579
100 *a*	9139.20	9143.09	9143.81	9139.92	9145.32	9145.32	9139.92
1000 *a*	128268	128272	128244	128240	128227	128294	128240

Mutual inductance between two rectangular busbars

If the length of two busbars is much greater than the other dimensions, in case of DC or low frequency, the mutual inductance between them expresses by formula [14]

$$M = \frac{\mu_0\, l}{2\,\pi}\left[\ln(2\,l) - 1 + G\right] \tag{12}$$

where

$$G = -\frac{1}{2\,a^2 b^2}\int_0^b\int_0^b\int_d^{d+a}\int_0^a \ln\left[(x_2-x_1)^2 + (y_2-y_1)^2\right]\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}y_1\mathrm{d}y_2 \tag{13}$$

(x_1, y_1) are coordinates of point of observation $X = X(x_1, y_1) \in A$ and (x_2, y_2) are coordinates of source point $Y_B = Y_B(x_2, y_2) \in B$.

Eq. (13) is a quadruple definite integral of four variables whose quadruple indefinite integral of variables $x = x_2 - x_1$ and $y = y_2 - y_1$ is given by (7).

Hence the mutual inductance between two rectangular busbars is given by following formula

$$M = \frac{\mu_0\, l}{2\,\pi}\left\{\ln(2\,l) - 1 + \left[\left[G(x,y)\right]\begin{matrix} -a-d,\, a-d \\ (x) \\ -d,-d \end{matrix}\right]\begin{matrix} -b,b \\ (y) \\ 0,0 \end{matrix}\right\} = \frac{\mu_0\, l}{2\,\pi}\left\{\ln(2\,l) - 1 + \sum_{i=1}^{i=4}\sum_{j=1}^{j=4}(-1)^{i+j} G(p_i, q_j)\right\} \tag{14}$$

On the basis of (14) we have the analytical formula for the mutual inductance between two long parallel straight busbars of rectangular cross section

$$M = \frac{\mu_0\, l}{2\,\pi}\left\{\begin{aligned} &\ln\frac{2\,l}{\sqrt[4]{\left[b^2+(a-d)^2\right]\left[b^2+(a+d)^2\right]}} + \frac{13}{12} - \frac{1}{3}\frac{(a-d)^3}{a^2 b}\tan^{-1}\frac{b}{a-d} - \frac{1}{3}\frac{b(a-d)}{a^2}\tan^{-1}\frac{a-d}{b} + \frac{2}{3}\frac{d^3}{a^2 b}\tan^{-1}\frac{b}{a} + \frac{2}{3}\frac{bd}{a^2}\tan^{-1}\frac{d}{b} - \\ &\frac{1}{3}\frac{(a+d)^3}{a^2 b}\tan^{-1}\frac{b}{a+d} - \frac{1}{3}\frac{b(a+d)}{a^2}\tan^{-1}\frac{a+d}{b} - \frac{1}{24}\frac{(a-d)^4}{a^2b^2}\ln\frac{(a-d)^2}{b^2+(a-d)^2} - \frac{1}{24}\frac{(a+d)^4}{a^2b^2}\ln\frac{(a+d)^2}{b^2+(a+d)^2} - \\ &\frac{1}{24}\frac{b^2-6d^2}{a^2}\ln\frac{b^2+d^2}{b^2+(a-d)^2} - \frac{1}{24}\frac{b^2-6d^2}{a^2}\ln\frac{b^2+d^2}{b^2+(a+d)^2} - \frac{1}{12}\frac{d^4}{a^2b^2}\ln\frac{b^2+d^2}{d^2} - \frac{1}{2}\frac{d}{a}\ln\frac{b^2+(a+d)^2}{b^2+(a-d)^2} \end{aligned}\right\} \tag{15}$$

For the chosen traverse dimensions and different lengths of two the same busbars the calculations of their mutual inductance have been made according to all previous, mentioned above, formulae – Table 2.

Table 2. Mutual inductance between two busbars of rectangular cross section for DC or low frequency

l (m)	Ruehli *L* (nH)	Grover *L* (nH)	Strunsky *L* (nH)	Hoer *L* (nH)	Eq. (15) *L* (nH)
1.00 *a*	3.922301	3.922301	3.921699	3.849786	negative
10.0 *a*	238.8215	238.8215	238.8005	236.2490	211.9011
100 *a*	5800.112	5800.112	5799.862	5769.184	5803.147
1000 *a*	94556.06	94556.06	94553.52	94240.43	94872.83

Inductance of a single-phase line

Using formulae (9) (or (10) and (11)) and (15) from (1) we calculate the inductance per unit length $\mathscr{L} = L/l$ of the single-phase line with chosen rectangular busbars. The results of calculations are shown in Table 3.

Table 3. Inductance of the single-phase line with rectangular busbars

Single phase line		MR busbar $a \times \mathrm{b}$ [15], $d = 2\,a$			
		7×7 mm	7×16 mm	7×36 mm	7×60 mm
$\mathscr{L}$	$\frac{\mathrm{nH}}{\mathrm{m}}$	599.498	430.637	265.429	182.376

Conclusions

Tables 1 and 2 show that we can use formulae for self and mutual inductance of long busbars in case when the normalized length l/a is greater than or equal to 10.

The self and mutual inductance per unit length of rectangular busbars depend on this length. So they may not be associated with a closed loop (according to the classical view of self and mutual inductance of a closed circuit) but they should be merely considered as a quantity helpful in calculating the inductances of real closed electrical circuits, for instance of a single-phase line – its inductance per unit length is independent of its length (Table 3).

Our formulae for self and mutual inductance of rectangular busbars have analytical forms, are analytically simple and can also replace the traditional tables and working ones.

These formulae can be used in the methods of numerical calculation of AC self and mutual inductance of rectangular conductors.

The derivations for the self and mutual inductance in a single-phase system can be extended to obtain the inductance per phase in a three-phase system – Fig. 2.

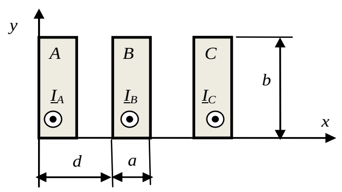

Fig. 2. A three phase system with rectangular busbars

Assuming a balanced three-phase system, where $I_A + I_B + I_C = 0$, the induction voltage drops across each phase are

$$\begin{bmatrix} \underline{U}_{LA} \\ \underline{U}_{LB} \\ \underline{U}_{LB} \end{bmatrix} = \mathrm{j}\,\omega \begin{bmatrix} L_{AA} & M_{AB} & M_{AC} \\ M_{BA} & L_{BB} & M_{BC} \\ M_{CA} & M_{CB} & L_{CC} \end{bmatrix} \begin{bmatrix} \underline{I}_A \\ \underline{I}_B \\ \underline{I}_C \end{bmatrix} \tag{16}$$

where $L_{AA} = L_{BB} = L_{CC} = L$ and are given by (9), (10) or (11), mutual inductions of flat three-phase line $M_{AB} = M_{BA} = M_{BC} = M_{CB} = M$ and are given by (15), mutual inductions $M_{AC} = M_{CA}$ are given by (15) after putting $2d$ under d in (15).

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