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Mutual inductance of long rectangular conductors

Abstract. *In this paper, using a definition of a mutual inductance for two conductors of any shape and finite lengths, the new exact closed formula for mutual inductance between two long rectangular bars is proposed. In case of direct current (DC) or low frequency (LF) this inductance is given by analytical formula. The mutual inductance between two long thin tapes is also presented.*

Streszczenie. *Stosując definicję indukcyjności wzajemnej między dwoma przewodami dowolnych kształtów i skończonej długości w pracy zaproponowano nowy dokładny wzór na obliczanie indukcyjności wzajemnej miedzy dwoma długimi przewodami o przekroju prostokątnym. W przypadku prądu stałego lub niskiej częstotliwości indukcyjność tę wyrażono wzorem analitycznym. Podano również wzór na indukcyjność wzajemną między dwoma długimi przewodami taśmowymi. (**Indukcyjność wzajemna długich przewodów o przekroju prostokątnym**)*

Key words: rectangular busbar, mutual inductance, electromagnetic field
Słowa kluczowe: prostokątny przewód szynowy, indukcyjność wzajemna, pole elektromagnetyczne

Introduction

Real system lumped conductors can be modeled by a connection of resistances, self and mutual inductances. The self and mutual inductances play an important role not only in power circuits, but also in printed circuit board (PCB) lands [1-3]. Formulae for the mutual inductances of set of conductors of rectangular cross-section are the subjects of many electrical papers and books. They are mathematically complex and their demonstrations are usually omitted and only the approximate formulae are given as thought they were exact. The most significant of them are: Grover's given in [2-5], Kalantarov and Tseitlin's presented in [6], Strunsky's shown in [7], Ruehli's presented in [5] and [8] as well as Hoer and Love's shown in [3], [5] and [9]. The mutual inductance can be calculate by many numerical methods. Zhong and Koh express in [10] the mutual inductance as a weighted sum of self inductances. C. R. Paul in [3] considers breaking the cross section of a rectangular conductor into "subbars" of rectangular cross section. Then the mutual inductance between the subbars is approximated as between filaments at the centers of the subbars. All the subbars are connected in parallel and from this circuit he determines the "self and impedance" matrix. Finally he obtains the effective impedance and mutual inductance between bars from it.

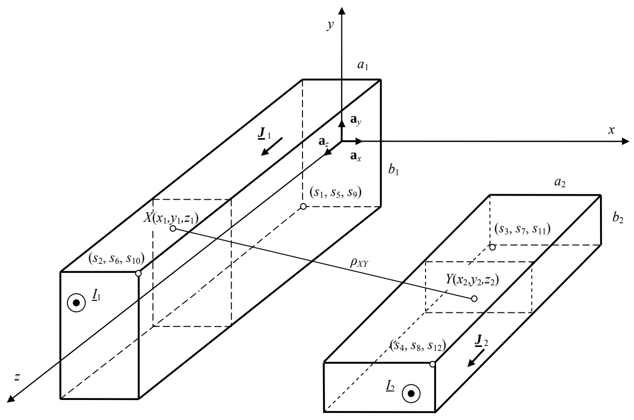

Fig. 1. Two parallel conductors of rectangular cross section with currents $\underline{I}_1$ and $\underline{I}_2$

In this paper a new method for calculating mutual inductance is presented. The method results in a system of two integral Fredholm's equations. We compare our analytical formulae with several well-known ones given in the literature for DC, low frequency or parallel thin tapes.

We consider a general case of two parallel conductors of rectangular cross section shown in Fig.1. The positions of conductors are determined by coordinates of diagonal corner points: (s_1, s_5, s_9), (s_2, s_6, s_{10}) of the first wire of dimensions $a_1 \times b_1 \times l_1$ and (s_3, s_7, s_{11}), (s_4, s_8, s_{12}) of the second wire of dimensions $a_2 \times b_2 \times l_2$. We assume the conductors to be parallel.

Definition of mutual inductance

The definition of mutual impedance between two straight conductors is given in [11, 12] i.a. by following formula

$$(1) \qquad \underline{Z}_{12} = \frac{\mathrm{j}\,\omega\,\mu_0}{4\pi\,\underline{I}_1^*\underline{I}_2} \int\limits_{v_1}\int\limits_{v_2} \frac{\underline{J}_{22}(Y)\underline{J}_{11}^*(X)}{\rho_{XY}}\,\mathrm{d}v\,\mathrm{d}v$$

where $\underline{J}_{22}(Y)$ is the complex current density at source point $Y = Y(x_2, y_2, z_2) \in S_2$, $\underline{J}_{11}^*(X)$ is the complex conjugate current density at point of observation $X = X(x_1, y_1, z_1) \in S_1$, v_1 and v_2 are conductors' volumes. Distance between the point of observation *X* and the source point *Y* (Fig.1) is $\rho_{XY} = \sqrt{r_{XY}^2 + (z_2 - z_1)^2}$ where $r_{XY} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. If conductors have a constant cross-sectional area S_1 and S_2 along theirs lengths, in case of DC, low frequency (for busbars of dimensions used in electrical power distribution system [1]) or for a thin strip conductors (in printed circuit board [2, 3]) we can assume that the current density is constant and given as $\underline{J}_{11}(X) = \underline{I}_1 / S_1$ and $\underline{J}_{22}(X) = \underline{I}_2 / S_2$ then, from the formulae (1), we obtain the mutual inductance between two straight parallel conductors

$$(2) \qquad M = M_{12} = M_{21} = \frac{\mu_0}{4\pi\,S_1\,S_2} \int\limits_{v_1}\int\limits_{v_2} \frac{1}{\rho_{XY}}\,\mathrm{d}\,v_1\,\mathrm{d}\,v_2$$

Mutual inductance between parallel conductors of rectangular cross section

The mutual inductance between two rectangular conductors of dimensions $a_1 \times b_1 \times l_1$ and $a_2 \times b_2 \times l_2$ shown in Fig. 1 is given by formula

(3) $$M = \frac{\mu_0}{4\pi} \frac{1}{a_1\, a_2\, b_1\, b_2} F$$

where

(3a) $$F = \int_{s_{11}}^{s_{12}} \int_{s_9}^{s_{10}} \int_{s_7}^{s_8} \int_{s_5}^{s_6} \int_{s_3}^{s_4} \int_{s_1}^{s_2} \frac{1}{\rho_{XY}} \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}z_1 \mathrm{d}z_2$$

The double definite integral

(4) $$f(x,y) = \int_0^l \int_0^l \frac{1}{\rho_{XY}} \mathrm{d}z_1 \mathrm{d}z_2 = \\ = 2\,l \left(\ln \frac{l + \sqrt{l^2 + r_{XY}^2}}{r_{XY}} - \frac{\sqrt{l^2 + r_{XY}^2}}{l} + \frac{r_{XY}}{l} \right)$$

If $l >> r_{XY}$ the function $f(x,y)$ becomes

(5) $$f(x,y) = 2\,l \left(\ln \frac{2\,l}{r_{XY}} - 1 \right)$$

and the mutual inductance between two rectangular conductors expresses by formula

(6) $$M = \frac{\mu_0\, l}{2\pi} \left[\ln(2\,l) - 1 + G \right]$$

where

(7) $$G = -\frac{1}{2\, a_1\, a_2\, b_1\, b_2} \times \\ \int_{s_7}^{s_8} \int_{s_5}^{s_6} \int_{s_3}^{s_4} \int_{s_1}^{s_2} \ln\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 \right] \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}y_1 \mathrm{d}y_2$$

is a quadruple definite integral of four variables. In general case this integral is very difficult to calculate. But if we substitute $x = x_2 - x_1$ and $y = y_2 - y_1$ then (7) becomes

(8)
$$G(x,y) = \frac{1}{288\, a_1\, a_2\, b_1\, b_2} \left\{ 150\, x^2 y^2 - 6 \left[8\, x\, y^3 \tan^{-1} \frac{x}{y} + 8\, x^3 y \tan^{-1} \frac{y}{x} - \left(x^4 - 6\, x^2 y^2 + y^4 \right) \ln\left(x^2 + y^2 \right) \right] \right\}$$

and now we determine the mutual inductance between two long parallel conductors of rectangular cross section

(9) $$M = \frac{\mu_0\, l}{2\pi} \left\{ \ln(2\,l) - 1 + \left[\left[G(x,y) \right]_{\substack{(x) \\ s_1 - s_3,\, s_2 - s_4}}^{s_1 - s_4,\, s_2 - s_3} \right]_{\substack{(y) \\ s_5 - s_7,\, s_6 - s_8}}^{s_5 - s_8,\, s_6 - s_7} \right\} = \frac{\mu_0\, l}{2\pi} \left\{ \ln(2\,l) - 1 + \sum_{i=1}^{i=4} \sum_{j=1}^{j=4} (-1)^{i+j} G(p_i, q_j) \right\}$$

In the particularly case if $s_1 = 0$, $s_2 = a = s_4 - s_3$, $s_5 = s_7 = 0$, $s_6 = s_8 = b$ and distance between bars is $d = s_3$ the conductor are the same and mutual inductance between them is given by following formula

(10) $$M = \frac{\mu_0\, l}{2\pi} \left\{ \ln(2\,l) - 1 + \left[\left[G(x,y) \right]_{\substack{(x) \\ -d,\, -d}}^{-a-d,\, a-d} \right]_{\substack{(y) \\ 0,\, 0}}^{-b,\, b} \right\} = \frac{\mu_0\, l}{2\pi} \left\{ \ln(2\,l) - 1 + \sum_{i=1}^{i=4} \sum_{j=1}^{j=4} (-1)^{i+j} G(p_i, q_j) \right\}$$

On the basis of (10) we have the analytical formula for the mutual inductance between two long parallel straight conductor of rectangular cross section

(11) $$M = \frac{\mu_0\, l}{2\pi} \left\{ \begin{array}{l} \ln \dfrac{2\,l}{\sqrt[4]{\left[b^2 + (a-d)^2 \right]\left[b^2 + (a+d)^2 \right]}} + \dfrac{13}{12} - \dfrac{1}{3} \dfrac{(a-d)^3}{a^2 b} \tan^{-1} \dfrac{b}{a-d} - \dfrac{1}{3} \dfrac{b(a-d)}{a^2} \tan^{-1} \dfrac{a-d}{b} + \dfrac{2}{3} \dfrac{d^3}{a^2 b} \tan^{-1} \dfrac{b}{a} + \dfrac{2}{3} \dfrac{bd}{a^2} \tan^{-1} \dfrac{d}{b} - \\ \dfrac{1}{3} \dfrac{(a+d)^3}{a^2 b} \tan^{-1} \dfrac{b}{a+d} - \dfrac{1}{3} \dfrac{b(a+d)}{a^2} \tan^{-1} \dfrac{a+d}{b} - \dfrac{1}{24} \dfrac{(a-d)^4}{a^2 b^2} \ln \dfrac{(a-d)^2}{b^2 + (a-d)^2} - \dfrac{1}{24} \dfrac{(a+d)^4}{a^2 b^2} \ln \dfrac{(a+d)^2}{b^2 + (a+d)^2} - \\ \dfrac{1}{24} \dfrac{b^2 - 6d^2}{a^2} \ln \dfrac{b^2 + d^2}{b^2 + (a-d)^2} - \dfrac{1}{24} \dfrac{b^2 - 6d^2}{a^2} \ln \dfrac{b^2 + d^2}{b^2 + (a+d)^2} - \dfrac{1}{12} \dfrac{d^4}{a^2 b^2} \ln \dfrac{b^2 + d^2}{d^2} - \dfrac{1}{2} \dfrac{d}{a} \ln \dfrac{b^2 + (a+d)^2}{b^2 + (a-d)^2} \end{array} \right\}$$

For the chosen traverse dimensions and different lengths of two the same busbars the calculations of their mutual inductance have been made according to all previous, shown above, formulae – Table 1.

Table 1. Mutual inductance between two busbars of rectangular cross section for DC or low frequency

Busbar: $a = 0.08\,\mathrm{m}$; $b = 0.007\,\mathrm{m}$; $d = 2\,a$						
	Ruehli	Grover	Strunsky	Hoer	Eq. (3)	Eq. (11)
l (m)	*L* (nH)	*L* (nH)	*L* (nH)	*L* (nH)	*L* (nH)	*L* (nH)
0.01 *a*	0.000399	0.000399	0.000399	0.000392	0.000392	negative
0.10 *a*	0.039991	0.039991	0.039985	0.039226	0.039226	negative
1.00 *a*	3.922301	3.922301	3.921699	3.849786	3.849786	negative
10.0 *a*	238.8215	238.8215	238.8005	236.2490	236.2490	211.9011
100 *a*	5800.112	5800.112	5799.862	5769.184	5769.184	5803.147
1000 *a*	94556.06	94556.06	94553.52	94240.43	94373.47	94872.83

Mutual inductance of thin tapes

For two parallel long thin tapes of widths a_1 and a_2, the same length l, thickness $b_1 \approx 0$ and $b_2 \approx 0$ and distance *d* between them the function $G(x,y)$ becomes

$$(12)\ G(x) = -\frac{1}{2\,a_1\,a_2}\iint \ln\left[d^2+x^2\right]\,\mathrm{dx}\,\mathrm{dx} = \frac{1}{4\,a_1\,a_2}\left[3\,x^2 - 4\,x\,d\,\tan^{-1}\frac{x}{d} - \left(x^2-d^2\right)\,\ln\left(x^2+d^2\right)\right]$$

In the particularly case if $s_1 = s_3 = 0$, $s_2 = s_4 = a$, $s_5 = s_6 = 0$ and distance between tapes is $d = s_7 - s_5 = s_8 - s_6$ the conductors are the same and mutual inductance between them is given by following formula

$$(13)\qquad M = \frac{\mu_0\,l}{2\,\pi}\left\{\ln(2\,l) - 1 + \left[G(x)\right]\substack{-a,a\\(x)\\0,0}\right\} = \frac{\mu_0\,l}{2\,\pi}\left\{\ln(2\,l) - 1 + \sum_{i=1}^{i=4}(-1)^{i+1}G(p_i)\right\}$$

On the basis of (13) we have the analytical formula for the mutual inductance between two long parallel straight thin tapes

$$(14)\qquad M = \frac{\mu_0\,l}{2\,\pi}\left[\ln\frac{2\,l}{d} + \frac{1}{2} - 2\,\frac{d}{a}\tan^{-1}\frac{a}{d} - \frac{1}{2}\left(1-\frac{d^2}{a^2}\right)\ln\left(1+\frac{a^2}{d^2}\right)\right]$$

For the mutual inductance of two the same thin tapes of width *a*, thickness *b* and length *l* above formulae give results sown in Table 2.

Table 2. Mutual inductance between two thin tapes of rectangular cross section for DC or low frequency

Thin tapes: $a = 0.5\,\mu$m; $b = 0.1\,\mu$m; $d = 2\,a$						
	Ruehli	Grover	Strunsky	Hoer	Eq. (3)	Eq. (14)
l (m)	*L* (pH)	*L* (pH)	*L* (pH)	*L* (pH)	*L* (pH)	*L* (pH)
0.01 *a*	0.000002	0.000002	0.000002	0.000002	0.000002	negative
0.10 *a*	0.000249	0.000249	0.000249	0.000245	0.000245	negative
1.00 *a*	0.024514	0.024514	0.024494	0.024088	0.024088	negative
10.0 *a*	1.492634	1.492634	1.491952	1.477150	1.477150	1.282709
100 *a*	36.25070	36.25070	36.24254	36.06343	36.06343	35.85295
1000 *a*	590.9754	590.9754	590.8924	589.0107	589.0030	588.7880

Conclusions

The mutual inductance between two long parallel conductors of rectangular cross section and of any dimensions including the long thin tapes are given by coordinates of diagonal corner points of the first conductor and the second one. These formulae can be used for any dimensions of long conductors and for any position between them.

In addition we have also obtained analytical forms for mutual inductance between long rectangular conductors as well as between the long thin tapes. Our formulae are analytically simple and can also replace the traditional tables and working ones.

These formulae can be used in the methods of numerical calculation of AC mutual inductance of rectangular conductors.

Acknowledgments

This work is financed by the National Science Centre, Poland as research project N N511 312540.

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