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Calculation of magnetic component of wire antenna electromagnetic field

***Abstract**. The possibility of analytical solution to calculation of azimuthal component of magnetic field intensity of wire antenna simulated by short-circuit current is demonstrated. For this purpose boundary condition on a cylindrical surface corresponding to the antenna surface presented by Fourier series is proposed.*

***Streszczenie.** W artykule opisano analityczną metodę wyznaczania charakterystyki natężenia pola magnetycznego anteny kablowej. W tym celu, korzystając z szeregu Fouriera, zaproponowano warunek brzegowy dla powierzchni cylindrycznej, adekwatnej do powierzchni anteny. (**Obliczenia składowej magnetycznej pola elektromagnetycznego anteny kablowej**).*

Keywords: antenna feeding, scalar Helmholtz equation, antenna parameters.
Słowa kluczowe: zasilanie anteny, równanie skalarne Helmholtza, parametry anteny.

Introduction

Wire antennas are widely used in practice due to their relatively simple design and possibility to provide the necessary technical characteristics. At present such antennas can be calculated by a number of analytical methods based on a simplified notion of the antenna as a dipole radiator [1, 2]. Application of another conventional analytical solution for the antenna of the considered type, used at great distances from it [2], is connected with evident errors in the calculation of antenna parameters. This conditioned the use of numerical methods, realized by means of relevant software products, for such calculations. In particular, [3] contains description of the methods and wire antenna parameters calculation results on the basis of application of the finite elements method.

The purpose of this paper is to indicate the possibility of analytical solution to the field problem posed and solved numerically in [3].

Problem statement

The paper deals with consideration of a wire antenna presenting an unbalanced vibrator situated over the ground plane; a coaxial circuit inner conductor is the body of the antenna (Fig.1).

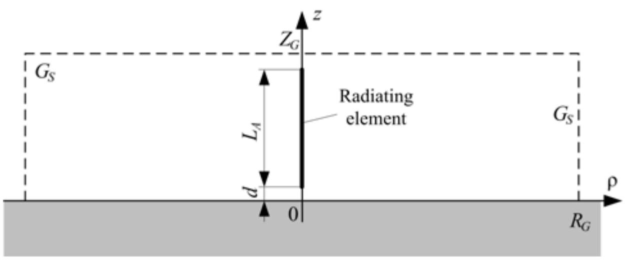


Fig. 1. A circuit of an unbalanced vibrator in the form of a wire antenna

In a general case electromagnetic field of such an antenna in a steady radiation mode is described by the following system of equations (Maxwell equations) [2]:

(1) $$rot\overline{H} = j\omega\varepsilon\overline{E} ,$$

(2) $$rot\overline{E} = -j\omega\mu\overline{H} ,$$

(3) $$div\overline{H} = 0 ,$$

(4) $$div\overline{E} = 0$$

where $\overline{H}$ and $\overline{E}$ are intensity vectors of electromagnetic field magnetic and electric components, respectively; ε and μ are electromagnetic field existence domain permittivity and permeability, respectively; ω – excitation angular frequency; j – imaginary unit ($j^2 = -1$).

If this system of equations is reduced to the equation just concerning magnetic field intensity $\overline{H}$ vector, it will be possible to write down equation [2, 3]

(5) $$rot\left(rot\overline{H}\right) - k_0^2\overline{H} = 0$$

where $k_0 = \omega\sqrt{\mu\varepsilon}$ – a wave number.

Taking into account the fact that the considered antenna is axially symmetrical, it can be stated that magnetic field has only one azimuthal component H_φ in the accepted cylindrical coordinate system (Fig. 1). It enables writing down a partial differential equation for H_φ (scalar Helmholtz equation) instead of (5)

(6) $$\Delta H_\varphi + k_0^2 H_\varphi = 0 ,$$

which should be solved for the upper half-space, taking into consideration the boundary conditions on the ground surface ($z = 0$), on the antenna symmetry axis ($\rho = 0$) and on the external boundary of the analyzed area designated as G_S in Fig.1.

In equation (6) Laplacian ΔH_φ for the accepted cylindrical coordinate system (Fig. 1) cannot be expanded just as scalar Laplacian of the form:

$$\Delta H_\varphi = \frac{\partial^2 H_\varphi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial H_\varphi}{\partial\rho} + \frac{\partial^2 H_\varphi}{\partial z^2} .$$

It should be taken into account that this Laplacian is obtained after the expansion of $rot(rot\overline{H})$ in (5), and, consequently, in the general case is the Laplacian of vector $\overline{H}$. I.e. in the considered case, when writing down Helmholtz equation for magnetic intensity H_φ in cylindrical coordinate system, after appropriate expansion of $rot(rot\overline{H})$, the following equation should be used instead of (6).

(7) $$\frac{\partial^2 H_\varphi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial H_\varphi}{\partial\rho} - \frac{H_\varphi}{\rho^2} + \frac{\partial^2 H_\varphi}{\partial z^2} + k_0^2 H_\varphi = 0 .$$

Further task consists in analytic solution to equation (7) in the designated analyzed area (Fig. 1).

It should be noted, that with $k_0 = 0$ (magnetic field is stationary and $\omega = 0$) expression (7) in its form is analogous to Poisson's equation for the azimuthal component of the vector magnetic potential described in [4, 5]. However, in this case $k_0 \neq 0$. Besides, physical nature of magnetic field intensity and vector magnetic potential is different. Therefore a direct use of the results of papers [4, 5] in this case is not possible. Although hereafter we follow the general approach described in [4, 5].

General solution to equation (7) by variable separation method

To solve equation (7) analytically a conventional variable separation method is used. It consists in representation of the required function (intensity H_φ) in the form of product

(8) $$H_\varphi = R(\rho)Z(z)$$

where $R(\rho)$ is a function depending only on coordinate ρ, and $Z(z)$ is a function depending only on coordinate z.

Substitution of (8) into (7) makes it possible, after simple transformations, to write down the following equality

(9) $$\frac{1}{R(\rho)}\frac{d^2R(\rho)}{d\rho^2}+\frac{1}{\rho R(\rho)}\frac{dR(\rho)}{d\rho}-\frac{1}{\rho^2}+\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=-k_0^2,$$

that can also be presented in the form

(10) $$\frac{1}{R(\rho)}\frac{d^2R(\rho)}{d\rho^2}+\frac{1}{\rho R(\rho)}\frac{dR(\rho)}{d\rho}-\frac{1}{\rho^2}+k_0^2=-\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}.$$

Then on the assumption of the fact that the left part of equation (10) only presents the function of coordinate ρ, and the right part of this equation only presents the function of coordinate *z*, it is possible to write down two equations instead of one (10)

(11) $$\frac{1}{R(\rho)}\frac{d^2R(\rho)}{d\rho^2}+\frac{1}{\rho R(\rho)}\frac{dR(\rho)}{d\rho}-\frac{1}{\rho^2}+k_0^2=\lambda^2,$$

(12) $$\frac{1}{Z(z)}\frac{d^2Z(z)}{dz^2}=-\lambda^2$$

where λ – is some arbitrary number.

Thus, due to application of variable separation method, solution (7) can be reduced to solution of two equations (11) and (12).

As to equation (12), it can be written down in the form

(13) $$\frac{\partial^2 Z(z)}{\partial z^2}+Z(z)\lambda^2=0,$$

for which its general solution is known and can be written down as [6]

(14) $$Z(z)=C_1\sin(z\lambda)+C_2\cos(z\lambda)$$

where C_1 and C_2 are some coefficients.

As to equation (11), it can also be presented as an equation with a known solution. To be exact, it is transformed to the form

$$\rho^2\frac{d^2R(\rho)}{d\rho^2}+\rho\frac{dR(\rho)}{d\rho}-R(\rho)(1-(k_0^2-\lambda^2)\rho^2)=0,$$

for which, according to [7], a general solution can be written down as [6]

$$R(\rho)=C_3J_1\left(\rho\sqrt{k_0^2-\lambda^2}\right)+C_4Y_1\left(\rho\sqrt{k_0^2-\lambda^2}\right)$$

where C_3 and C_4 are coefficients; and $J_1\left(\rho\sqrt{k_0^2-\lambda^2}\right)$ and $Y_1\left(\rho\sqrt{k_0^2-\lambda^2}\right)$ are first-order Bessel functions of the first and the second kind, respectively.

A general solution to equation (7) as product of equations (11) and (12) solutions, according to (13) and (14), can be finally written down in the following form

(15) $$H_\varphi(\rho,z)=\sum_\lambda\left[C_1\sin(z\lambda)+C_2\cos(z\lambda)\right]\times \\ \times\left[C_3J_1\left(\rho\sqrt{k_0^2-\lambda^2}\right)+C_4Y_1\left(\rho\sqrt{k_0^2-\lambda^2}\right)\right]$$

where it is taken into account that λ may be of arbitrary values and the subscript under the index of summation denotes that summing up should be done on all possible values of λ.

It is clear that expression (15) provides the solution to the initial equation (7) in the most general form and requires determination of coefficients C_1, C_2, C_3, C_4 and number λ in order to be practically used in the calculation of intensity H_φ distribution. With this purpose in view it is necessary to use boundary conditions describing the behavior of intensity H_φ or its derivatives at the boundary of analyzed area: ground surface, external boundary G_S and symmetry axis (Fig. 1).

About boundary conditions

First of all it should be pointed out that in the considered case the ground surface can be assumed to be the surface with reflective properties, which conditions the symmetry of intensity H_φ scalar field about this surface. It means that the following boundary condition is true for plane $z=0$ (Fig.1)

(16) $$\left.\frac{\partial H_\varphi}{dz}\right|_{z=0}=0.$$

As to external boundary, there are two surfaces for it: an end (at a height of $z=Z_G$) and side (at the radius of $\rho=R_G$) surfaces of the cylinder limiting the analyzed area (Fig. 1). Accordingly, there are two boundary conditions for the external boundary.

In this case it is assumed that on the said end surface ($z=Z_G$) the intensity vector tangent component of electromagnetic field electric component $\overline{E}$ is equal to zero (vector $\overline{E}$ is orthogonal to this surface). Then, taking into consideration the connection between vectors $\overline{H}$ and $\overline{E}$, assigned by equation (1) for intensity H_φ on the analyzed surface, the following boundary condition can be written down

(17) $$\left.\frac{1}{j\omega\varepsilon_0}\frac{\partial H_\varphi}{\partial z}\right|_{z=Z_G}=0.$$

By analogy it is assumed that on the said side surface of the cylinder limiting the analyzed area ($\rho=R_G$, Fig. 1) the intensity $\overline{E}$ vector tangent component is also equal to zero (vector $\overline{E}$ is orthogonal to this surface). Then, taking into consideration (1), the following boundary condition can be written down for the considered surface

(18) $$\left.\frac{1}{j\omega\varepsilon_0}\left(\frac{\partial H_\varphi}{\partial\rho}+\frac{H_\varphi}{\rho}\right)\right|_{\rho=R_G}=0.$$

Speaking about boundary condition on symmetry axis $\rho = 0$, as $J_1(0) = 0$ and $Y_1(0) = \infty$ [7], its application in the considered case for specification of constants being part of (15) is not directly possible. So, it is necessary to determine boundary conditions at minimally possible approximation to the axis ("isolated axis"). It is discussed later.

Use of boundary condition for ground surface

This boundary condition is written above as relation (16). Substitution of intensity H_φ, according to (15), into this boundary condition makes it possible to write down the equation

$$\sum_\lambda \lambda[C_1 \cos(z\lambda) - C_2 \sin(z\lambda)] \times \left. \times\left[C_3 J_1\left(\rho\sqrt{k_0^2-\lambda^2}\right) + C_4 Y_1\left(\rho\sqrt{k_0^2-\lambda^2}\right)\right]\right|_{z=0} = 0,$$

or (taking into account that $\sin 0 = 0$, and $\cos 0 = 1$)

$$\sum_\lambda \lambda C_1 \left[C_3 J_1\left(\rho\sqrt{k_0^2-\lambda^2}\right) + C_4 Y_1\left(\rho\sqrt{k_0^2-\lambda^2}\right)\right] = 0$$

which can be met at $C_1 = 0$.

It allows rewriting (15) in the form

$$H_\varphi(\rho, z) = \sum_\lambda \cos(z\lambda)\left[C_3^* J_1\left(\rho\sqrt{k_0^2-\lambda^2}\right) + C_4^* Y_1\left(\rho\sqrt{k_0^2-\lambda^2}\right)\right] \tag{19}$$

where there are the following designations: $C_3^* = C_2 C_3$ and $C_4^* = C_2 C_4$.

Use of conditions for the external boundary of the analyzed area

According to the above assumed, there are two surfaces for the external boundary: an end and a side surface of the cylinder limiting the analyzed area (Fig. 1). Accordingly, there are two boundary conditions (17) and (18).

Use of condition (17) with substitution of magnetic field intensity H_φ expression into it, according to (19), provides the following equation

$$\sum_\lambda \sin(Z_G\lambda)\left[C_3^* J_1\left(\rho\sqrt{k_0^2-\lambda^2}\right) + C_4^* Y_1\left(\rho\sqrt{k_0^2-\lambda^2}\right)\right] = 0,$$

which can be met if it is assumed that

$$\sin(Z_G\lambda) = 0,$$

whence the following relation to all λ numbers (considering their dependence on n, hereafter they are designated as λ_n) is obtained

$$\lambda_n = \frac{n\pi}{Z_G} \tag{20}$$

where n = 1, 2, 3…

As to boundary condition (18), substitution of (19) into it, after appropriate transformations, makes it possible to obtain the following equation

$$\sum_{n=1}^{\infty} \cos(z\lambda_n)\sqrt{k_0^2-\lambda_n^2}\left[C_3^* J_0\left(R_G\sqrt{k_0^2-\lambda_n^2}\right) + C_4^* Y_0\left(R_G\sqrt{k_0^2-\lambda_n^2}\right)\right] = 0 \tag{21}$$

where J_0 and Y_0 are zero-order Bessel functions of the first and second kind, respectively.

When writing (21) it was taken into account, that in accordance with (20), the summation in (19) is made according to *n*.

Equation (21) can be met if it is assumed that

$$C_4^* = -C_3^*\left(J_0\left(R_G\sqrt{k_0^2-\lambda_n^2}\right)\right) \Big/ \left(Y_0\left(R_G\sqrt{k_0^2-\lambda_n^2}\right)\right). \tag{22}$$

Thus, according to (22) and (20), instead of (19) it is possible to write down the following expression for intensity H_φ

$$H_\varphi(\rho, z) = \sum_{n=1}^{\infty} C_3^* \cos((n\pi/Z_G)z) \times \left[J_1\left(\rho\sqrt{k_0^2-(n\pi/Z_G)^2}\right) - Y_1\left(\rho\sqrt{k_0^2-(n\pi/Z_G)^2}\right) \times J_0\left(R_G\sqrt{k_0^2-(n\pi/Z_G)^2}\right) Y_0^{-1}\left(R_G\sqrt{k_0^2-(n\pi/Z_G)^2}\right)\right]. \tag{23}$$

About boundary condition near axis *z* ($\rho = 0$)

As it is mentioned above, in the considered case it is impossible to directly assume any boundary condition on axis *z* when $\rho = 0$. If axis *z* is "isolated" with a cylindrical surface $\rho = a$ (surface AB in Fig. 2), this surface can be used to assign the last boundary condition with the aim of determination of coefficient C_3^* in (23) with its help.

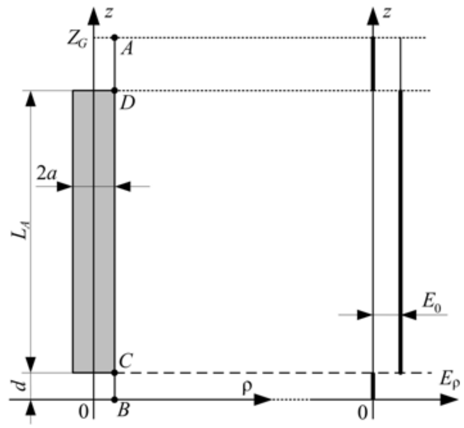


Fig. 2. Concerning determination of the boundary condition for "isolated axis", when $\rho = a$

For this purpose it will be taken into consideration that on antenna DC metal surface ($d \le z \le L_A$, Fig. 2) intensity $\overline{E}$ vector tangent component is equal to zero ($E_\varphi = 0$), and the normal component (component E_ρ), as a first approximation, can be assumed to be equal to some constant value E_0.

It is also taken into account, that considering the small radius *a* of antenna, condition $E_\rho = 0$ (line of force), typical of the symmetry axis ($\rho = 0$) below and above the antenna (Fig.2), can be accepted for surfaces AD ($L_A \le z \le Z_G$) and CB ($0 \le z \le d$).

Thus, for the boundary condition on surface AB ($\rho = a$, $L_A \le z \le Z_G$, Fig. 2) the following expression can be written

$$-(1/j\omega\varepsilon_0)(\partial H_\varphi/\partial z)\Big|_{\rho=a} = E_\rho(z)\Big|_{\rho=a} \tag{24}$$

where $E_\rho(z)$ is the function of distribution of vector $\overline{E}$ normal component on surface AB, the view of which is represented in the right part of Fig.2.

Function $E_\rho(z)$ can be expanded into Fourier series

$$E_\rho(z) = 2E_0 \sum_{n=1}^{\infty} \frac{1}{(n\pi)^2} \sin\left(\frac{n\pi}{Z_G} z\right) \times$$
$$\times \left[\cos\left(\frac{n\pi}{Z_G} d\right) - \cos\left(\frac{n\pi}{Z_G}(d + L_A)\right)\right].$$

Substituting this expansion into (24) and considering (23), after appropriate transformations a certain expression can be obtained for coefficient C_3^*. Substitution of this expression into (23) makes it possible to write down the final desired solution to equation (7)

$$H_\varphi(\rho, z) = 2 j\varepsilon_0\omega E_0 \sum_{n=1}^{\infty} \frac{Z_G}{(n\pi)^2} \cos\left((n\pi/Z_G)z\right) \times$$
$$\times \left[J_1\left(\rho\sqrt{k_0^2 - (n\pi/Z_G)^2}\right) - \right.$$
$$(25) \qquad \left. - Y_1\left(\rho\sqrt{k_0^2 - (n\pi/Z_G)^2}\right) Q(k_0, Z_G, R_G, n) \right] \times$$
$$\times \frac{\cos\left((n\pi/Z_G)d\right) - \cos\left((n\pi/Z_G)(d + L_A)\right)}{J_1\left(a\sqrt{k_0^2 - (n\pi/Z_G)^2}\right) -} \to$$
$$\to \frac{}{- Y_1\left(a\sqrt{k_0^2 - (n\pi/Z_G)^2}\right) Q(k_0, Z_G, R_G, n)}$$

where the following is designated

$$Q(k_0, Z_G, R_G, n) = J_0\left(R_G\sqrt{k_0^2 - (n\pi/Z_G)^2}\right) \times$$
$$\times Y_0^{-1}\left(R_G\sqrt{k_0^2 - (n\pi/Z_G)^2}\right).$$

Direct calculation showed good convergence of series in (25), which, depending on calculation parameter values, is achieved when the number of summands is about 50-100.

Fig. 3 shows dependences of $H_\varphi(z, \rho)$ obtained by calculation according to (25) with $L_A = 50$ mm; $d = 5$ mm; $a = 1.5$ mm; $R_G = 3L_A$; $Z_G = 3L_A$; $\omega = 1000$ H. The form of these dependences corresponds to the known [2] fact that field intensity is weaker when the distance from antenna is greater.

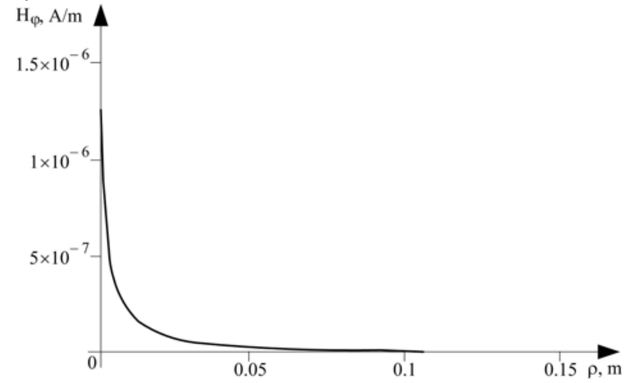
The approach used to obtain expression (25), due to its generality, can also be the base for determination of wire antenna electromagnetic field when boundary conditions are taken into account more accurately.

In the conclusion it should be mentioned that the known distribution of intensity H_φ also allows determining the distribution of antenna electromagnetic field electrical component ($\overline{E} = (1/j\omega\varepsilon_0) rot\overline{H}$). In its turn, it provides the possibility to find the distribution of Umov-Pointing vector on the external boundary of the considered area. Using this distribution it is possible to calculate antenna radiation full power and antenna input impedance [2, 3].

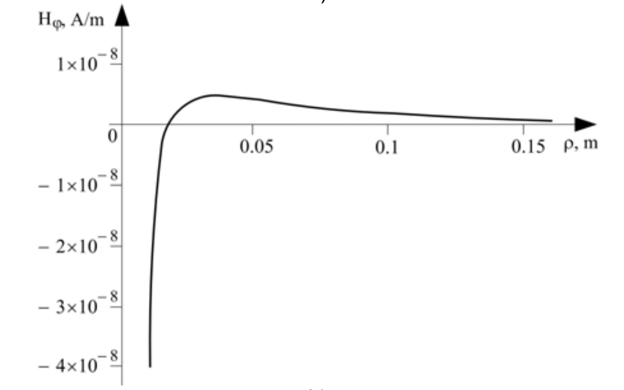
Conclusion:

The use of variable separation method makes it possible to obtain an analytical solution in the form of an infinite sum for Helmholtz equation with respect to azimuthal component of a wire antenna magnetic field intensity.

It enables determination of such important measured parameters of antenna as full power of radiation and input impedance.



a)



b)

Fig.3 Calculation dependences of intensity $H_\varphi(z, \rho)$, when $z = d + 0.5L_A$ (a) and $z = d + L_A$ (b)

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