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Mutual inductance of two thin tapes with perpendicular widths

Abstract. *In this paper, using a definition of a mutual inductance for two conductors of any shape and finite lengths, the new exact closed formula for mutual inductance between two thin tapes whose axes are parallel and whose widths are perpendicular is proposed. In case of direct current (DC) or low frequency (LF) this inductance is given by analytical formula. The mutual inductance between two long thin tapes whose plans are perpendicular to one another is also presented.*

Streszczenie. *Stosując definicję indukcyjności wzajemnej między dwoma przewodami dowolnych kształtów i skończonej długości w pracy zaproponowano nowy dokładny wzór na obliczanie indukcyjności wzajemnej miedzy dwoma równoległymi cienkimi przewodami taśmowymi o szerokościach wzajemnie prostopadłych. W przypadku prądu stałego lub niskiej częstotliwości indukcyjność tę wyrażono wzorem analitycznym. Podano również wzór na indukcyjność wzajemną między dwoma długimi cienkimi przewodami taśmowymi o płaszczyznach wzajemnie prostopadłych. (**Indukcyjność wzajemna równoległych cienkich przewodów taśmowych o prostopadłych płaszczyznach**)*

Key words: rectangular busbar, thin tape, mutual inductance, electromagnetic field
Słowa kluczowe: prostokątny przewód szynowy, przewód taśmowy, indukcyjność wzajemna, pole elektromagnetyczne

Introduction

The self and mutual inductances play an important role not only in power circuits, but also in printed circuit board (PCB) lands [1-2]. The mutual inductance between two thin tapes whose axes are parallel but whose widths are perpendicular to one another such as shown in Fig. 1 can be calculated from general formulae given for rectangular busbars [3-8] by assuming the thickness to be very small. In [7] Hoer and Love present special formula for the mutual inductance between two parallel thin tapes of zero thickness.

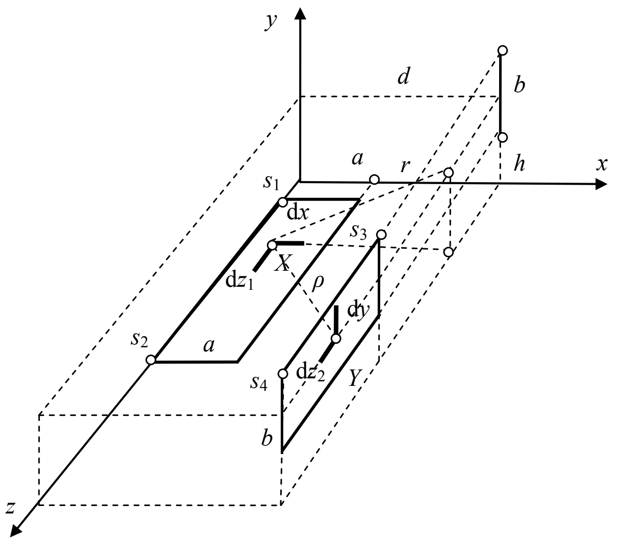

Fig. 1. Two parallel thin tapes with perpendicular widths

In this paper a new method for calculating mutual inductance of this set of tapes is presented. The method results in a system of two integral Fredholm's equations. We compare our analytical formulae with several well-known ones given in the literature for DC, low frequency or parallel thin tapes.

We consider the mutual inductance between two parallel thin tapes of widths a and b, lengths $l_1 = s_2 - s_1$ and $l_2 = s_4 - s_3$ respectively and of zero thickness such as shown in Fig. 1.

Definition of mutual inductance

The definition of mutual impedance between two straight conductors is given in [9-11] by following formula

(1) $$\underline{Z}_{12} = \frac{\mathrm{j}\,\omega\,\mu_0}{4\pi\,\underline{I}_1^*\underline{I}_2}\int_{v_1}\int_{v_2}\frac{\underline{J}_{22}(Y)\underline{J}_{11}^*(X)}{\rho}\,\mathrm{d}\,v\,\mathrm{d}\,v$$

where $\underline{J}_{22}(Y)$ is the complex current density at source point $Y = Y(x_2, y_2, z_2) \in S_2$, $\underline{J}_{11}^*(X)$ is the complex conjugate current density at point of observation $X = X(x_1, y_1, z_1) \in S_1$, v_1 and v_2 are conductors' volumes. Distance between the point of observation *X* and the source point *Y* (in general case) is $\rho = \sqrt{r^2 + (z_2 - z_1)^2}$ where $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. If conductors have a constant cross-sectional area S_1 and S_2 along theirs lengths, in case of DC, low frequency or for a thin strip conductors (in printed circuit board [1, 2]) we can assume that the current density is constant and given as $\underline{J}_{11}(X) = \underline{I}_1 / S_1$ and $\underline{J}_{22}(X) = \underline{I}_2 / S_2$ then, from the formulae (1), we obtain the mutual inductance between two straight parallel conductors

(2) $$M = M_{12} = M_{21} = \frac{\mu_0}{4\pi\,S_1\,S_2}\int_{v_1}\int_{v_2}\frac{1}{\rho}\,\mathrm{d}\,v_1\,\mathrm{d}\,v_2$$

Mutual inductance between two thin tapes

The mutual inductance between two thin tapes whose axes are parallel and whose widths are perpendicular to one another such as shown in Fig. 1 from (2) is given by formula

(3) $$M = \frac{\mu_0}{4\,\pi}\frac{1}{a\,b}F$$

where

(4) $$F = \int_h^{h+b}\int_0^a\int_{s_3}^{s_4}\int_{s_1}^{s_2}\frac{\mathrm{d}z_1\mathrm{d}z_2\;\mathrm{d}x\,\mathrm{d}y}{\sqrt{r^2 + (z_2 - z_1)^2}}$$

where $r^2 = (d - x)^2 + y^2$.

is a quadruple definite integral of four variables (x, y, z_1, z_2) into which the distance h is measured from xOz plan to the bottom of the second tape.

Now we can put $g - x = u \;\Rightarrow\; \mathrm{d}x = -\mathrm{d}u \;\left(u\big|_d^{d-a}\right)$ and $z = z_2 - z_1$ and first to calculate a quadruple indefinite integral

$$F(u,y,z)=\iint\iint\frac{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}z}{\sqrt{u^2+y^2+z^2}} \tag{5}$$

once with respect to u, once with respect to y and twice with respect to z. Finally, after a lengthy integration, formula (5) yields an expression for quadruple indefinite integral

$$\begin{aligned}F(u,y,z)=&\frac{u\,y}{3}\sqrt{u^2+y^2+z^2}+\frac{z^3}{6}\tan^{-1}\frac{u\,y}{z\sqrt{u^2+y^2+z^2}}+\frac{y^2z}{2}\tan^{-1}\frac{u\,z}{y\sqrt{u^2+y^2+z^2}}+\frac{u^2z}{2}\tan^{-1}\frac{yz}{u\sqrt{u^2+y^2+z^2}}+\\&\left(\frac{y^2}{6}-\frac{z^2}{2}\right)y\ln\left(u+\sqrt{u^2+y^2+z^2}\right)+\left(\frac{u^2}{6}-\frac{z^2}{2}\right)u\ln\left(y+\sqrt{u^2+y^2+z^2}\right)-u\,y\,z\ln\left(z+\sqrt{u^2+y^2+z^2}\right)\end{aligned} \tag{6}$$

Hence the mutual inductance between two parallel thin tapes is given by following formula

$$\begin{aligned}M&=\frac{\mu_0}{4\pi}\frac{1}{a\,b}F=\frac{\mu_0}{4\pi}\frac{1}{a\,b}\left[\left[\left[F(u,y,z)\right]_{d}^{d-a}(u)\right]_{h}^{h+b}(y)\right]_{s_1-s_3,s_2-s_4}^{s_1-s_4,s_2-s_3}(z)=\\&=\frac{\mu_0}{4\pi}\frac{1}{a\,b}\left[\left[\left[F(u,y,z)\right]_{p_2}^{p_1}(u)\right]_{q_2}^{q_1}(y)\right]_{r_2,r_4}^{r_1,r_3}(z)=\frac{\mu_0}{4\pi}\frac{1}{a\,b}\sum_{i=1}^{i=2}\sum_{j=1}^{j=2}\sum_{k=1}^{k=4}(-1)^{i+j+k+1}F(p_i,q_j,r_k)\end{aligned} \tag{7}$$

It is exactly the Hoer's formula given in [8]. For the same two tapes of width *a* and length l, distance *d* between them and without displacements along *y* axis ($h=0$) and along *z* axis ($s_1=s_3=0$) the mutual inductance is

$$M=\frac{\mu_0}{4\pi}\frac{1}{a^2}F=\frac{\mu_0}{4\pi}\frac{1}{a^2}\left[\left[\left[F(u,y,z)\right]_{d}^{d-a}(u)\right]_{0}^{a}(y)\right]_{0,0}^{-l,l}(z) \tag{8}$$

where

$$\begin{aligned}F=&\frac{2}{3}ad\sqrt{a^2+d^2}-\frac{2}{3}a(d-a)\sqrt{2a^2-2ad+d^2}-\frac{2}{3}ad\sqrt{a^2+d^2+l^2}+\frac{2}{3}a(d-a)\sqrt{2a^2-2ad+d^2+l^2}-\\&-\frac{1}{3}s^3\left(\tan^{-1}\frac{ad}{l\sqrt{a^2+d^2+s^2}}+\tan^{-1}\frac{a(a-d)}{l\sqrt{2a^2-2ad+d^2+l^2}}\right)-d^2l\tan^{-1}\frac{al}{d\sqrt{a^2+d^2+l^2}}-\\&-a^2l\left(\tan^{-1}\frac{dl}{a\sqrt{a^2+d^2+l^2}}+\tan^{-1}\frac{(a-d)l}{a\sqrt{2a^2-2ad+d^2+l^2}}\right)-(a-d)^2l\tan^{-1}\frac{al}{(a-d)\sqrt{2a^2-2ad+d^2+l^2}}+\\&+\frac{1}{3}d^3\ln\frac{a+\sqrt{a^2+d^2}}{d}+\frac{1}{3}a^3\ln\frac{d+\sqrt{a^2+d^2}}{d-a+\sqrt{2a^2-2ad+d^2}}-\frac{1}{3}(d-a)^3\ln\frac{a+\sqrt{2a^2-2ad+d^2}}{d-a}+\\&+\frac{1}{6}(a-d)\left[(a-d)^2-3l^2\right]\ln\frac{(a-d)^2+l^2}{a+\sqrt{2a^2-2ad+d^2+l^2}}+\frac{1}{6}d(d^2-3l^2)\ln\frac{d^2+l^2}{a+\sqrt{a^2+d^2+l^2}}+\\&+adl\ln\frac{l+\sqrt{a^2+d^2+l^2}}{-l+\sqrt{a^2+d^2+l^2}}+\frac{1}{3}a(a^2-3l^2)\ln\frac{d-a+\sqrt{2a^2-2ad+d^2+l^2}}{d+\sqrt{a^2+d^2+l^2}}+a(d-a)l\ln\frac{l+\sqrt{2a^2-2ad+d^2+l^2}}{-l+\sqrt{2a^2-2ad+d^2+l^2}}\end{aligned} \tag{9}$$

Mutual inductance between two long thin tapes

The double definite integral

$$\begin{aligned}f(x,y)&=\int_0^l\int_0^l\frac{1}{\rho_{XY}}\mathrm{d}z_1\mathrm{d}z_2=\\&=2\,l\left(\ln\frac{l+\sqrt{l^2+r_{XY}^2}}{r_{XY}}-\frac{\sqrt{l^2+r_{XY}^2}}{l}+\frac{r_{XY}}{l}\right)\end{aligned} \tag{10}$$

If $l \gg r_{XY}$ the function $f(x,y)$ becomes

$$f(x,y)=2\,l\left(\ln\frac{2\,l}{r_{XY}}-1\right) \tag{11}$$

and the mutual inductance between two long thin tapes expresses by formula

$$M=\frac{\mu_0\,l}{2\pi}\left[\ln(2\,l)-1+G\right] \tag{12}$$

where

$$G=-\frac{1}{2\,a\,b}\int_d^{d-a}\int_h^{h+b}\ln\left[u^2+y^2\right]\mathrm{d}u\mathrm{d}y \tag{13}$$

is a double definite integral of two variables. A double indefinite integral of (13) is

$$\begin{aligned}G(u,y)&=-\frac{1}{2\,a\,b}\iint\ln\left[u^2+y^2\right]\mathrm{d}u\,\mathrm{d}y=\\&-\frac{1}{2\,a\,b}\left[-3uy+y^2\tan^{-1}\frac{u}{y}+u^2\tan^{-1}\frac{y}{u}+uy\ln\left[u^2+y^2\right]\right]\end{aligned} \tag{14}$$

Hence the mutual inductance between two long thin tapes is

$$M=\frac{\mu_0\,l}{2\pi}\left\{\ln(2\,l)-1+\left[\left[G(u,y)\right]_{d}^{d-a}(u)\right]_{h}^{h+b}(y)\right\}=\frac{\mu_0\,l}{2\pi}\left\{\ln(2\,l)-1+\left[\left[G(u,y)\right]_{p_2}^{p_1}(u)\right]_{q_2}^{q_1}(y)\right\}=\frac{\mu_0\,l}{2\pi}\left\{\ln(2\,l)-1+\sum_{i=1}^{i=2}\sum_{j=1}^{j=2}(-1)^{i+j}G(p_i,q_j)\right\} \tag{15}$$

For the same two tapes of width *a*, distance *d* between them and without displacements along *y* axis ($h = 0$) and along *z* axis ($s_1 = s_3 = 0$) the mutual inductance is given by following formula

$$M = \frac{\mu_0\, l}{2\,\pi}\left\{\ln(2\,l) - 1 + \left[\left[G(u,y)\right]_{d}^{d-a}(u)\right]_{0}^{a}(y)\right\} \tag{16}$$

On the basis of (16) we have the analytical formula for the mutual inductance between two long parallel straight thin tapes with perpendicular plans

$$M = \frac{\mu_0\, l}{2\,\pi}\left[\begin{array}{l}\ln\frac{2\,l}{d} + \frac{1}{2} - \frac{(a-d)^2}{2a^2}\tan^{-1}\frac{a}{a-d} - \frac{d^2}{2a^2}\tan^{-1}\frac{a}{d} - \frac{1}{2}\tan^{-1}\frac{d}{a} - \frac{1}{2}\tan^{-1}\frac{a-d}{a} - \\ - \frac{d}{2a}\ln\frac{a^2+d^2}{2a^2-2ad+d^2} - \frac{1}{2}\ln\frac{2a^2-2ad+d^2}{d^2}\end{array}\right] \tag{17}$$

Computational results

In this section, we present the evaluation results for all the above mentioned mutual inductance formulae in the range of VLSI applications. For the mutual inductance of two identical real thin tapes of width *a*, thickness *t* and length *l* above formulae give results sown in Table 1.

Table 1. Mutual inductance between two real thin tapes for DC or low frequency

l (m)	Hoer *L* (pH)	Eq. (9) *L* (pH)	Eq. (18) *L* (pH)
0.01 *a*	0.000002	0.000003	negative
0.10 *a*	0.000207	0.000321	negative
1.00 *a*	0.020406	0.031088	negative
10.0 *a*	1.336459	1.691528	1.537306
100 *a*	34.33413	38.55992	38.39981
1000 *a*	571.3152	614.4173	614.2566

Conclusions

The mutual inductance between two thin tapes of zero thickness whose widths are perpendicular is given by the analytical formula for any length of tapes and for any position between them.

In addition we have also obtained analytical form for mutual inductance between long thin tapes for any position between them. Our formulae are analytically simple and can also replace the traditional tables and working ones.

Table 1 shows that we can use formula for mutual inductance of long thin tapes in case when the normalized length l/a is larger than 10.

These formulae can be used in the methods of numerical calculation of AC mutual inductance of rectangular conductors.

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