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# A non-interior-point algorithm predictor-corrector algorithm for Variational Inequality Problem with equality linear constraints

***Abstract**. We present a noninterior-point predictor-corrector algorithm for variational inequality Problem with equality linear constraints based on Chen-Kanzow-Smale smoothing techniques. This method is based upon a modified predictor-corrector interior-point algorithm. It is established the global linear convergence.*

***Streszczenie.** W artykule przedstawiono metodę punktów nie-wewnętrznych predykator-korektor, uwzględniający nierówności wariancyjne z liniowym ograniczeniem liniowości. Jego działanie oparto na technice Chen-Kanzow-Smale oraz zmodyfikowanej metodzie punktów wewnętrznych predykator-korektor. Analizie poddano liniową zbieżność algorytmu. (**Metoda punktów nie-wewnętrznych predykator-korektor w rozwiązywaniu nierówności wariancyjnych z liniowym ograniczeniem równościowym**).*

**Keywords:** Variational inequality problem; Predictor-corrector; Non-interior-point algorithm; Equality linear constraints.
**Słowa kluczowe:** nierówność wariancyjna, predyktor-korektor, metoda punktu nie wewnętrznego, liniowe ograniczenie równościowe.

## Introduction

The finite-dimensional variational inequality (VI) provides a broad unifying setting for the study of optimization and equilibrium problems and serves as the main computational framework for the practical solution of a host of continuum problems in the mathematical sciences. Extensive studies on theoretical aspects and computational of VI have been done. For a detailed review, we refer to the monographs[1-4].

In this paper, we consider the following variational inequalities problem(VIP):

Problem1.1 Find $x \in K$ such that

(1) $$(t-x)^T f(x) \ge 0 \quad \forall t \in K$$

Where $K = \left\{ Ax=b, x \in R^n \right\}$, $A \in R^{m \times n}$ and $R(A) = m$, $f(x): R^n \to R^n$ is a vector-valued functions and is $\xi$ -monotone on $R^n$.

The KKT system of the problem1.1 can be cased as NCPs as follows

Problem1.2

(2) $$\begin{cases} Ax=b \\ f(x)-A^T y-z=0 \\ x^T z=0 \end{cases}$$

Note $S = \left\{ (x,y,z) \middle| Ax=b, f(x)-A^T y-z=0, x\ge 0, z\ge 0 \right\}$, we assume the set $S^+ = \left\{ (x,y,z) \middle| (x,y,z)\in S, x>0, z>0 \right\}$ is nonempty. The path to be followed is the central path

$$C = \left\{ (x,y,z): Ax=b, f(x)-A^T y-z=0, Xz=\mu^2 e, x>0, z>0, \mu>0 \right\}$$

We denote by $e \in R^n$ the vector each of whose components is 1 and by $X$ the diagonal matrix whose diagonal entries are given by the vector $x \in R^n$ .The algorithm is based on Chen--Harker---Kanzow---Smale smoothing techniques [5-7] and as such relies on the function

(3) $$\phi(a,b,\mu) = a + b - \sqrt{(a-b)^2 + 4\mu^2}$$

This function is a member of the Chen—Mangasarian class of smoothing functions for the problem LCP( $q, M$ ). It is easily verified that for $\mu \ge 0$ :

(4) $$\phi(a,b,\mu) = 0 \text{ if and only if } a \ge 0, b \ge 0, ab = \mu^2$$

We develop a predictor-corrector path following algorithm for solving Problem1.2, based on the idea which is first non-interior predictor—corrector strategy proposed for following the central path. The central idea is to apply Newton's method to equations of the form $F(x,y,z,\mu) = v$ for appropriate values for $v$ where the function $F: R^n \times R^m \times R^n \times R_{++} \to R^m \times R^n \times R^n \times R_{++}$ is given by:

(5) $$F(x,y,z,\mu) = \begin{bmatrix} Ax-b \\ f(x)-A^T y-z \\ \phi(x,z,\mu) \\ \mu \end{bmatrix}$$

with

(6) $$\phi(x,z,\mu) := \begin{bmatrix} \phi(x_1, z_1, \mu) \\ \vdots \\ \phi(x_n, z_n, \mu) \end{bmatrix}$$

Note that $F(x,y,z,\mu) = 0$ if and only if $x$ solves Problem1.2. We take the set:

(7) $$N(\beta) = \left\{ (x,y,z) \middle| \begin{matrix} Ax-b=0, f(x)-A^T y-z=0 \\ \phi(x,z,\mu)\le 0, \|\phi(x,z,\mu)\| \le \beta\mu \text{ for some } \mu>0 \end{matrix} \right\}$$

as our neighborhood for the central path, where $\beta > 0$ is given. For $\mu > 0$ , this neighborhood can be viewed as the union of the slices

(8) $$N(\beta,\mu) = \left\{ (x,y,z) \middle| \begin{matrix} Ax-b=0, f(x)-A^T y-z=0 \\ \phi(x,z,\mu)\le 0, \|\phi(x,z,\mu)\| \le \beta\mu \end{matrix} \right\}$$

**A Predictor-Corrector Algorithm**

In this section, we state our predictor—corrector algorithm and show that it is well-defined. The algorithm:

Step0 (Initialization) Choose $\left(x^0,y^0,z^0\right)\in R^n\times R^m\times R^n$, set $k=0$, $Ax^0=b,\ \nabla f\left(x^0\right)^T x^0-A^T y^0-z^0=0$ and let $\mu_0>0$ be such that $\phi\left(x^0,y^0,z^0,\mu_0\right)<0$ .choose $\beta$ so that $\left\|\phi\left(x^0,y^0,z^0,\mu_0\right)\right\|\le\beta\mu_0$ .we now have $\left(x^0,y^0,z^0\right)\in N\left(\beta,\mu_0\right)$ .choose $\gamma,\alpha_1$ and $\alpha_2$ from $(0,1)$.

Step1(the predictor step) Let $\left(\Delta x^k,\Delta y^k,\Delta z^k\right)$ solve the equation:

(9)
$$F\left(x^k,y^k,z^k,\mu_k\right)+\nabla F\left(x^k,y^k,z^k,\mu_k\right)^T\begin{bmatrix}\Delta x^k\\ \Delta y^k\\ \Delta z^k\\ \Delta\mu_k\end{bmatrix}=0$$

If $\left\|\phi\left(x^k+\Delta x^k,z^k+\Delta z^k,0\right)\right\|=0$，then stop and $\left(x^k+\Delta x^k,y^k+\Delta y^k,z^k+\Delta z^k\right)$ solves Problem1.1.

else let $\tau_k$ be the first element in the sequence $\left\{1,\alpha_1,\alpha_1^2,\cdots\right\}$ such that

(10) $$\left\|\phi\left(x^k+\tau_k\Delta x^k,z^k+\tau_k\Delta z^k,\tau_k\mu_k\right)\right\|\le\tau_k\beta\mu_k$$

and set $\hat{x}:=x^k+\tau_k\Delta x^k,\ \hat{y}^k=y^k+\tau_k\Delta y^k$,

(11) $$\hat{z}^k=z^k+\tau_k\Delta z^k,\ \hat{\mu}_k=\tau_k\mu_k$$

Step2 (the corrector step) Let $\left(\Delta\hat{x}^k,\Delta\hat{y}^k,\Delta\hat{z}^k,\Delta\hat{\mu}_k\right)$ solve the equation

(12)
$$F\left(\hat{x}^k,\hat{y}^k,\hat{z}^k,\hat{\mu}_k\right)+\nabla F\left(\hat{x}^k,\hat{y}^k,\hat{z}^k,\hat{\mu}_k\right)^T\begin{bmatrix}\Delta\hat{x}^k\\ \Delta\hat{y}^k\\ \Delta\hat{z}^k\\ \Delta\hat{\mu}_k\end{bmatrix}=\begin{bmatrix}0\\0\\0\\(1-\gamma)\hat{\mu}_k\end{bmatrix}$$

and let $\lambda_k$ be the maximum of the values $1,\alpha_2,\alpha_2^2,\cdots$ such that

(13) $$\left\|\phi\left(\hat{x}^k+\lambda_k\Delta x^k,\hat{z}^k+\lambda_k\Delta z^k,\left(1-\gamma\lambda_k\right)\hat{\mu}_k\right)\right\|\le\left(1-\gamma\lambda_k\right)\beta\hat{\mu}_k$$

Set $x^{k+1}=\hat{x}^k+\lambda_k\Delta x^k,\ y^{k+1}=\hat{y}^k+\lambda_k\Delta y^k$,

$$z^{k+1}=\hat{z}^k+\lambda_k\Delta z^k,\ \mu_{k+1}=\left(1-\gamma\lambda_k\right)\hat{\mu}_k \quad (14)$$

and set k=k+1 return to step 1.

**Theorem1** nonlinear system of equations（9）and（12）has a unique solution，i.e. the matrix

$$\nabla F\left(x,y,z,\mu\right)^T=\begin{bmatrix}A & O & O & O\\ \nabla f(x)^T & -A^T & -I & O\\ \nabla_x\phi(x,z,\mu) & O & \nabla_z\phi(x,z,\mu) & \nabla_\mu\phi(x,z,\mu)\\ O & O & O & I\end{bmatrix}$$

is nonsingular, where

$$\nabla_x\phi\left(x,z,\mu\right)=diag\left(\left.\frac{\partial\phi(a,b,\mu)}{\partial a}\right|_{\substack{a=x\\ b=z\\ \mu=\mu}}\right)=diag\left(1-\frac{x_i-z_i}{\sqrt{\left(x_i-z_i\right)^2+4\mu^2}}\right),$$

$$\nabla_z\phi\left(x,z,\mu\right)=diag\left(\left.\frac{\partial\phi(a,b,\mu)}{\partial b}\right|_{\substack{a=x\\ b=z\\ \mu=\mu}}\right),\quad \nabla_\mu\phi\left(x,z,\mu\right)=diag\left(\left.\frac{\partial\phi(a,b,\mu)}{\partial \mu}\right|_{\substack{a=x\\ b=z\\ \mu=\mu}}\right).$$

**Convergence Analysis**

We are now ready to establish the global linear convergence of the algorithm.

Theorem 2（global linear convergence）Given $\beta>0$ and $\mu_0>0$ We assume that $\left\|\nabla F_{(x,y,z)}\left(\hat{x},\hat{y},\hat{z},\mu\right)^{-1}\right\|\le C$ for all $0<\mu\le\mu_0$ and $\left(\hat{x}^k,\hat{y}^k,\hat{z}^k\right)\in N\left(\beta,\mu_k\right)$ .Let $\left(x^k,y^k,z^k,\mu_k\right)$ be the sequence generated by the algorithm. If the algorithm does not terminate finitely at the unique solution to Problem1.2, then for $k=0,1,\cdots$,

(15) $$\left(x^k,y^k,z^k\right)\in N\left(\beta,\mu_k\right)$$

(16) $$\left(1-\gamma\hat{\lambda}_{k-1}\right)\tau_{k-1}\cdots\left(1-\gamma\hat{\lambda}_0\right)\tau_0\mu_0=\mu_k$$

with

(17) $$\lambda_k\ge\bar{\lambda}=\min\left\{1,\frac{(1-\gamma)\beta}{C^2\left(\beta+2\sqrt{n}\gamma\right)^2+\sqrt{n}\gamma^2+\gamma(1-\gamma)\beta}\right\}$$

where $C$ is the constant. Therefore $\mu_k$ converges to 0 at a global linear rate. In addition, the sequence $\left(x^k,y^k,z^k\right)$ converges to the unique solution of Problem1.2.

Proof. Since $A\left(x^k+\Delta x^k\right)=Ax^k+A\Delta x=b$

$$\nabla f\left(x^k\right)^T\left(x+\Delta x^k\right)-A^T\left(y^k+\Delta y^k\right)-\left(z^k+\Delta z^k\right)$$
$$=\nabla f\left(x^k\right)^T x^k+-A^T y^k-z^k+\left(\nabla f\left(x^k\right)^T\Delta x^k-A^T\Delta y^k-\Delta z^k\right)=0$$

First note that the componentwise concavity of $\phi$ implies that for any $(x,z)\in R^{2n}$ with $\mu>0$ ,and $(\Delta x,\Delta z)\in R^{2n}$ one has

$$\phi\left(x+\Delta x,z+\Delta z\right)\le\phi\left(x,z\right)+\nabla\phi\left(x,z\right)^T\begin{bmatrix}\Delta x\\ \Delta z\end{bmatrix}$$

Look at the Newton equations (9) and (12). In (13), we have $\Delta\hat{\mu}_k=-\left(1-\gamma\lambda_k\right)\hat{\mu}_k$ ,so (23) reducing to the system

$$\begin{cases}A\Delta\hat{x}^k=0\\ \nabla f\left(\hat{x}^k\right)^T\Delta\hat{x}^k-A^T\Delta\hat{y}^k-\Delta\hat{z}^k=0\\ \nabla_x\phi\left(\hat{x}^k,\hat{z}^k,\hat{\mu}_k\right)^T\Delta x+\nabla_z\phi\left(\hat{x}^k,\hat{z}^k,\hat{\mu}_k\right)^T\Delta z\\ =-\phi\left(\hat{x}^k,\hat{z}^k,\hat{\mu}_k\right)+\left(1-\lambda_k\gamma\right)\hat{\mu}_k\nabla_\mu\phi\left(\hat{x}^k,\hat{z}^k,\hat{\mu}_k\right)^T\end{cases}$$

In either case, $\phi\left(x^{k+1},z^{k+1},\mu_{k+1}\right) \le 0$ . For sake of simplicity, set $(x,y,z,\mu) = (\hat{x},\hat{y},\hat{z},\hat{\mu})$ and $(\Delta x,\Delta y,\Delta z) = (\Delta\hat{x},\Delta\hat{y},\Delta\hat{z})$ , then for $i \in \{1,2,\cdots,n\}$ and $\lambda \in [0,1]$ , then,

$$\left\|\phi\left(x+\lambda\Delta x,z+\lambda\Delta z,(1-\gamma\lambda)\mu\right)\right\|$$
$$\le(1-\lambda)\left\|\phi(x,z,\mu)\right\|+\frac{\lambda^2}{(1-\gamma\lambda)\mu}\left\|(\Delta x,\Delta z,-\gamma\mu)\right\|^2$$
$$\le(1-\lambda)\left\|\phi(x,z,\mu)\right\|+\frac{\lambda^2}{(1-\gamma\lambda)\mu}\left[\left\|(\Delta x,\Delta z)\right\|^2+\sqrt{n}\gamma^2\mu^2\right]$$
$$\le(1-\lambda)\left\|\phi(x,z,\mu)\right\|+\frac{\lambda^2}{(1-\gamma\lambda)\mu}\left[\left\|(\Delta x,\Delta y,\Delta z)\right\|^2+\sqrt{n}\gamma^2\mu^2\right]$$
$$\le(1-\lambda)\beta\mu+\frac{\lambda^2}{(1-\gamma\lambda)}\left[C^2\left(\beta+2\sqrt{n}\gamma\right)^2+\sqrt{n}\gamma^2\right]\mu$$

Where the last inequality follows from (17) and part (iv) of Lemma 2 in [11] .This yields the bound

$$\left\|(\Delta x,\Delta y,\Delta z)\right\| \le \left\|\nabla_{(x,y,z)}F(x,y,z,\mu)\right\|$$
$$\left[\left\|\phi(x,z,\mu)\right\|+\gamma\mu\left\|\nabla_{\mu}\phi(x,z,\mu)\right\|\right] \le C\left(\beta+2\sqrt{n}\gamma\right)\mu$$

It is easily verified that

$$(1-\lambda)\beta\mu + \frac{\lambda^2}{(1-\gamma\lambda)}\left[C^2\left(\beta+2\sqrt{n}\gamma\right)^2+\sqrt{n}\gamma^2\right]\mu \le (1-\gamma\lambda)\beta\mu$$

Whenever

$$\lambda \le \frac{(1-\gamma)\beta}{C^2\left(\beta+2\sqrt{n}\gamma\right)^2+\sqrt{n}\gamma^2+\gamma(1-\gamma)\beta}$$

Therefore

$$\lambda_k \ge \min\left\{1,\frac{(1-\gamma)\beta}{C^2\left(\beta+2\sqrt{n}\gamma\right)^2+\sqrt{n}\gamma^2+\gamma(1-\gamma)\beta}\right\}$$

To conclude, just as in (15),the relations (16) and (17) combined with part (iv) of Lemma2 in [11] yield the bounds

$$\left\|\left(\Delta x^k,\Delta y^k,\Delta z^k\right)\right\| \le C\left(\beta+2\sqrt{n}\right)\mu_k$$

and

$$\left\|\left(\Delta\hat{x}^k,\Delta\hat{y}^k,\Delta\hat{z}^k\right)\right\| \le C\left(\beta+2\sqrt{n}\right)\mu_k$$

Since $0 < \gamma < 1$ and $0 < \lambda_k \le 1$ for all $k$ . therefore ,

$$\left\|\left(x^{k+1},y^{k+1},z^{k+1}\right)-\left(x^k,y^k,z^k\right)\right\| \le \left\|\left(\Delta x^k,\Delta y^k,\Delta z^k\right)\right\| +$$
$$\hat{\lambda}_k\left\|\left(\Delta\hat{x}^k,\Delta\hat{y}^k,\Delta\hat{z}^k\right)\right\| \le 2C\left(\beta+2\sqrt{n}\right)\mu_k \le 2C\left(\beta+2\sqrt{n}\right)\left(1-\gamma\bar{\lambda}\right)^k \mu_0$$

Hence, $\left(x^k,y^k,z^k\right)$ is a Cauchy sequence and so must converge to the unique solution of Problem1.2.


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