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Diffraction of Longitudinal Shear Waves on a Thin Piezoelectric Inclusion of Low Rigidity

Abstract. *A thin plane piezoelectric inclusion is located in a uniform background (in two-dimensional Euclidean space) and subjected to an incident plane longitudinal shear wave. Using the representation theorem, the problem is reduced to solving of the singular integral equation. Numerical solution and high frequency asymptotic for far field are presented.*

Streszczenie. *Cienkie płaskie piezoelektryczne włączenie jest umieszczone w jednorodnym ośrodku (w dwuwymiarowej przestrzeni Euklidesowej) i jest poddane działaniu płaskiej fali ścinania. Odpowiednie zagadnienie brzegowe jest sprowadzone do rozwiązywania osobliwego równania całkowego. Przedstawiono numeryczne rozwiązanie zagadnienia w strefie falowej włączenia oraz zaprezentowano odpowiednie asymptotyczne rozwiązanie dla zakresu wysokich częstotliwości.(**Dyfrakcja fal ścinania na cienkim piezoelektrycznym włączeniu o małej sztywności**)*

Keywords: piezoelectric, diffraction
Słowa kluczowe: Piezoelektryki, dyfrakcja

Introduction

Today, the electromechanical systems with piezoelectric patches or layers find increasingly wide application. For example, such systems are used in ultrasonic diagnostics in medicine, in ultrasonic non-destructive testing of materials and constructions, in ultrasonic welding, cleaning of surfaces, sputtering, drilling, etc. [1]. In the design of smart materials and corresponding electromechanical systems, an important part is the development of mathematical models adequately describing the wave processes in these objects [2-5]. In these cases the thickness of the piezoelectric layer is usually thin (in comparison to the relevant wavelengths) as is the structure to which it is embedded. In this paper, the problem of diffraction of longitudinal shear waves on a thin piezoelectric inclusion of low rigidity is considered. Using a known model of the interaction of inclusion with an elastic medium [6] the corresponding boundary-value problem is reduced to the singular integral equation. This equation is solved numerically by reducing it to an infinite algebraic system, which is regularized using the information on the stress singularity at the patch edges. A comparison of the numerical results with those obtained by the analytical method is done.

Problem formulation

Let us consider an elastic uniform medium characterized by the shear modulus μ and the mass density ρ, in which there is a piezoelectric inclusion in conditions of the ideal mechanical contact. The longitudinal shear of the elastic system is assumed. The inhomogenity occupies the region $S = \{|x_1| < a, |x_3| < h/2\}$, $|x_2| < \infty$, where h is the thickness, (x_1, x_2, x_3) are the Cartesian coordinates and the quantity $\varepsilon = h/a$ is the small parameter. The material of the heterogeneity with an elastic constant c_{44} falls into the crystallographic class $6mm$, and the axis of symmetry of the sixth order is perpendicular to the plane x_1x_3.

A plane, incident longitudial shear wave of the form

(1)
$$u^i(x) = exp\left[ik(l, x)\right], \ x = (x_1, x_3), \ k = \omega/c, \ c^2 = \mu/\rho$$

impinges on the inclusion (the time factor of the form $e^{-i\omega t}$ is omitted throughout the analysis). Here $\mathbf{l} = (\sin\theta_0, -\cos\theta_0)$ is the direction of sounding, k is the wave number and typical wavelength kh satisfies the condition $kh \ll 1$.

The total wave $u = u^i + u^s$ is decomposed into the given incident wave u^i and the unknown scattered wave u^s which is required to satisfy the Sommerfeld radiation condition at infinity, from which it follows that

(2)
$$u^s(\mathbf{x}) = \frac{\mathrm{e}^{\mathrm{i}k\mathbf{x}+\mathrm{i}\pi/4}}{\sqrt{8\pi \mathrm{k}|\mathbf{x}|}}\mathrm{f}(\omega; \mathbf{l}, \boldsymbol{\nu}), \ |\mathbf{x}| \to \infty,$$

where $f(\omega, \mathbf{l}, \boldsymbol{\nu})$ is the complex amplitude or far-field pattern of the scattering wave, $\boldsymbol{\nu} = x/|x| = (\sin\theta, \cos\theta)$ is the direction of observation, ω is the circular frequency. Herein $u(x)$ is the displacement in the x_2-direction and for the only non vanishing stress components the notation $\sigma_{\beta 2} = \mu\frac{\partial u}{\partial x_\beta}$ with $\beta = 1, 3$ is introduced. Let $u^0(x), \ x \in S$ and $k_0 = \omega/c_0$ denote the displacement field and wave number in the inclusion, respectively, and $\phi(\mathbf{x})$ is the electric potential for the heterogeneity.

The scattering problem of time harmonic waves is described by the wave equations

(3)
$$(\Delta + k^2)u(\mathbf{x}) = 0, \ \mathbf{x} \in R^2/S,$$

$$(\Delta + k_0^2)u^0(\mathbf{x}) = 0, \ \mathbf{x} \in S,$$

$$\Delta u^0(\mathbf{x}) - \Delta\phi_p(\mathbf{x}) = 0, \ \phi_p(\mathbf{x}) = \frac{\varepsilon_{11}}{e_{15}}\phi(\mathbf{x}), \ \mathbf{x} \in S,$$

and following conditions along the boundary ∂S of the inclusion:

(4)
$$u(\mathbf{x}) = u^0(\mathbf{x}), \ \frac{\partial u(\mathbf{x})}{\partial n} = \gamma_e\frac{\partial}{\partial n}[u^0(\mathbf{x}) + \eta^2\phi_p(\mathbf{x})],$$

$$\phi(\mathbf{x}) = 0, \ \mathbf{x} \in \partial S,$$

where n denotes the outer normal direction, e_{15} and ε_{11} are the piezoelectric constant and the permittivity of the material of the heterogeneity, respectively, η is the electromechanical coupling coefficient and ρ_0 is the mass density for the inclusion,

$$c_0^2 = \frac{c_{44}}{\rho_0}(1 + \eta^2), \ \gamma_e = \frac{c_{44}}{\mu}.$$

Our objective is to find the numerical and asymptotic representations of the solution to the problem (1)-(4) for the small but non vanishing ε. As follow from results obtained previously in [6], the field $u(x)$ satisfies in the domain R^2/S the Helmholtz equation and the following effective boundary conditions on the interval $|x_1| < a$

(5)
$$\Phi(x_1) = u(x_1, +0) - u(x_1, -0) = 2k^{-1}Z\frac{\partial u(x_1, 0)}{\partial x_3},$$

$$Z=\frac{kh}{2\gamma},\ \gamma=\gamma_e(1+\eta^2),\ \eta=\frac{e_{15}}{\sqrt{c_{44}\varepsilon_{11}}},$$

$$\frac{\partial u(x_1,+0)}{\partial x_3}=\frac{\partial u(-x_1,0)}{\partial x_3},\ |x_1|<a,\ x_3=0,$$

where $\Phi(x_1)$ is the displacements jump across the inclusion.

Solution

Using the Green theorem, from Eqs. (3), (5), we take the singular integral equation for determining the jump displacement $\Phi\left(x_1\right)$:

(6) $$\Phi\left(x_1\right)+Z\,k\int\limits_{-a}^{a}\Phi\left(p\right)K\left(k\left|x_1-p\right|\right)dp=$$

$$=q\exp\left(ikx_1\sin\theta_0\right),\ |x_1|<a,$$

$$K\left(|z|\right)=\frac{1}{2\pi}\int\limits_{\Gamma}\gamma\left(\alpha\right)\exp\left(i\alpha z\right)d\alpha,$$

$$\gamma\left(\alpha\right)=\sqrt{\alpha^2-1},\ q=-2A_0 i\,Z\cos\theta_0,$$

where the contour Γ coincides with the real axis everywhere except for the branching points $\alpha=\pm1$. The contour Γ passes these points below in the right-hand semi-plane of complex variable α, and above, in the left-hand one, according to the limiting absorption principle. In addition, the point $\alpha=0$ is situated below the contour Γ, and for $|\alpha|<1$ the radical $\gamma\left(\alpha\right)$ is defined by the condition $\mathrm{Im}\gamma<0$.

The principal term of the asymptotic representation for $x>>1$ ($x=ka$) of the solution to Eq. (6) is sought in the form [7]

(7) $$\Phi\left(x_1\right)=\Phi^{+}\left(x_1\right)+\Phi^{-}\left(x_1\right)-v\left(x_1\right),\ |x_1|<a,$$

where the functions $v\left(\eta\right)=\Phi\left(\eta k^{-1}\right)$ and

$$\Phi^{\pm}\left(\eta\right)=\Phi\left(\mp a\pm\eta k^{-1}\right)\exp\left(\pm ixl_1\right)$$

satisfy the corresponding convolution-type equation:

(8) $$v\left(\eta\right)+Z\int\limits_{-\infty}^{\infty}v\left(\zeta\right)K\left(\left|\eta-\zeta\right|\right)d\zeta=$$

$$=q\left(\theta_0\right)\exp\left(i\eta l_1\right),\ -\infty<\eta<\infty,$$

(9) $$\Phi^{\pm}\left(\eta\right)+Z\int\limits_{0}^{\infty}\Phi^{\pm}\left(\zeta\right)K\left(\left|\eta-\zeta\right|\right)d\zeta=$$

$$=q\left(\theta_0\right)\exp\left(i\eta l_1\right),\ 0<\eta<\infty.$$

It is noted that the expression given by the Equation (7) is motivated by the general scheme of the method of the the compound asymptotic expansions where a stretching factor is the wave number.

The equation (8) is specified over the entire axis. Then, applying the exponential Fourier transformation to the equation (8), we have

(10)
$$v\left(\eta\right)=q\left(\theta_0\right)R^{-1}\left(l_1\right)\exp\left(i\eta l_1\right),\ R\left(\alpha\right)=1+Z\,\gamma\left(\alpha\right),$$

where the function $R\left(\alpha\right)$ is a regular function in the plane of the complex variable, with the cuts $\mathrm{Re}\gamma=0$, and has no zeroes in this plane.

Applying the Wiener-Hopf technique [8] to solving the equation (9), we obtain

(11) $$\Phi^{\pm}\left(\eta\right)=\frac{q\left(l_1\right)}{R\left(l_1\right)}\exp\left(\pm i\eta l_1\right)-$$

$$-2A_0\frac{\left(1\mp l_1\right)^{1/2}}{\pi r^{-}\left(\mp l_1\right)}g^{\pm}\left(\eta;l_1\right)+\Phi^{0}\left(\eta\right),$$

$$\Phi^{0}\left(\eta\right)=-A_*\frac{2^{1/2}}{\pi}r^{+}\left(1\right)\ g^{-}\left(\eta;1\right),$$

$$g^{\pm}\left(\eta;l_1\right)\approx\frac{r^{+}\left(1\right)}{1\mp l_1}\left(\frac{\pi}{\eta}\right)^{1/2}e^{i\eta}\,e^{i\pi/4}\,g_0\left(\eta\right),$$

$$g_0\left(\eta\right)=1-Z^{-1}\left(\pi\eta/2\right)^{1/2}\,e^{i\pi/4}\,\exp\left[i\eta/\left(2Z^2\right)\right]\times$$

$$\times\mathrm{erfc}\left[Z^{-1}\left(\eta/2\right)^{1/2}\,e^{i\pi/4}\right],$$

$$\mathrm{erfc}\left(\eta\right)=2\pi^{-1/2}\int\limits_{\eta}^{\infty}\exp\left(-t^2\right)dt,$$

$$r(\alpha)=1+\left[Z\gamma(\alpha)\right]^{-1}=r^{+}(\alpha)r^{-}(\alpha),$$

$$r^{\pm}(\alpha)=\exp\left[\chi^{\pm}(\alpha)\right],\ r^{+}(\alpha)=r^{-}(-\alpha),$$

$$\chi^{\pm}(\alpha)=\frac{1}{\pi}\int_0^{\infty}\arctan\left[Z^{-1}(\tau^2+1)^{-1/2}\right]\frac{idu}{i\tau\pm u}-$$

$$-\frac{1}{\pi}\int_0^{1}\arctan\left[Z^{-1}(1-\sigma^2)^{-1/2}\right]\frac{d\sigma}{\sigma\pm u},$$

where the function $\Phi^0(\eta)=A_*\exp^{-i\eta},\ \eta<0$ corresponds to the incident rays, whose fields are the product of subsequent diffractions at two inclusion tips.

From the equation (11), we observe that the primary diffracted fields decay as $\eta^{-1/2}$. On the basis of (7), (10) and (11), including only one secondary diffraction, we can then write approximately

(12) $$\Phi\left(x_1\right)\approx\frac{q\left(l_1\right)}{R\left(l_1\right)}e^{ikl_1x_1}-$$

$$-2A_0\sum_{\pm}D^{\pm}\frac{e^{ik(a\pm x_1)}}{\left(a\pm x_1\right)^{1/2}}g_0\left[k\left(a\pm x_1\right)\right],$$

$$D^{\pm}=\frac{r^{+}\left(1\right)e^{i\pi/4}\,e^{\mp ixl_1}}{r^{-}\left(\mp l_1\right)\left(1\mp l_1\right)^{1/2}}\left(\pi k\right)^{-1/2},$$

$$k\left(a\pm x_1\right)>>1,\ |l_1|<1,$$

where $D^{\pm}$ is the diffraction coefficient at the left and the right inclusion tip, respectively. Consequently, the expression

(13) $$f(\omega,\mathbf{l},\boldsymbol{\nu})=-\mathrm{ik}\nu_3\int_{-\mathrm{a}}^{\mathrm{a}}\Phi(\mathrm{p})\mathrm{e}^{-\mathrm{ik}\nu_1\mathrm{p}}\mathrm{dp}$$

and the approximation (12), give the solution of the problem in the wave zone of a scatterer.

Numerical formulation

With the purpose of the numerical solution of the integral equation (6), we use the Galerkin method. In view of the edge condition, we represent $\Phi\left(x_1\right)$ in the complete sequence of the Chebyshev polynomials

$$(14)\quad \Phi\left(x_1\right)=q\left[1-\left(x_1/a\right)^2\right]^{1/2}\sum_{m=1}^{\infty} a_m U_{m-1}\left(x_1/a\right),$$

where $U_{m-1}\left(x_1\right)$ are the Chebyshev polynomials of the second kind and a_m are the unknown expansion coefficients. Substituting the equation (14) into the equation (6), we obtain an infinite system of linear algebraic equations for a_m

$$(15)\qquad \sum_{m=1}^{\infty} m\,n\,a_m A_{mn}=b_n,\ n=1,2,...,$$

$$A_{mn}=\left[1+(-1)^{m+n}\right]\times$$

$$\times\left\{\frac{2}{\pi}\left[1-(m-n)^2\right]^{-1}\left[(m+n)^2-1\right]^{-1}-\right.$$

$$\left.-\frac{Z}{2x}\exp\left(\frac{m-n}{2}i\pi\right)I_{nm}\right\},$$

$$I_{nm}=\frac{1}{2}\int\limits_{\Gamma}\gamma\left(\alpha\right)J_n\left(\alpha x\right)J_m\left(\alpha x\right)\alpha^{-2}d\alpha,$$

$$b_n=n\exp\left(\frac{n-1}{2}i\pi\right)J_n\left(x\sin\theta_0\right)\left(x\sin\theta_0\right)^{-1},$$

where $J_n\left(x\right)$ is the Bessel function of the first kind.

The scattering amplitude (13) is connected with the coefficients a_m as follows:

$$(16)\qquad f(\omega;\mathbf{l},\boldsymbol{\nu})=\pi\mathrm{qx}\sum_{\mathrm{m}=1}^{\infty}\mathrm{a_m m}(-1)^{\mathrm{m}}\frac{\mathrm{J_m(x}\nu_1)}{\mathrm{x}\nu_1}\,.$$

It is obvious that in numerical calculations, m and n in the equations (15), (16) are limited and the quantities A_{nm} and, consequently, a_m can be calculated with a sufficient accuracy by appropriate numerical procedures. Indeed, an accuracy of one percent is obtained if $m=n\approx 2x$.

The total scattering cross-section $\sigma(\theta_0)$ of the inclusion is defined as the outward power-flow of the scattered field normalized at the intensity of the incident field. When the incident wave is the plane wave (1), $\sigma(\theta_0)$ is directly related to the equation

$$\sigma(\theta_0)=\frac{1}{A_0 k}\mathrm{Im}f(\omega;\mathbf{l},\mathbf{l}).$$

Conclusions

On the Fig.1 the dimensionless cross-section $\tau(0)$ is plotted as a function of the dimensionless wave size x for the incident angles $\theta_0=0^0$ and $\gamma=0$, $\gamma=0.1$, $\gamma=0.2$, where $\tau(\theta_0)=\sigma(\theta_0)/2a$. Here and hereafter the calculations are carried out with $\varepsilon=0.04$ for the inclusion. Circles on this figure denote the results obtained using the formulae (12)-(13). It is noted that the agreement between the exact and asymptotic values is excellent. As it can be seen from the Fig.1, the decrease of the inclusion material stiffness yields the scattering level increase. Fig.2 depicts the normalized amplitude far-field patterns $f_0=|f(\omega;\mathbf{l},\boldsymbol{\nu})|/4\mathrm{A}_0$ for $\theta_0=0^0$ and

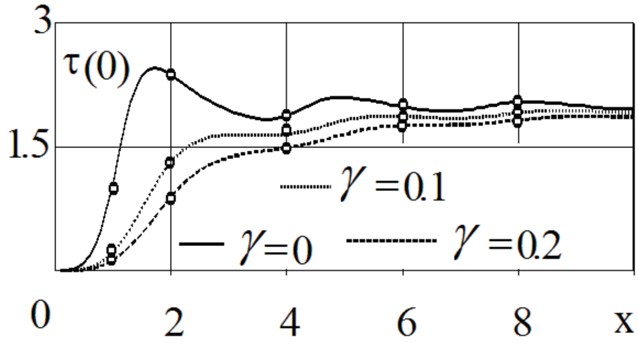

Fig. 1. The dimensionless cross-section $\tau(0)$ for $\theta_0=0^0$ and $\gamma=0$, $\gamma=0.1$, $\gamma=0.2$.

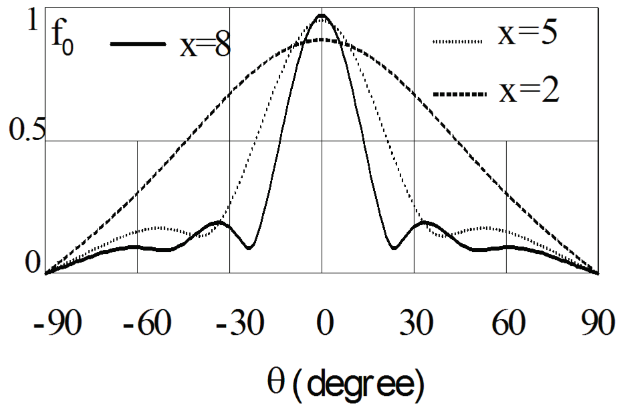

Fig. 2. The normalized amplitude far-field patterns $f_0=|f(\omega;\mathbf{l},\boldsymbol{\nu})|/4\mathrm{A}_0$ for $\theta_0=0^0$ and $\gamma=0.2$.

$\gamma=0.2$. As it follows from the numerical results presented, the influence of the dimensionless wave number on the angular distributions of the displacements in wave zone is significant.

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