Zygmunt PIĄTEK[1], Bernard BARON[2], Tomasz SZCZEGIELNIAK[1], Dariusz KUSIAK[1], Artur PASIERBEK[2]

Czestochowa University of Technology (1), The Silesian University of Technology (2)

Mutual inductance of two thin tapes with parallel widths

Abstract. *In this paper, using a definition of a mutual inductance for two conductors of any shape and finite lengths, the new exact closed formula for mutual inductance between two thin tapes whose axes are parallel and whose widths are also parallel is proposed. In case of direct current (DC) or low frequency (LF) this inductance is given by analytical formula. The mutual inductance between two long thin tapes is also presented.*

Streszczenie. *Stosując definicję indukcyjności wzajemnej między dwoma przewodami dowolnych kształtów i skończonej długości w pracy zaproponowano nowy dokładny wzór na obliczanie indukcyjności wzajemnej miedzy dwoma równoległymi cienkimi przewodami taśmowymi. W przypadku prądu stałego lub niskiej częstotliwości indukcyjność tę wyrażono wzorem analitycznym. Podano również wzór na indukcyjność wzajemną między dwoma długimi cienkimi przewodami taśmowymi. (**Indukcyjność wzajemna równoległych cienkich przewodów taśmowych**)*

Key words: rectangular busbar, thin tape, mutual inductance, electromagnetic field
Słowa kluczowe: prostokątny przewód szynowy, przewód taśmowy, indukcyjność wzajemna, pole elektromagnetyczne

Introduction

Real system lumped conductors can be modeled by a connection of resistances, self and mutual inductances. The self and mutual inductances play an important role not only in power circuits, but also in printed circuit board (PCB) lands [1-2]. Formulae for the mutual inductances of set of conductors of rectangular cross-section are the subjects of many electrical papers and books. They are mathematically complex and their demonstrations are usually omitted and only the approximate formulae are given as though they were exact. The most significant of them are: Grover's given in [1-4], Kalantarov and Tseitlin's presented in [5], Strunsky's shown in [6], Ruehli's presented in [4] and [7] as well as Hoer and Love's shown in [2], [4] and [8]. The mutual inductance can be calculated by many numerical methods. Zhong and Koh express in [9] the mutual inductance as a weighted sum of self inductances.

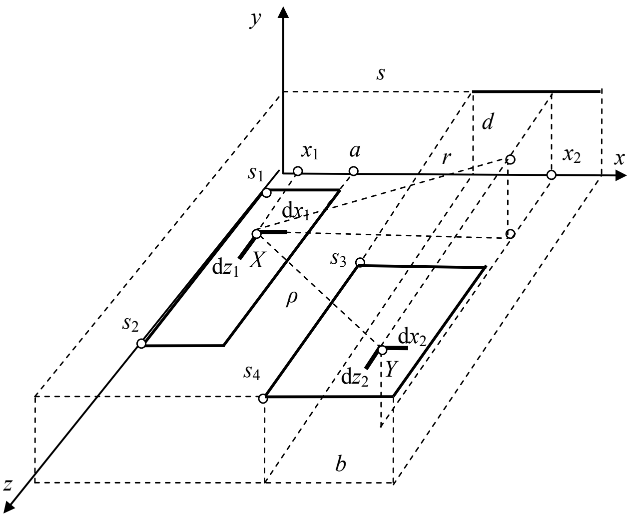

Fig. 1. Two parallel thin tapes with parallel widths

In this paper a new method for calculating mutual inductance is presented. The method results in a system of two integral Fredholm's equations. We compare our analytical formulae with several well-known ones given in the literature for DC, low frequency or parallel thin tapes.

We consider the mutual inductance between two parallel thin tapes of widths a and b, lengths $l_1 = s_2 - s_1$ and $l_2 = s_4 - s_3$ respectively and of zero thickness such as shown in Fig. 1.

Definition of mutual inductance

The definition of mutual impedance between two straight conductors is given in [10-12] by following formula

$$\underline{Z}_{12} = \frac{\mathrm{j}\,\omega\,\mu_0}{4\pi\,\underline{I}_1^*\underline{I}_2}\int\limits_{v_1}\int\limits_{v_2}\frac{\underline{J}_{22}(Y)\underline{J}_{11}^*(X)}{\rho}\,\mathrm{d}\,v\,\mathrm{d}\,v \tag{1}$$

where $\underline{J}_{22}(Y)$ is the complex current density at source point $Y = Y(x_2, y_2, z_2) \in S_2$, $\underline{J}_{11}^*(X)$ is the complex conjugate current density at point of observation $X = X(x_1, y_1, z_1) \in S_1$, v_1 and v_2 are conductors' volumes. Distance between the point of observation *X* and the source point *Y* (in general case) is $\rho = \sqrt{r^2 + (z_2 - z_1)^2}$ where $r = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. If conductors have a constant cross-sectional area S_1 and S_2 along theirs lengths, in case of DC, low frequency or for a thin strip conductors (in printed circuit board [1, 2]) we can assume that the current density is constant and given as $\underline{J}_{11}(X) = \underline{I}_1 / S_1$ and $\underline{J}_{22}(X) = \underline{I}_2 / S_2$ then, from the formulae (1), we obtain the mutual inductance between two straight parallel conductors

$$M = M_{12} = M_{21} = \frac{\mu_0}{4\pi\,S_1\,S_2}\int\limits_{v_1}\int\limits_{v_2}\frac{1}{\rho}\,\mathrm{d}\,v_1\,\mathrm{d}\,v_2 \tag{2}$$

Mutual inductance between two thin tapes

The mutual inductance between two thin tapes whose axes are parallel and whose widths are also parallel to one another such as shown in Fig. 1 from (2) is given by formula

$$M = \frac{\mu_0}{4\,\pi}\frac{1}{a\,b}F \tag{3}$$

where

$$F = \int\limits_{s}^{s+b}\int\limits_{0}^{a}\int\limits_{s_3}^{s_4}\int\limits_{s_1}^{s_2}\frac{\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}z_1\mathrm{d}z_2}{\sqrt{d^2 + (x_2 - x_1)^2 + (z_2 - z_1)^2}} \tag{4}$$

is a quadruple definite integral of four variables (x_1, x_2, z_1, z_2) into which the distance d is measured from the plan of the first tape to the plan of the second one.

If two variables, for example z_1 and z_2 can be replaced with only one variable $z = z_2 - z_1$ then a double definite integral can be calculated from following formula

$$F(x)=\int_{s_3}^{s_4}\int_{s_1}^{s_2} f(x,z_2-z_1)\,\mathrm{d}z_2\,\mathrm{d}z_1=$$

(5) $$=F(x,s_1-s_4)-F(x,s_1-s_3)+F(x,s_2-s_3)-F(x,s_2-s_4)=$$

$$=\left[F(x,z)\right]_{s_1-s_3,\,s_2-s_4}^{s_1-s_4,\,s_2-s_3}(z)=\left[F(x,z)\right]_{r_2,r_4}^{r_1,r_3}(z)=\sum_{k=1}^{k=4}(-1)^{k+1}F(x,r_k)$$

where

(5a) $$F(x,z)=\iint f(x,z)\,\mathrm{d}z\,\mathrm{d}z$$

is an indefinite integral of $f(x,z)$.

So we can put $x=x_2-x_1$ and $z=z_2-z_1$ and first calculate a quadruple indefinite integral

(6) $$F(x,z)=\iint\iint\frac{\mathrm{d}x\,\mathrm{d}x\,\mathrm{d}z\,\mathrm{d}z}{\sqrt{d^2+x^2+z^2}}$$

twice with respect to *x* and twice with respect to *z*.

Finally, after a lengthy integration, formula (6) yields an expression for quadruple indefinite integral

(7) $$F(x,z)=\frac{x^2-d^2}{2}z\ln\left(z+\sqrt{d^2+x^2+z^2}\right)+\frac{z^2-d^2}{2}x\ln\left(x+\sqrt{d^2+x^2+z^2}\right)-\frac{1}{6}\left(x^2-2d^2+z^2\right)\sqrt{d^2+x^2+z^2}-d\,x\,z\tan^{-1}\frac{x\,z}{d\sqrt{d^2+x^2+z^2}}$$

Hence the mutual inductance between two parallel thin tapes is given by following formula

(8) $$M=\frac{\mu_0}{4\pi}\frac{1}{a\,b}\left[\left[F(x,z)\right]_{p_2,p_4}^{p_1,p_3}(x)\right]_{r_2,r_4}^{r_1,r_3}(z)=\sum_{i=1}^{i=4}\sum_{k=1}^{k=4}(-1)^{i+k}F(p_i,r_k)$$

where $p_1=s-a$, $p_2=s+b-a$, $p_3=s+b$, $p_4=s$, $r_1=s_1-s_4$, $r_2=s_1-s_3$, $r_3=s_2-s_3$ and $r_4=s_2-s_4$.

It is exactly the Hoer's formula given in [8]. For the same two tapes of width *a*, distance *d* between them and without displacements along *x* axis ($s=0$) and along *z* axis ($s_1=s_3=0$) the mutual inductance is given by following formula

(9) $$M=\frac{\mu_0}{4\pi}\frac{1}{a^2}F$$

where

(10) $$F=\frac{4}{3}d^3+\frac{2}{3}\left(a^2-2d^2\right)\sqrt{a^2+d^2}+\frac{2}{3}\left(l^2-2d^2\right)\sqrt{l^2+d^2}-\frac{2}{3}\left(l^2-2d^2+a^2\right)\sqrt{a^2+l^2+d^2}-4adl\tan^{-1}\frac{a\,l}{d\sqrt{a^2+l^2+d^2}}+ad^2\ln\frac{\left(a+\sqrt{a^2+d^2}\right)\left(-a+\sqrt{a^2+l^2+d^2}\right)}{\left(-a+\sqrt{a^2+d^2}\right)\left(a+\sqrt{a^2+l^2+d^2}\right)}+ld^2\ln\frac{\left(l+\sqrt{a^2+d^2}\right)\left(-l+\sqrt{a^2+l^2+d^2}\right)}{\left(-l+\sqrt{a^2+d^2}\right)\left(l+\sqrt{a^2+l^2+d^2}\right)}+al^2\ln\frac{\left(a+\sqrt{a^2+l^2+d^2}\right)}{\left(-a+\sqrt{a^2+l^2+d^2}\right)}+a^2l\ln\frac{\left(l+\sqrt{a^2+l^2+d^2}\right)}{\left(-l+\sqrt{a^2+l^2+d^2}\right)}$$

Mutual inductance between two long thin tapes

The double definite integral

(11) $$f(x,y)=\int_0^l\int_0^l\frac{1}{\rho_{XY}}\,\mathrm{d}z_1\mathrm{d}z_2=2\,l\left(\ln\frac{l+\sqrt{l^2+r_{XY}^2}}{r_{XY}}-\frac{\sqrt{l^2+r_{XY}^2}}{l}+\frac{r_{XY}}{l}\right)$$

If $l>>r_{XY}$ the function $f(x,y)$ becomes

(12) $$f(x,y)=2\,l\left(\ln\frac{2\,l}{r_{XY}}-1\right)$$

and the mutual inductance between two long thin tapes expresses by formula

(13) $$M=\frac{\mu_0\,l}{2\pi}\left[\ln(2\,l)-1+G\right]$$

where

(14) $$G=-\frac{1}{2\,a\,b}\int_s^{s+b}\int_0^a\ln\left[d^2+(x_2-x_1)^2\right]\mathrm{d}x_1\mathrm{d}x_2$$

is a double definite integral of two variables. Now we substitute $x=x_2-x_1$ and calculate double indefinite integral

(15) $$G(x)=-\frac{1}{2\,a\,b}\iint\ln\left[d^2+x^2\right]\mathrm{d}x\,\mathrm{d}x=\frac{1}{4\,a\,b}\left[3\,x^2-4\,x\,d\tan^{-1}\frac{x}{d}-\left(x^2-d^2\right)\ln\left(x^2+d^2\right)\right]$$

Hence the mutual inductance between two long thin tapes of widths a and b, the same length l, zero thickness and distance *d* between them is

(16) $$M=\frac{\mu_0\,l}{2\pi}\left\{\ln(2\,l)-1+\left[G(x)\right]_{s+b-a,\,s}^{s-a,\,s+b}(x)\right\}=\frac{\mu_0\,l}{2\pi}\left\{\ln(2\,l)-1+\left[G(x)\right]_{p_2,\,p_4}^{p_1,\,p_3}(x)\right\}=\frac{\mu_0\,l}{2\pi}\left\{\ln(2\,l)-1+\sum_{i=1}^{i=4}(-1)^{i+1}G(p_i)\right\}$$

For the same two tapes of width *a*, distance *d* between them and without displacements along *x* axis ($s = 0$) and along *z* axis ($s_1 = s_3 = 0$) the mutual inductance is given by following formula

(17) $$M = \frac{\mu_0\, l}{2\,\pi}\left\{\ln(2\,l) - 1 + \left[G(x)\right]_{0,0}^{-a,a}(x)\right\} = \frac{\mu_0\, l}{2\,\pi}\left\{\ln(2\,l) - 1 + \sum_{i=1}^{i=4}(-1)^{i+1} G(p_i)\right\}$$

On the basis of (17) we have the analytical formula for the mutual inductance between two long parallel straight thin tapes

(18) $$M = \frac{\mu_0\, l}{2\,\pi}\left[\ln\frac{2\,l}{d} + \frac{1}{2} - 2\frac{d}{a}\tan^{-1}\frac{a}{d} - \frac{1}{2}\left(1 - \frac{d^2}{a^2}\right)\ln\left(1 + \frac{a^2}{d^2}\right)\right]$$

Computational results

In this section, we present the evaluation results for all the above mentioned mutual inductance formulae in the range of VLSI applications. For the mutual inductance of two identical real thin tapes of width *a*, thickness *t* and length *l* above formulae give results shown in Table 1.

Table 1. Mutual inductance between two real thin tapes for DC or low frequency

Thin tapes: $a = 0.5\ \mu$m; $t = 0.1\ \mu$m; $d = 2\,a$						
	Ruehli	Grover	Strunsky	Hoer	Eq. (9)	Eq. (18)
l (m)	*L* (pH)	*L* (pH)	*L* (pH)	*L* (pH)	*L* (pH)	*L* (pH)
0.01 *a*	0.000002	0.000002	0.000002	0.000002	0.000002	negative
0.10 *a*	0.000249	0.000249	0.000249	0.000245	0.000245	negative
1.00 *a*	0.024514	0.024514	0.024494	0.024088	0.024054	negative
10.0 *a*	1.492634	1.492634	1.491952	1.477150	1.476415	1.282709
100 *a*	36.25070	36.25070	36.24254	36.06343	36.05597	35.85295
1000 *a*	590.9754	590.9754	590.8924	589.0107	588.9919	588.7880

Conclusions

The mutual inductance between two thin tapes of zero thickness whose widths are parallel is given by distance from the plane of the first tape to the plane of the second one and by displacements along *x* axis and along *z* axis. This analytical formula can be used for any length of tapes and for any position between them.

In addition we have also obtained analytical form for mutual inductance between long thin tapes for any position between them. Our formulae are analytically simple and can also replace the traditional tables and working ones.

Table 1 shows that we can use formula for mutual inductance of long thin tapes in case when the normalized length l/a is larger than 10.

These formulae can be used in the methods of numerical calculation of AC mutual inductance of rectangular conductors.

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Authors: *prof. dr hab. inż. Zygmunt Piątek, Politechnika Częstochowska, Wydział Inżynierii i Ochrony Środowiska, Instytut Inżynierii Środowiska, e-mail: zygmunt.piatek@interia.pl*

prof. dr hab. inż. Bernard Baron, Politechnika Śląska, Wydział Elektryczny, Instytut Elektrotechniki i Informatyki, e-mail: bernard.baron@polsl.pl

dr inż. Tomasz Szczegielniak, Politechnika Częstochowska, Wydział Inżynierii i Ochrony Środowiska, Instytut Inżynierii Środowiska, e-mail: szczegielniak@interia.pl

dr inż. Dariusz Kusiak, Politechnika Częstochowska, Wydział Elektryczny, Instytut Elektrotechniki Przemysłowej, e-mail: dariuszkusiak@wp.pl

dr inż. Artur Pasierbek, Politechnika Śląska, Wydział Elektryczny, Inst. Elektrotechniki i Informatyki, e-mail: artur.pasierbek@polsl.pl