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Exact closed form formula for mutual inductance of conductors of rectangular cross section

Abstract. *In this paper, using a definition of a mutual inductance for two conductors of any shape and lengths, the new exact closed formula for mutual inductance between two rectangular bars of any lengths is proposed. In the case of direct current (DC) or low frequency (LF) this inductance is given by analytical formula. The mutual inductance between two thin tapes of any lengths is also presented.*

Streszczenie. *Stosując definicję indukcyjności wzajemnej między dwoma przewodami dowolnych kształtów i długości w pracy zaproponowano nowy dokładny wzór na obliczanie indukcyjności wzajemnej między dwoma przewodami o przekroju prostokątnym i dowolnej długości. W przypadku prądu stałego lub niskiej częstotliwości indukcyjność tę wyrażono wzorem analitycznym. Podano również wzór na indukcyjność wzajemną między dwoma przewodami taśmowymi o dowolnej długości. (**Dokładny wzór na indukcyjność wzajemną przewodów o przekroju prostokątnym**)*

Key words: rectangular busbar, mutual inductance, electromagnetic field
Słowa kluczowe: prostokątny przewód szynowy, indukcyjność wzajemna, pole elektromagnetyczne

Introduction

The real system lumped conductors can be modeled by a connection of resistances, self and mutual inductances. The self and mutual inductances play an important role not only in power circuits, but also in printed circuit board (PCB) lands [1-8]. Formulae for the mutual inductances of set of conductors of rectangular cross-section are the subjects of many electrical papers and books. They are mathematically complex and their demonstrations are usually omitted and only the approximate formulae are given as thought they were exact. The most significant of them are: Grover's given in [2, 3, 9, 10], Kalantarov and Tseitlin's presented in [11], Strunsky's shown in [12], Ruehli's presented in [10] and [13] as well as Hoer and Love's shown in [3], [10] and [14]. The formulae for the mutual inductances between two long rectangular conductors of any transversal dimensions and thin tapes are given in [15]. The mutual inductance can be calculated by many numerical methods. Zhong and Koh express in [16] the mutual inductance as a weighted sum of self inductances.C. R. Paul in [3] considers breaking the cross section of a rectangular conductor into "subbars" of rectangular cross section. Then the mutual inductance between the subbars is approximated as between filaments at the centers of the subbars. All the subbars are connected in parallel and from this circuit he determines the "self and impedance" matrix. Finally he obtains the effective impedance and mutual inductance between bars from it.

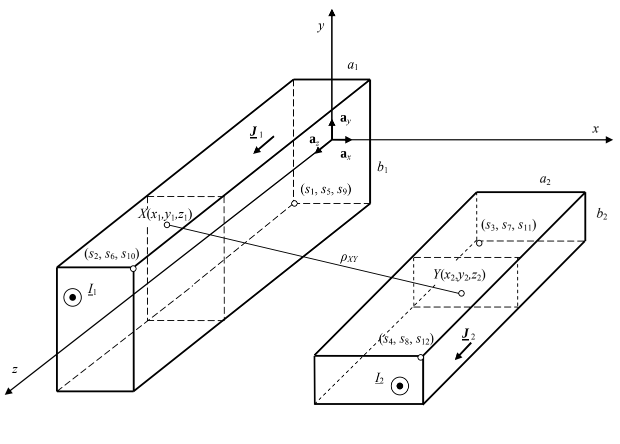

Fig. 1. Two parallel conductors of rectangular cross section with currents $\underline{I}_1$ and $\underline{I}_2$

In this paper a new method for calculating mutual inductance between two rectangular conductors of any transversal dimensions and any lengths is presented. The method results in a system of two integral Fredholm's equations. We compare our analytical formulae with several well-known ones given in the literature for DC, low frequency or parallel thin tapes.

We consider a general case of two parallel conductors of rectangular cross section shown in Fig.1. The positions of conductors are determined by coordinates of diagonal corner points: (s_1, s_5, s_9), (s_2, s_6, s_{10}) of the first wire of dimensions $a_1 \times b_1 \times l_1$ and (s_3, s_7, s_{11}), (s_4, s_8, s_{12}) of the second wire of dimensions $a_2 \times b_2 \times l_2$. We assume the conductors to be parallel.

Definition of mutual inductance

The definition of mutual impedance between two straight conductors is given in [17, 18] i.a. by following formula

$$\underline{Z}_{12} = \frac{\mathrm{j}\,\omega\,\mu_0}{4\pi\,\underline{I}_1^*\underline{I}_2} \int_{v_1}\int_{v_2} \frac{\underline{J}_{22}(Y)\underline{J}_{11}^*(X)}{\rho_{XY}}\,\mathrm{d}\,v_1\,\mathrm{d}\,v_2 \tag{1}$$

where $\underline{J}_{22}(Y)$ is the complex current density at source point $Y = Y(x_2, y_2, z_2) \in S_2$, $\underline{J}_{11}^*(X)$ is the complex conjugate current density at point of observation $X = X(x_1, y_1, z_1) \in S_1$, v_1 and v_2 are conductors' volumes. Distance between the point of observation *X* and the source point *Y* (Fig.1) is $\rho_{XY} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$. If conductors have a constant cross-sectional area S_1 and S_2 along theirs lengths, in case of DC, low frequency (for busbars of dimensions used in electrical power distribution system [1]) or for a thin strip conductors (in printed circuit board [2-8]) we can assume that the current density is constant and given as $\underline{J}_{11}(X) = \underline{I}_1 / S_1$ and $\underline{J}_{22}(Y) = \underline{I}_2 / S_2$ then, from the formulae (1), we obtain the mutual inductance between two straight parallel conductors

$$M = M_{12} = M_{21} = \frac{\mu_0}{4\pi\,S_1\,S_2} \int_{v_1}\int_{v_2} \frac{1}{\rho_{XY}}\,\mathrm{d}\,v_1\,\mathrm{d}\,v_2 \tag{2}$$

Mutual inductance between parallel conductors of rectangular cross section

The mutual inductance between two rectangular conductors of dimensions $a_1 \times b_1 \times l_1$ and $a_2 \times b_2 \times l_2$ shown in Fig. 1 is given by formula

$$M = \frac{\mu_0}{4\pi} \frac{1}{a_1\, a_2\, b_1\, b_2} F \tag{3}$$

where

$$F = \int_{s_{11}}^{s_{12}} \int_{s_9}^{s_{10}} \int_{s_7}^{s_8} \int_{s_5}^{s_6} \int_{s_3}^{s_4} \int_{s_1}^{s_2} \frac{1}{\rho_{XY}} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}z_1 \mathrm{d}z_2 \tag{3a}$$

is a sixtuple definite integral of an integrable function ρ_{XY}^{-1} of six variables $(x_1, x_2, y_1, y_2, z_1, z_2)$.

In general case this integral is very difficult to calculate. But in our case we can put $x = x_2 - x_1$, $y = y_2 - y_1$, $z = z_2 - z_1$ and first to calculate a sixtuple indefinite integral

$$F(x,y,z) = \iiiiiint \frac{1}{\rho_{XY}(x,y,z)} \mathrm{d}x\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}z \tag{4}$$

of a function

$$\rho_{XY}^{-1}(x,y,z) = \left(x^2 + y^2 + z^2\right)^{-\frac{1}{2}} \tag{4a}$$

of three variables (x, y, z) - twice with respect to *x,* twice with respect to *y* and twice with respect to *z*. After each double integration we omit terms which depend only on one or two variables - they are constants with respect to the considered variable. We can also omit terms proportional to one variable like $H(x,y,z) = x\, g(y,z)$.

Finally, after a lengthy integration, formula (4) yields an expression for sixtuple indefinite integral

$$F(x,y,z) = \frac{1}{72} \left\{ \begin{array}{l} \frac{6}{5}\left(x^4 + y^4 + z^4 - 3x^2y^2 - 3x^2z^2 - 3y^2z^2\right)\sqrt{x^2+y^2+z^2} - \\ 12xyz\left(z^2 \tan^{-1}\frac{xy}{z\sqrt{x^2+y^2+z^2}} + y^2 \tan^{-1}\frac{xz}{y\sqrt{x^2+y^2+z^2}} + x^2 \tan^{-1}\frac{yz}{x\sqrt{x^2+y^2+z^2}} \right) - \\ 3x(y^4 - 6y^2z^2 + z^4)\ln\left(x + \sqrt{x^2+y^2+z^2} \right) - 3y(x^4 - 6x^2z^2 + z^4)\ln\left(y + \sqrt{x^2+y^2+z^2} \right) - \\ 3z(x^4 - 6x^2y^2 + y^4)\ln\left(z + \sqrt{x^2+y^2+z^2} \right) \end{array} \right\} \tag{5}$$

Now, at the first step, we calculate double definite integral with respect to *x* variable receiving following function

$$F(y,z) = \left[F(x,y,z)\right]_{s_1-s_3,\, s_2-s_4}^{s_1-s_4,\, s_2-s_3} (x) = \left[F(x,y,z)\right]_{p_2,\, p_4}^{p_1,\, p_3} (x) = \sum_{i=1}^{i=4} (-1)^{i+1} F(p_i, y, z) \tag{6}$$

The second step gives the function

$$F(z) = \left[F(y,z)\right]_{s_5-s_7,\, s_6-s_8}^{s_5-s_8,\, s_6-s_7} (y) = \left[F(y,z)\right]_{q_2,\, q_4}^{q_1,\, q_3} (y) = \sum_{j=1}^{j=4} (-1)^{j+1} F(q_j, z) = \left[\left[F(x,y,z)\right]_{p_2,p_4}^{p_1,p_3}(x)\right]_{q_2,q_4}^{q_1,q_3}(y) = \sum_{i=1}^{i=4}\sum_{j=1}^{j=4} (-1)^{i+j} F(p_i, q_j, z) \tag{7}$$

And, finally we have

$$F = \left[F(z)\right]_{s_9-s_{11},\, s_{10}-s_{12}}^{s_9-s_{12},\, s_{10}-s_{11}} (z) = \left[F(z)\right]_{r_2, r_4}^{r_1, r_3} (z) = \sum_{k=1}^{k=4} (-1)^{k+1} F(r_k) = \left[\left[\left[F(x,y,z)\right]_{p_2,p_4}^{p_1,p_3}(x)\right]_{q_2,q_4}^{q_1,q_3}(y)\right]_{r_2,r_4}^{r_1,r_3}(z) = \sum_{i=1}^{i=4}\sum_{j=1}^{j=4}\sum_{k=1}^{k=4} (-1)^{i+j+k+1} F(p_i, q_j, r_k) \tag{8}$$

Hence the mutual inductance between two parallel conductors of rectangular cross section is given by following formula

$$M = \frac{\mu_0}{4\pi} \frac{1}{a_1\, a_2\, b_1\, b_2} \left[\left[\left[F(x,y,z)\right]_{q_2,p_4}^{p_1,p_3}(x)\right]_{q_2,q_4}^{q_1,q_3}(y)\right]_{r_2,r_4}^{r_1,r_3}(z) = \frac{\mu_0}{4\pi} \frac{1}{a_1\, a_2\, b_1\, b_2} \sum_{i=1}^{i=4}\sum_{j=1}^{j=4}\sum_{k=1}^{k=4} (-1)^{i+j+k+1} F(p_i, q_j, r_k) \tag{9}$$

where $a_1 = s_2 - s_1$, $b_1 = s_6 - s_5$, $a_2 = s_4 - s_3$, $b_2 = s_8 - s_7$, $l_1 = s_{10} - s_9$ and $l_2 = s_{12} - s_{11}$.

Mutual inductance between two parallel thin tapes

The mutual inductance between two parallel thin tapes of widths a_1 and a_2, lengths l_1 and l_2 respectively, thickness $b_1 \approx 0$ and $b_2 \approx 0$ and distance *d* between them (Fig.2) is given by formula

$$M = \frac{\mu_0}{4\pi} \frac{1}{a_1\, a_2} F \tag{10}$$

where

$$F = \int_{s_{11}}^{s_{12}} \int_{s_9}^{s_{10}} \int_{s_3}^{s_4} \int_{s_1}^{s_2} \frac{\mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}z_1 \mathrm{d}z_2}{\sqrt{d^2 + \left(x_2 - x_1\right)^2 + \left(z_2 - z_1\right)^2}}$$

is a quadruple definite integral of four variables (x_1, x_2, z_1, z_2) into which the distance $d = s_7 - s_5$ is measured from the plan of the first tape to the plan the second one. Now we can put $x = x_2 - x_1$ and $z = z_2 - z_1$ and first to calculate a quadruple indefinite integral

$$F(x,z) = \iiiint \frac{\mathrm{d}x\, \mathrm{d}x\, \mathrm{d}z\, \mathrm{d}z}{\sqrt{d^2 + x^2 + z^2}} \tag{11}$$

twice with respect to *x* and twice with respect to *z*. Finally, after a lengthy integration, formula (11) yields an expression for quadruple indefinite integral

$$F(x,z)=\frac{x^2-d^2}{2}z\ln\left(z+\sqrt{d^2+x^2+z^2}\right)+\frac{z^2-d^2}{2}x\ln\left(x+\sqrt{d^2+x^2+z^2}\right)-\frac{1}{6}\left(x^2-2\,d^2+z^2\right)\sqrt{d^2+x^2+z^2}-d\,x\,z\tan^{-1}\frac{x\,z}{d\sqrt{d^2+x^2+z^2}} \tag{12}$$

Hence the mutual inductance between two thin tapes is given by following formula

$$M=\frac{\mu_0}{4\pi}\frac{1}{a_1\,a_2}\left[\left[F(x,z)\right]_{p_2,p_4}^{p_1,p_3}(x)\right]_{r_2,r_4}^{r_1,r_3}(z)=\frac{\mu_0}{4\pi}\frac{1}{a_1\,a_2}\sum_{i=1}^{i=4}\sum_{k=1}^{k=4}\left(-1\right)^{i+k}F(p_i,r_k) \tag{13}$$

where $p_1=s-a_1$, $p_2=s+a_2-a_1$, $p_3=s+a_2$, $p_4=s$, $r_1=s_1-s_4$, $r_2=s_1-s_3$, $r_3=s_2-s_3$ and $r_4=s_2-s_4$.

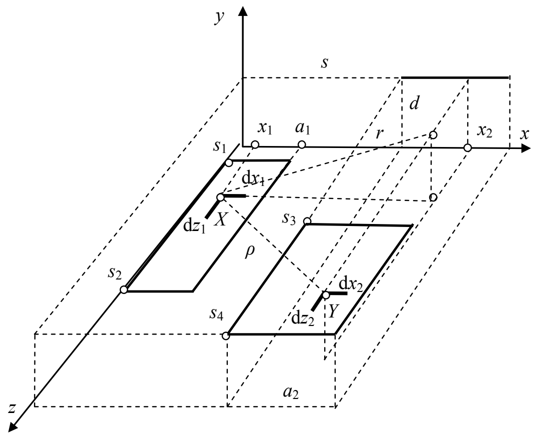

Fig. 2. Two parallel thin tapes with parallel widths

It is exactly the Hoer's formula given in [14]. For the same two tapes of width $a=a_1=a_2$, distance *d* between them and without displacement ($s=s_3=s_1=0$) the mutual inductance is given by following formula

$$M=\frac{\mu_0}{4\pi}\frac{1}{a^2}\left[\begin{array}{l}\frac{4}{3}d^3+\frac{2}{3}\left(a^2-2d^2\right)\sqrt{a^2+d^2}+\frac{2}{3}\left(l^2-2d^2\right)\sqrt{l^2+d^2}-\frac{2}{3}\left(l^2-2d^2+a^2\right)\sqrt{a^2+l^2+d^2}-\\ 4adl\tan^{-1}\frac{a\,l}{d\sqrt{a^2+l^2+d^2}}+ad^2\ln\frac{\left(a+\sqrt{a^2+d^2}\right)\left(-a+\sqrt{a^2+l^2+d^2}\right)}{\left(-a+\sqrt{a^2+d^2}\right)\left(a+\sqrt{a^2+l^2+d^2}\right)}+\\ ld^2\ln\frac{\left(l+\sqrt{a^2+d^2}\right)\left(-l+\sqrt{a^2+l^2+d^2}\right)}{\left(-l+\sqrt{a^2+d^2}\right)\left(l+\sqrt{a^2+l^2+d^2}\right)}+\\ al^2\ln\frac{\left(a+\sqrt{a^2+l^2+d^2}\right)}{\left(-a+\sqrt{a^2+l^2+d^2}\right)}+a^2l\ln\frac{\left(l+\sqrt{a^2+l^2+d^2}\right)}{\left(-l+\sqrt{a^2+l^2+d^2}\right)}\end{array}\right] \tag{14}$$

Computational results

Table 1. Mutual inductance between two busbars of rectangular cross section for DC or low frequency.

Two busbars: $a=0.08\,\mathrm{m}$; $b=0.007\,\mathrm{m}$; $d=2\,a$					
l (m)	Ruehli *L* (nH)	Grover *L* (nH)	Strunsky *L* (nH)	Hoer *L* (nH)	Eq. (9) *L* (nH)
0.10 *a*	0.03999	0.03999	0.03998	0.03922	0.03922
1.00 *a*	3.92230	3.92230	3.92169	3.84978	3.84978
10.0 *a*	238.821	238.821	238.800	236.249	236.249
100 *a*	5800.11	5800.11	5799.86	5769.18	5769.18
1000 *a*	94556.0	94556.0	94553.5	94240.4	94373.4

Table 2. Mutual inductance between two thin tapes of rectangular cross section for DC or low frequency.

Two thin tapes: $a=0.5\,\mu\mathrm{m}$; $b=0.1\,\mu\mathrm{m}$; $d=2\,a$					
l (m)	Ruehli *L* (pH)	Grover *L* (pH)	Strunsky *L* (pH)	Eq. (9) *L* (pH)	Eq. (14) *L* (pH)
0.10 *a*	0.00024	0.00024	0.00024	0.00024	0.00024
1.00 *a*	0.02451	0.02451	0.02449	0.02408	0.02408
10.0 *a*	1.49263	1.49263	1.49195	1.47715	1.47715
100 *a*	36.2507	36.2507	36.2425	36.0634	36.0634
1000 *a*	590.975	590.975	590.892	589.003	589.010

For the chosen transversal dimensions and different lengths of a busbars the calculations of their mutual inductance have been made according to all previous, shown above, formulae – Table 1.

For the mutual inductance of thin tapes of width *a*, thickness *b* and length *l* above formulae gives results sown in Table 2.

Conclusions

We have defined the mutual inductance of two conductors of any shape and any length given by sixtuple definite integral. For rectangular conductors the limits of this integral are given by coordinates of diagonal corner points of the first conductor and the second one. In the case of DC or low frequency we have given general formulae for the mutual inductance of conductors of rectangular cross section of any dimensions including the thin tapes. These formulae can be used for any dimensions of conductors and for any position between them.

By computations we have shown that our formulae give the same results as first of all Hoer's ones for all dimensions of conductors. In addition we have also obtained analytical forms for the mutual inductance between the thin tapes. Of course they give the same results as the general formulae. Our formulae are analytically simple and can also replace the traditional tables and working ones.

These formulae can be used in the methods of numerical calculation of AC mutual inductance of rectangular conductors. Then the cross sections of the conductors are divided into rectangular subbars (elementary bars) in which the current is assumed to be uniformly distributed over the cross section of each subbar.

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REFERENCES

[1] Balzer G. et al: *Switcher Manual.* 10th edn., ABB Calor Emag Mittelspannung GmbH, Ratingen 2004.

[2] Kazimierczuk M. K.: *High-Frequency Magnetic Components.* J Wiley & Sons, Chichester, 2009.

[3] Paul C. R.: *Inductance: Loop and Partial.* J Wiley & Sons, New Jersey, 2010.

[4] Piątek Z. and Baron B.: *Exact closed form formula for self inductance of conductor of rectangular cross section.* Progress in Electromagnetics Research M. Vol. 26, 2012, pp. 225-236.

[5] Piątek Z. et al: *Self inductance of long conductor of rectangular cross section.* Przegląd Elektrotechniczny (Electrical Review), R. 88, No. 8, 2012, pp. 323-326.

[6] Broydé, F., Clavelier E. and Broydé L.: *A direct current per-unit-length inductance matrix computation using modified partial inductance*. Proc. Of the CEM 2012 Int. Symp. on Electromagnetic Compatibility, Rouen, 25-27 April, 2012.

[7] Hashemi-Nasad M. and Cheldavi A.: *Coupling model for the two orthogonal microstrip lines in two layer PCB board (quasi-tem approach)*. Progress in Electromagnetics Research, PIER 60, 2006, pp. 153–163.

[8] Koledintseva M. Y. et al: *Method of edge currents for calculating mutual external inductance in a microstrip structure*. Progress in Electromagnetics Research, PIER 80, 2008, pp. 197–224.

[9] Grover F. W. *Inductance Calculations.* Dover Publications, Inc., New York, 1973.

[10] Kim H. and Chen C. C. P.: *Be Careful of Self and Mutual Inductance Formulae*. Online: http:// ccf.ee. ntu.edu.tw/~cchen/research/CompInduct9.pdf), 2009.

[11] Kalantarov P. L. and Tseitlin L. A.: *Inductance Calculations* (in Russian). Energiya, Saint Petersburg 1970.

[12] Strunsky B. M.: *Short Electric Network of Electric Furnaces* (in Russian). GN-TIL, Moskow 1962.

[13] Ruehli A. E.: *Inductance Calculations in a Complex Integrated Circuit Environment*. IBM J. Res. Develop. Sept. 1972, pp. 470-481.

[14] Hoer C. and Love C.: *Exact Inductance Equations for Rectangular Conductors with Application to More Complicated Geometries*. J. Res. N. B. S., No. 2, 1965, pp. 127-137.

[15] Piątek Z. et al: *Mutual inductance of long rectangular conductors.* Przegląd Elektrotechniczny (Electrical Review), R. 88, No. 9a, 2012, pp. 175-177.

[16] Zhong G. and Koh C-K.: *Exact Form Formula for Mutual Inductance of On-Chip Interconnects*. IEEE Trans. Circ. and Sys., I:FTA, No. 10, 2003, pp. 1349-1353.

[17] Piatek Z.: *Self and Mutual Impedances of a Finite Length Gas Insulated Transmission Line (GIL)*. Elec. Pow. Syst. Res., No. 77, 2007, pp. 191-203.

[18] Piątek Z.: *Impedances of Tubular High Current Busducts*. Polish Academy of Sciences. Warsaw 2008.

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