Orest HAMOLA
Lviv Polytechnic National University

Application of integral equations for analysis of electric circuit transients

Abstract. *This paper discusses the particularities of integral equations for calculating the electric circuits transients with multiple switching. Each switching in some way affects constraining forces that is taken into account in certain correction of the right side of integral equation. The numerical method is used to solve integral equations.*

Streszczenie. *W pracy zbadano właściwości równań całkowych opisujących stan nieustalony w obwodach z wielokrotnym przełączaniem. Każde przełączenie powoduje modyfikację prawej strony równań. Pokazano metodę numeryczną rozwiązywania równań całkowych opisujących obwody. (**Zastosowanie równań całkowy do analizy stanów nieustalonych w obwodach elektrycznych**)*

Keywords: transient**,** integral equations, equation kernel.
Słowa kluczowe: stan nieustalony, równania całkowe, jądro równań całkowych.

Introduction

Analysis of transients in electric and magnetic circuits is the most common electrical problem. Models of dynamic processes in the electrical circuits are mainly presented as a system of differential equations. However, the models formation of transients of electric and magnetic circuits in the form of integral equations' system is worth of attention. It is appropriate in description of many tasks of electrotechnics to use Volterra equations with its advantages comparing to differential equations. The advantages include convenience and compact of description of transients. Numerical implementation of integrate dependencies is highly stable on principle. Models based on the integral equations do not impose restrictions on the smoothness of the functions that describe the circuit parameters and allow the existence of discontinuous solutions.

In the case of linear electric and magnetic circuits with lumped parameters the problem of transients' analysis lies in the solution of Volterra integral equations of the second kind or systems of equations with powermode or exponential kernels. Linear one-dimensional (scalar) Volterra equation of the second kind has the form [1]

(1) $$y(t)-\int_a^t K(t,\tau)y(\tau)d\tau = f(t), \quad t\in[a,b]$$

where $K(t,\tau)$ is integral equation's kernel; $f(t)$ – constraining force.

The mathematical model formation of transients of electric and magnetic circuits in the form of integral equations' system is presented in references [1,2] in general. In the article [3] the method of calculation of transients is examined in linear circuits with permanent parameters with integral equations, which applies the same approach as classical method of solution of the Cauchy problem. This paper discusses the application particularities of integral equations for calculating the dynamic processes in electric circuits.

The mathematical formulation and solution of the problem

Transients in a linear stationary electric circuit are described as a linear system of n Volterra integral equations (SVIE) of the second kind with n state variables of circuit as an unknown functions of time

$$\sum_{j=1}^{m} a_{1j}y_j(t)-\sum_{j=1}^{m}\int_a^t K_{1j}(t,\tau)y_j(\tau)d\tau = f_1(t),$$

(2) $$\sum_{j=1}^{m} a_{2j}y_j(t)-\sum_{j=1}^{m}\int_a^t K_{2j}(t,\tau)y_j(\tau)d\tau = f_2(t),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\sum_{j=1}^{m} a_{mj}y_j(t)-\sum_{j=1}^{m}\int_a^t K_{mj}(t,\tau)y_j(\tau)d\tau = f_m(t).$$

It is convenient to choose currents in branches with inductive elements i_L and voltage on capacitors u_C for state variables of circuit

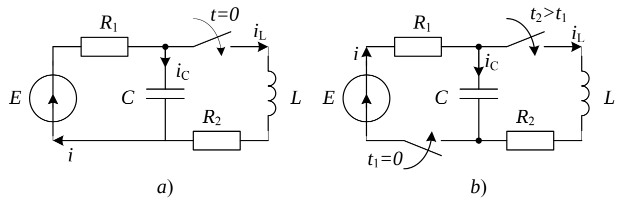

Fig.1. Electric circuit scheme: a is one commutation; b – two commutations

The mathematical model formation of transients of electric circuit in the form of Volterra integral equations' system needs the describing of equations' system in relation to state variables of this circuit based on the Kirchhoff's first and second laws and equations of elements

$u_R = Ri$, $\quad u_L = L\,di_L/dt$, $\quad i_L = L^{-1}\int_0^t u_L d\tau + i_L(0)$,

$i_C = C\,du_C/dt$, $\quad u_C = C^{-1}\int_0^t i_C d\tau + u_C(0)$. For example, such system of equations in relation to two state variables i_L, u_C for electrical circuit on fig.1, a is presented

$$i - i_C - i_L = 0, \quad i = i_L + C\frac{du_C}{dt},$$

(3) $$u_C - u_{R_2} - u_L = u_C - R_2 i_L - L\frac{di_L}{dt} = 0,$$

$$u_{R_1} + u_C = R_1 i_L + R_1 C\frac{du_C}{dt} + u_C = E.$$

Further integrating the equations of this system on an interval from 0 to *t* leads to SVIE after some transformations

$$i_L(t) - \int_0^t \left(-\frac{R_2}{L}\right) i_L d\tau - \int \frac{1}{L} u_C d\tau = i_L(0),$$

(4) $$u_C(t) - \int_0^t \left(-\frac{1}{C}\right) i_L d\tau - \int_0^t \left(-\frac{1}{R_1 C}\right) u_C d\tau =$$

$$= \int_0^t \frac{E}{R_1 C} d\tau + u_C(0).$$

Most numerical methods for solution of the Volterra equations of second kind are based on integral replacement by quadrature formulas. Numerical realization of SVIE (2) by the quadrature method bases on bringing it to the systems of algebraic equations on discrete set of points and further solving the obtained systems. Transformed system of equations is

(5) $$\begin{gathered}
\sum_{j=1}^{m} y_j^{(i)}\left(a_{1j} - K_{1ji}^{(i)} A_i\right) = f_1^{(i)} + \sum_{j=1}^{m}\sum_{l=1}^{i-1} K_{1ji}^{(l)} y_j^{(l)} A_l, \\
\sum_{j=1}^{m} y_j^{(i)}\left(a_{2j} - K_{2ji}^{(i)} A_i\right) = f_2^{(i)} + \sum_{j=1}^{m}\sum_{l=1}^{i-1} K_{2ji}^{(l)} y_j^{(l)} A_l, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\sum_{j=1}^{m} y_j^{(i)}\left(a_{mj} - K_{mji}^{(i)} A_i\right) = f_m^{(i)} + \sum_{j=1}^{m}\sum_{l=1}^{i-1} K_{mji}^{(l)} y_j^{(l)} A_l,
\end{gathered}$$

Where A_i are weight coefficients determined by the chosen quadrature formula.

Quadrature formula choosing is an ambiguous task, however mostly used are formulas of rectangles $A_i = (b-a)/(n-1)$, $i = 1,2,\ldots,n$ and of trapezoids $A_1 = A_n = h/2$, $A_2 = A_3 = \ldots = A_{n-1} = h$, $h = (b-a)/(n-1)$.

Matrix of coefficients of equations' system for two state variables has the appearance of fig.3

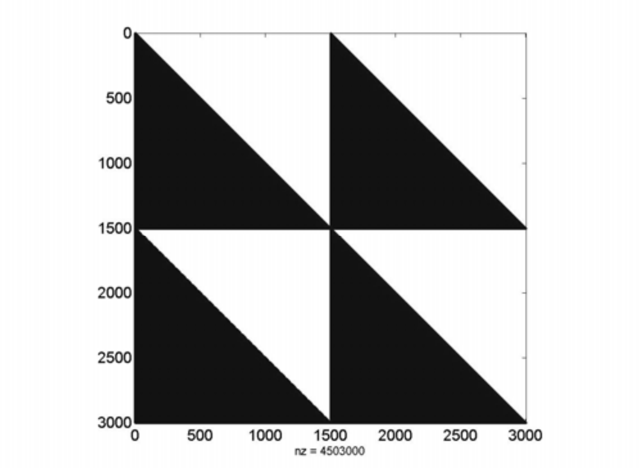

Fig.2. Matrix of coefficients of equations' system

Referring the stability of the numerical solution of SVIE, quadrature formula of rectangles provides the greatest firmness. It can be shown on the example of electric circuit of fig.1, a. If the size of inductance is less then $L = 0{,}0000017\,\mathrm{H}$, ($E = 50\,\mathrm{V}$, $R_1 = R_2 = 15\,\Omega$, $C = 20\,\mu F$), the trapezoidal formula does not provide stable solution fig.2. Current curve $i_L(t)$ in the beginning is fuzzy.

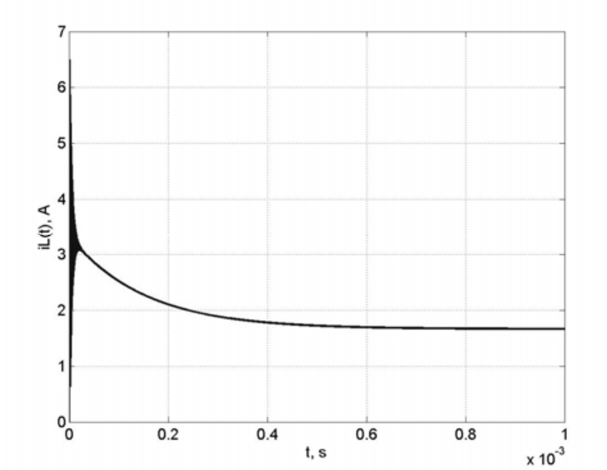

Fig.3. Transient current $i_L(t)$ with the trapezoid formula

The application of rectangle quadrature formula under the same conditions provides stable solution fig. 3.

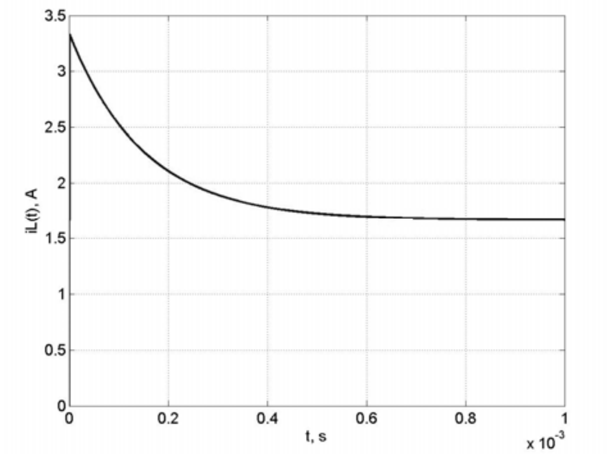

Fig.4. Transient current $i_L(t)$ with the rectangle formula

During the calculation of transients in electrical circuits with multiple switchings it is necessary to consider the changes of constraining force in the moments of switchings, exactly

for constant voltage sources $e(t) = E = const$

(6) $$\begin{gathered}
\sum_{j=1}^{m} a_{1j} y_j(t) - \sum_{j=1}^{m}\int_a^t K_{1j}(t,\tau) y_j(\tau) d\tau = \\
= E_{1,1} t + \sum_{i=2}^{n} E_{1,i}(t - t_i) + \sum_{j=1}^{m} b_{1j} y_j(a), \\
\sum_{j=1}^{m} a_{2j} y_j(t) - \sum_{j=1}^{m}\int_a^t K_{2j}(t,\tau) y_j(\tau) d\tau = \\
= E_{2,1} t + \sum_{i=2}^{n} E_{2,i}(t - t_i) + \sum_{j=1}^{m} b_{2j} y_j(a), \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\sum_{j=1}^{m} a_{mj} y_j(t) - \sum_{j=1}^{m}\int_a^t K_{mj}(t,\tau) y_j(\tau) d\tau = \\
= E_{m,1} t + \sum_{i=2}^{n} E_{m,i}(t - t_i) + \sum_{j=1}^{m} b_{mj} y_j(a).
\end{gathered}$$

for harmonic voltage sources $e(t) = E_m \sin(\omega t)$

$$
\begin{aligned}
&\sum_{j=1}^{m} a_{1j} y_j(t) - \sum_{j=1}^{m} \int_a^t K_{1j}(t,\tau) y_j(\tau) d\tau = E_{1,1}\cos(\omega t) + \\
&+ \sum_{i=2}^{n} E_{1,i} \cos(\omega(t - t_i)) + \sum_{j=1}^{m} b_{1j} y_j(a), \\
&\sum_{j=1}^{m} a_{2j} y_j(t) - \sum_{j=1}^{m} \int_a^t K_{2j}(t,\tau) y_j(\tau) d\tau = E_{2,1}\cos(\omega t) + \\
(7)\quad &+ \sum_{i=2}^{n} E_{2,i} \cos(\omega(t - t_i)) + \sum_{j=1}^{m} b_{2j} y_j(a), \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\sum_{j=1}^{m} a_{mj} y_j(t) - \sum_{j=1}^{m} \int_a^t K_{mj}(t,\tau) y_j(\tau) d\tau = E_{m,1}\cos(\omega t) + \\
&+ \sum_{i=2}^{n} E_{m,i} \cos(\omega(t - t_i)) + \sum_{j=1}^{m} b_{mj} y_j(a).
\end{aligned}
$$

where E_{mi} are the amplitudes of constraining forces on the corresponding time fragments between switchings; *n* – the amount of switchings; $y_j(t)$, $y_j(a)$ – state variables (currents $i_L(t)$ and voltage $u_C(t)$) and its initial conditions.

The application of expressions (7) is shown on the example of calculation of electric circuit transient fig.1, b.

This electric circuit's transient according to two switchings in the time moments t_1 and t_2 is described by the following system of integral equations

$$
(8)\quad
\begin{aligned}
&i_L(t) - \int_{t_1}^{t} K_1(t,\tau) i_L(\tau) d\tau - \int_{t_1}^{t} K_2(t,\tau) u_C(\tau) d\tau = \\
&= f_1(t), \\
&u_C(t) - \int_{t_1}^{t} K_3(t,\tau) i_L(\tau) d\tau - \int_{t_2}^{t} K_4(t,\tau) u_C(\tau) d\tau = \\
&= f_2(t).
\end{aligned}
$$

where interval $t_1 \le t \le t_2$

$$K_1(t,\tau) = 0;\ K_2(t,\tau) = 0;\ K_3(t,\tau) = 0;\ K_4(t,\tau) = -\frac{1}{R_1 C};$$

$$f_1(t) = 0;\quad f_2(t) = \frac{E}{R_1 C} t + u_C(t_1);$$

and interval $t_2 \le t \le t$

$$K_1(t,\tau) = -\frac{R_2}{L};\ K_2(t,\tau) = \frac{1}{L};\ K_3(t,\tau) = -\frac{1}{C};$$

$$K_4(t,\tau) = -\frac{1}{R_1 C};\ f_1(t) = i_L(t_2);\quad f_2(t) = \frac{E}{R_1 C} t + u_C(t_1);$$

$E = 50\,\text{V}$; $R_1 = R_2 = 15\,\Omega$; $C = 20\,\mu F$; $t_1 = 0,\ t_2 = 0{,}00025\,s$.

The obtained transient current $i_L(t)$ and transient voltage $u_C(t)$ are shown on fig.4 and fig.5 appropriately.

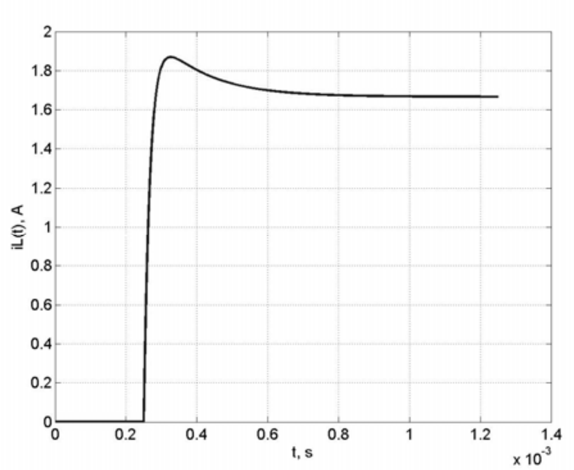

Fig.5. Transient current $i_L(t)$

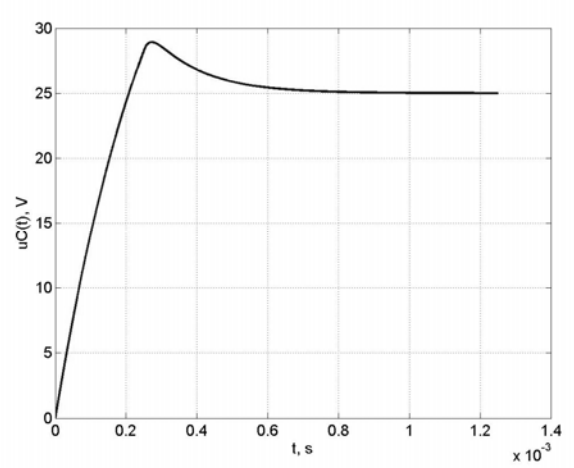

Fig.6. Transient voltage $u_C(t)$

Conclusion

The mathematical models of transients in electric circuits based on integral equations had to be used widely. It provides comfortable description of processes, stability of numerical solution, assumes the existence of discontinuous solutions.

REFERENCES

[1] Верлань А.Ф., Сизиков В.С. Интегральные уравнения: методы, алгоритмы, программы. – К.: *Наукова думка.* – 1986. – С. 542.

[2] Манжиров А.В., Полянин А.Д. Методы решения интегральных уравнений: Справочник. – М.: Факториал. – 1999. – С.272.

[3] Фільц Р.В. Метод розрахунку перехідних процесів у лінійних електричних колах з постійними параметрами за інтегральними рівняннями, Вісник НУ "Львівська політехніка" Електроенергетичні та електромеханічні системи, (2005), №544, 137-145.

Author*: Dr. Orest Hamola Lviv Polytechnic National University, 12 St.Bandera st, 79013 Lviv, E-mail: oresthamola@polynet.lviv.ua.*