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Fast algorithms to compute matrix-vector products for Toeplitz and Hankel matrices

***Abstract**. The paper presents practical and effective algorithms to calculate the Toeplitz/Hankel matrix by a vector product that are recursiveless modification of Karatsuba's method. Unlike traditional algorithms, in this case using the FFT is not required. Realization of the developed algorithms involves the use of unconventional ways of choosing the elements of the initial transformation matrix during the formation of an array of data to be processed. We have called these methods respectively "7-order" technique and "mirrored 7-order" technique. This approach allows us to calculate the vector-matrix products in parallel way with a reduced number of hardware multipliers and adders.*

***Streszczenie.** W artykule zaprezentowano praktyczne i efektywne algorytmy obliczania iloczynu macierzy Toeplitza/Hankela przez wektor będące pozbawionymi rekursji modyfikacjami metody Karatsuby. W odróżnieniu od tradycyjnych algorytmów, stosowanie FFTw tym przypadku nie jest konieczne. Realizacja opracowanych algorytmów wykorzystuje niekonwencjonalny sposób doboru elementów macierzy transformacji podczas tworzenia tablicy danych do przetworzenia(nazwane odpowiednio „7-order" i „mirrored 7-order"). Podejście to pozwala na wyznaczanie iloczynów macierzowo-wektorowych równolegle przy jednoczesnej redukcji mnożarek i sumatorów. (**Szybki algorytm obliczania iloczynu macierzy Toeplitza/Hankela**)*

Słowa kluczowe: Szybki algorytm, iloczyn macierzowo-wektorowy, macierz Toeplitz'a, macierz Hankel'a.
Keywords: Fast algorithms, matrix-vector product, Toeplitzmatrix, Hankel matrix.

Introduction

Matrices with special structure play a fundamental role in systems and control [1-7]. In digital signal processing, data coding and statistics is often necessary to compute the product of a Toeplitz matrix $\mathbf{T}_N = \|t_{i,j}\|$ by arbitrary vector $\mathbf{X}_{N\times1} = [x_0, x_1, ..., x_{N-1}]^{\mathrm{T}}$:

(1) $$\mathbf{Y}_{N\times1}^{(T)} = \mathbf{T}_N \mathbf{X}_{N\times1}$$

where $\mathbf{Y}_{N\times1}^{(T)} = [\breve{y}_0, \breve{y}_1, ..., \breve{y}_{N-1}]^{\mathrm{T}}$ is a vector which elements are the result of multiplication, and Toeplitz matrix is defined as:

(2) $$\mathbf{T}_N = \begin{bmatrix} t_0 & t_1 & \cdots & t_{N-1} \\ t_{-1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_1 \\ t_{-(N-1)} & \cdots & t_{-1} & t_0 \end{bmatrix}$$

where t_i are scalars, or their block equivalent, where t_i are then $k \times k$ matrices [3].

In control, one often has to deal with Hankel matrix. Then the product of a Hankel matrix $\mathbf{H}_N = \|h_{i,j}\|$ by vector $\mathbf{X}_{N\times1}$ can be represented as follows:

(3) $$\mathbf{Y}_{N\times1}^{(H)} = \mathbf{H}_N \mathbf{X}_{N\times1}$$

where $\mathbf{Y}_{N-1}^{(H)} = [\hat{y}_0, \hat{y}_1, ..., \hat{y}_{N-1}]^{\mathrm{T}}$ is a vector which elements are the result of multiplication, and Hankel matrix is defined as [3]:

(4) $$\mathbf{H}_N = \begin{bmatrix} h_0 & h_1 & \cdots & h_{N-1} \\ h_1 & \iddots & \iddots & \vdots \\ \vdots & \iddots & \iddots & h_1 \\ h_{N-1} & \cdots & \cdots & h_{2N-2} \end{bmatrix}$$

where h_i are scalars, or their block equivalent, where h_i are then $k \times k$ matrices. Naive ways of implementing the calculations in accordance with (1) and (3) requires N^2 multiplications and $N(N-1)$ additions. The Karatsuba's approach [8,9] provides significant reduction of these operations but at a cost of parallel computation and need of using unwanted recursion, there are also no actual algorithms defined for matrices of given order. For large N the calculating duration of the vector-matrix product can be so long that it will not give the result in the allotted time. In this case, there is a need to develop new parallel algorithmic solutions to reduce the number of arithmetic operations and/or time of execution required for realization of calculations [4-6]. There are large number of scientific publications, which declare the existence of ways for fast evaluation of products of Toeplitz and Hankel matrices by vectors. Known fast algorithms are based on embedding the Toeplitz matrix into a circulant matrix of larger size, and calculating the product of the resultant matrix by a vector. Hence the Toeplitz (Hankel) matrix-vector multiplication can be computed as circulant matrix-vector product. This product can be computed using fast Fourier transform (FFT) or fast convolution algorithms. Using these algorithms leads to data redundancy and requires *O*(2*N*log2*N*) operations [10]. In this paper we proposed and described in detail the new fast algorithms for multiplying Toeplitz and Hankel matrices by vector with reduced number of arithmetical operations or hardware resources.

Synthesis of computational procedures

It is clear that Hankel matrix can be easily transformed into Toeplitz matrix and vice versa by a simple rows or columns reversal, so only one algorithm could be used. However we decided to describe two separate approaches so that additional expenses for matrix shuffle could be avoided. If there is no such a need, there is no problem to use only one of proposed solution to both matrices types.

Let $N = 2^m$, $m = 1, 2, ..., n$.

Initially, we introduce a few matrices:

$$\mathbf{A}_{3^k 2^{m-k} \times 3^{k-1} 2^{m-k+1}}^{(k)} = \mathbf{I}_{3^{k-1}} \otimes \mathbf{P}_{3\times2} \otimes \mathbf{I}_{2^{m-k}} ,$$

$$\widetilde{\mathbf{A}}^{(k)}_{3^{k-1}2^{m-k+1}\times 3^k 2^{m-k}} = \mathbf{I}_{3^{k-1}} \otimes \mathbf{L}_{2\times 3} \otimes \mathbf{I}_{2^{m-k}} ,$$

$$\mathbf{B}^{(k)}_{3^k 2^{m-k}\times 3^{k-1}2^{m-k+1}} = \mathbf{I}_{3^{k-1}} \otimes \mathbf{G}_{3\times 2} \otimes \mathbf{I}_{2^{m-k}} ,$$

$$k = 1,2,...,m ,$$

$$\mathbf{P}_{3\times 2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 1 \end{bmatrix},\ \mathbf{L}_{2\times 3} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix},\ \mathbf{G}_{3\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Here, the symbol $\otimes$ - denotes the Kronecker product of two matrices [11].

Next, we define the diagonal matrices $\mathbf{S}^{(T)}_{3^m}$, $\mathbf{S}^{(H)}_{3^m}$. The elements of these matrices are elements of the vectors $\mathbf{S}^{(T)}_{3^m\times 1} = [s'_0, s'_1,..., s'_{3^m-1}]^{\mathrm{T}}$, and $\mathbf{S}^{(H)}_{3^m\times 1} = [s''_0, s''_1,..., s''_{3^m-1}]^{\mathrm{T}}$, respectivilly, located on its diagonals in ascending indexes. In turn, the elements of the vectors $\mathbf{S}^{(T)}_{3^m\times 1}$ $\mathbf{S}^{(H)}_{3^m\times 1}$ can be computed as follows:

(5) $$\mathbf{S}^{(T)}_{3^m\times 1} = \mathbf{C}^{(m)}_{3^m} \cdots \mathbf{C}^{(2)}_{3^m} \mathbf{C}^{(1)}_{3^m} \mathbf{Z}^{(T)}_{3^m\times 1}$$

(6) $$\mathbf{S}^{(H)}_{3^m\times 1} = \mathbf{C}^{(m)}_{3^m} \cdots \mathbf{C}^{(2)}_{3^m} \mathbf{C}^{(1)}_{3^m} \mathbf{Z}^{(H)}_{3^m\times 1}$$

where

$$\mathbf{C}^{(k)}_{3^m} = \mathbf{I}_{3^{k-1}} \otimes \mathbf{\Gamma}_3 \otimes \mathbf{I}_{3^{m-k}},\ \mathbf{\Gamma}_3 = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

and $\mathbf{Z}^{(T)}_{3^m\times 1} = [\breve{z}_0, \breve{z}_1,..., \breve{z}_{3^m-1}]^{\mathrm{T}}$, $\mathbf{Z}^{(H)}_{3^m\times 1} = [\widehat{z}_0, \widehat{z}_1,..., \widehat{z}_{3^m-1}]^{\mathrm{T}}$ - are vectors formed from elements of the matrix $\mathbf{T}_N$ and $\mathbf{H}_N$ selected on the basis of the "7-order" technique (or "mirrored 7-order" technique - for Hankel matrix). The essence of the mentioned techniques (for $N = 8$) is clear by considering figure 1 (for Toeplitz matrix) and figure 2 (for Hankel matrix).

Using above defined matrices, we can write procedures for fast computation of the matrix-vector products for Toeplitz and Hankel matrices:

(7) $$\mathbf{Y}^{(T)}_{2^m\times 1} = \widetilde{\mathbf{A}}^{(1)}_{2^m\times 3\cdot 2^{m-1}} \widetilde{\mathbf{A}}^{(2)}_{2^{m-1}\times 3\cdot 2^{m-2}} \cdots \widetilde{\mathbf{A}}^{(m)}_{2^m\times 3\cdot 2^{m-1}} \times \\ \times \mathbf{S}^{(T)}_{3^m} \mathbf{A}^{(m)}_{2^m\times 3\cdot 2^{m-1}} \mathbf{A}^{(m-1)}_{2^m\times 3\cdot 2^{m-1}} \cdots \mathbf{A}^{(1)}_{2^m\times 3\cdot 2^{m-1}} \mathbf{X}_{2^m\times 1}$$

(8) $$\mathbf{Y}^{(H)}_{2^m\times 1} = \widetilde{\mathbf{A}}^{(1)}_{2^m\times 3\cdot 2^{m-1}} \widetilde{\mathbf{A}}^{(2)}_{2^{m-1}\times 3\cdot 2^{m-2}} \cdots \widetilde{\mathbf{A}}^{(m)}_{2^m\times 3\cdot 2^{m-1}} \times \\ \times \mathbf{S}^{(H)}_{3^m} \mathbf{B}^{(m)}_{2^m\times 3\cdot 2^{m-1}} \mathbf{B}^{(m-1)}_{2^m\times 3\cdot 2^{m-1}} \cdots \mathbf{B}^{(1)}_{2^m\times 3\cdot 2^{m-1}} \mathbf{X}_{2^m\times 1}$$

Let us consider the synthesis of fast algorithms for computing Toeplitz/Hankel matrix-vector products for $N = 8$. In this case the corresponding matrices take the following form:

$$\mathbf{T}_8 = \begin{bmatrix} t_0 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 & t_7 \\ t_{-1} & t_0 & t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\ t_{-2} & t_{-1} & t_0 & t_1 & t_2 & t_3 & t_4 & t_5 \\ t_{-3} & t_{-2} & t_{-1} & t_0 & t_1 & t_2 & t_3 & t_4 \\ t_{-4} & t_{-3} & t_{-2} & t_{-1} & t_0 & t_1 & t_2 & t_3 \\ t_{-5} & t_{-4} & t_{-3} & t_{-2} & t_{-1} & t_0 & t_1 & t_2 \\ t_{-6} & t_{-5} & t_{-4} & t_{-3} & t_{-2} & t_{-1} & t_0 & t_1 \\ t_{-7} & t_{-6} & t_{-5} & t_{-4} & t_{-3} & t_{-2} & t_{-1} & t_0 \end{bmatrix},$$

$$\mathbf{H}_8 = \begin{bmatrix} h_0 & h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 \\ h_1 & h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 \\ h_2 & h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 \\ h_3 & h_4 & h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} \\ h_4 & h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} \\ h_5 & h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} \\ h_6 & h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} & h_{13} \\ h_7 & h_8 & h_9 & h_{10} & h_{11} & h_{12} & h_{13} & h_{14} \end{bmatrix},$$

$$\mathbf{A}^{(1)}_{12\times 8} = \mathbf{P}_{3\times 2} \otimes \mathbf{I}_4 = \left[\begin{array}{c:c} \mathbf{0}_4 & \mathbf{I}_4 \\ \hdashline \mathbf{I}_4 & \mathbf{0}_4 \\ \hdashline \mathbf{I}_4 & \mathbf{I}_4 \end{array}\right],$$

$$\mathbf{A}^{(2)}_{18\times 12} = \mathbf{I}_3 \otimes \mathbf{P}_{3\times 2} \otimes \mathbf{I}_2 =$$

$$= \left[\begin{array}{cc:cc:cc} \mathbf{0}_2 & \mathbf{I}_2 & & & & \\ \mathbf{I}_2 & \mathbf{0}_2 & \multicolumn{2}{c:}{\mathbf{0}_{6\times 4}} & \multicolumn{2}{c}{\mathbf{0}_{6\times 4}} \\ \mathbf{I}_2 & \mathbf{I}_2 & & & & \\ \hdashline & & \mathbf{0}_2 & \mathbf{I}_2 & & \\ \multicolumn{2}{c:}{\mathbf{0}_{6\times 4}} & \mathbf{I}_2 & \mathbf{0}_2 & \multicolumn{2}{c}{\mathbf{0}_{6\times 4}} \\ & & \mathbf{I}_2 & \mathbf{I}_2 & & \\ \hdashline & & & & \mathbf{0}_2 & \mathbf{I}_2 \\ \multicolumn{2}{c:}{\mathbf{0}_{6\times 4}} & \multicolumn{2}{c:}{\mathbf{0}_{6\times 4}} & \mathbf{I}_2 & \mathbf{0}_2 \\ & & & & \mathbf{I}_2 & \mathbf{I}_2 \end{array}\right],$$

$$\mathbf{A}^{(3)}_{27\times 18} = \mathbf{I}_9 \otimes \mathbf{P}_{3\times 2} = diag\left[\begin{array}{c} \mathbf{P}_{3\times 2} \\ \hdashline \mathbf{P}_{3\times 2} \\ \hdashline \vdots \\ \hdashline \mathbf{P}_{3\times 2} \end{array}\right],$$

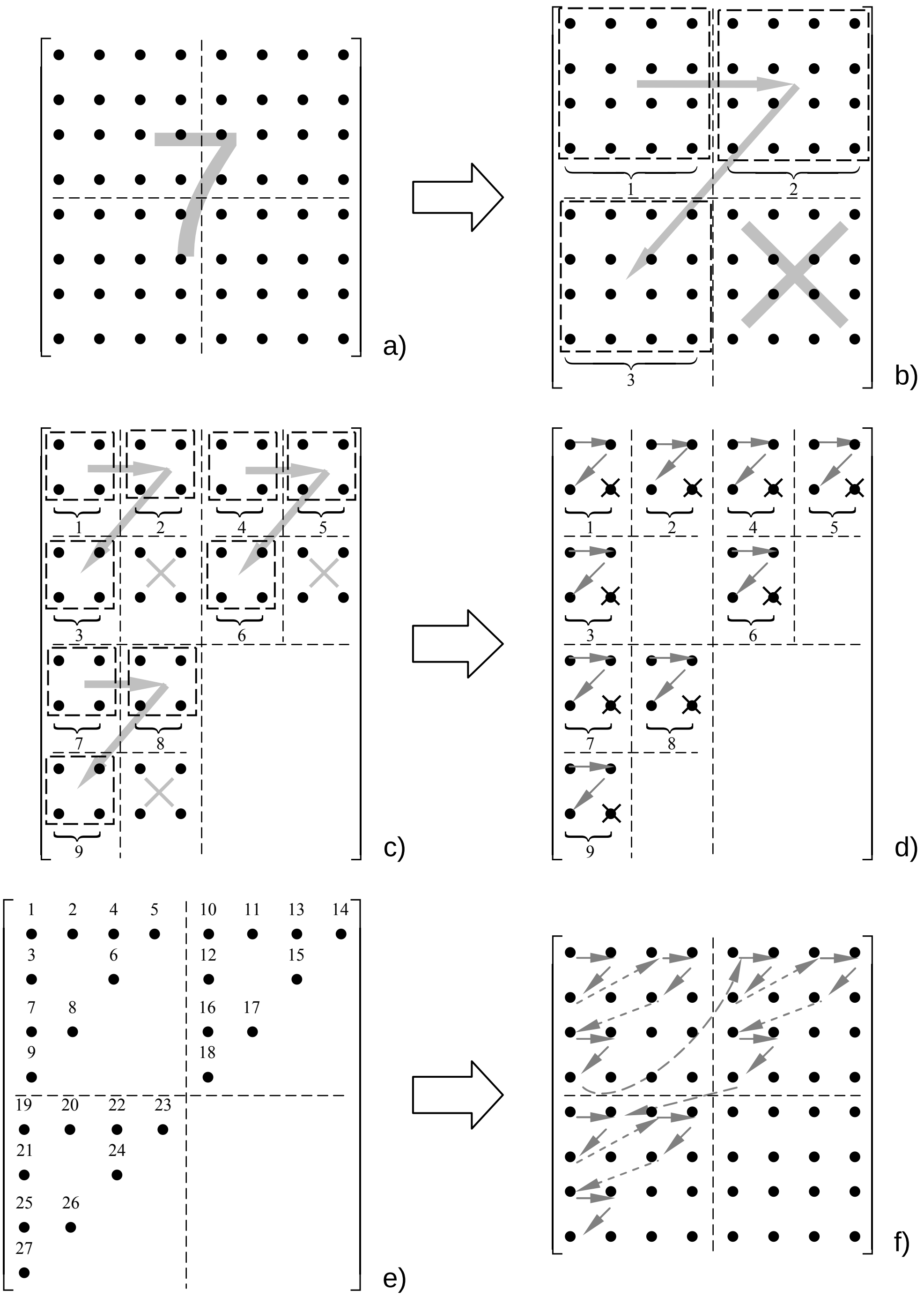
$$\widetilde{\mathbf{A}}^{(1)}_{8\times 12} = \mathbf{L}_{2\times 3} \otimes \mathbf{I}_4 = \left[\begin{array}{c:c:c} \mathbf{I}_4 & \mathbf{0}_4 & \mathbf{I}_4 \\ \hdashline \mathbf{0}_4 & \mathbf{I}_4 & \mathbf{I}_4 \end{array}\right],$$

$$\widetilde{\mathbf{A}}^{(2)}_{12\times 18} = \mathbf{I}_3 \otimes \mathbf{L}_{2\times 3} \otimes \mathbf{I}_2 =$$

$$= \left[\begin{array}{ccc:ccc:ccc} \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \multicolumn{3}{c:}{\mathbf{0}_{4\times 6}} & \multicolumn{3}{c}{\mathbf{0}_{4\times 6}} \\ \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{I}_2 & & & & & & \\ \hdashline \multicolumn{3}{c:}{\mathbf{0}_{4\times 6}} & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{I}_2 & \multicolumn{3}{c}{\mathbf{0}_{4\times 6}} \\ & & & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{I}_2 & & & \\ \hdashline \multicolumn{3}{c:}{\mathbf{0}_{4\times 6}} & \multicolumn{3}{c:}{\mathbf{0}_{4\times 6}} & \mathbf{I}_2 & \mathbf{0}_2 & \mathbf{I}_2 \\ & & & & & & \mathbf{0}_2 & \mathbf{I}_2 & \mathbf{I}_2 \end{array}\right],$$

$$\widetilde{\mathbf{A}}^{(3)}_{27\times 18} = \mathbf{I}_9 \otimes \mathbf{L}_{2\times 3} = diag\left[\begin{array}{c} \mathbf{L}_{2\times 3} \\ \hdashline \mathbf{L}_{2\times 3} \\ \hdashline \vdots \\ \hdashline \mathbf{L}_{2\times 3} \end{array}\right],$$

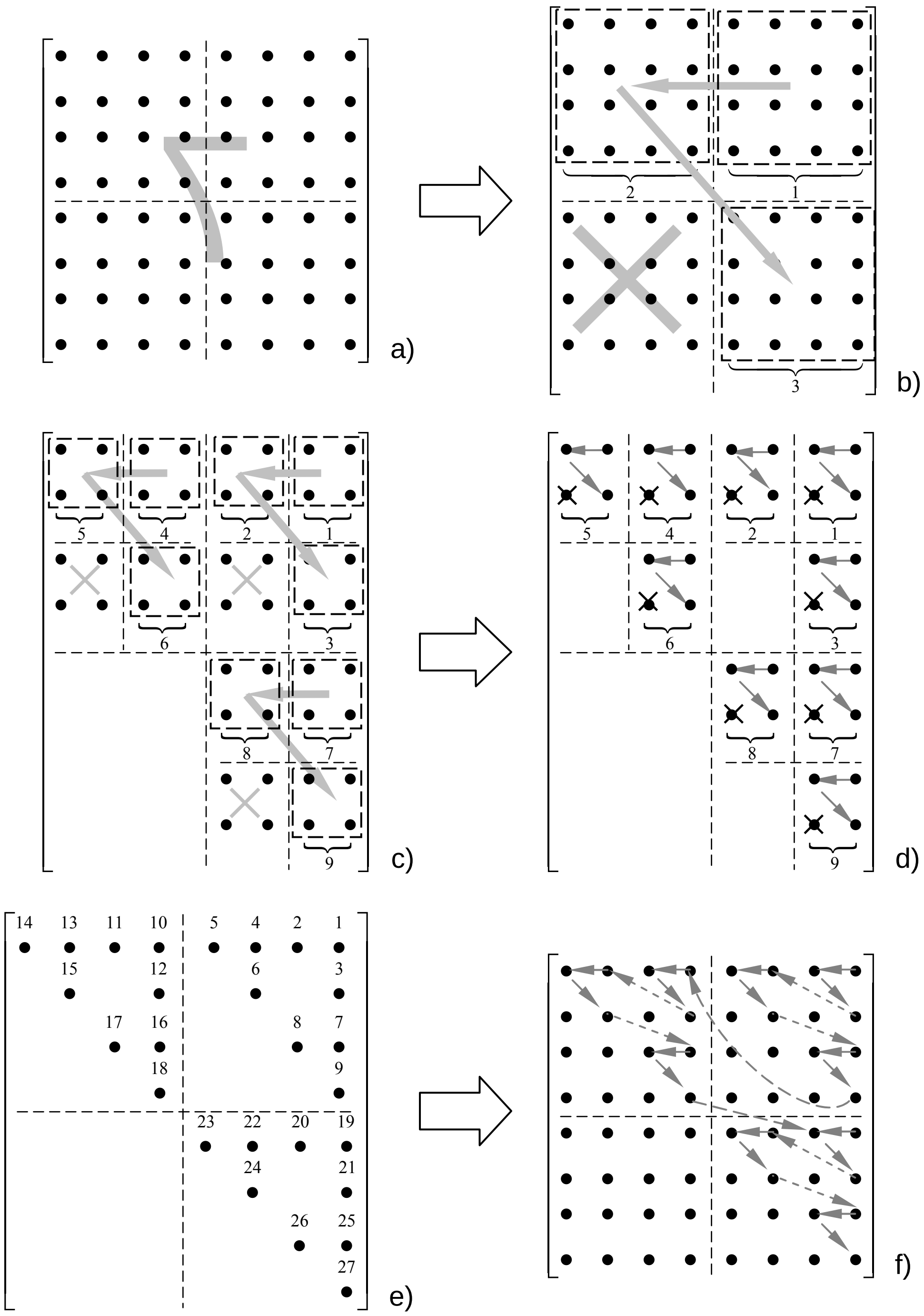
$$\mathbf{B}^{(1)}_{12\times 8} = \mathbf{G}_{3\times 2} \otimes \mathbf{I}_4 = \left[\begin{array}{c:c} \mathbf{I}_4 & \mathbf{0}_4 \\ \hdashline \mathbf{0}_4 & \mathbf{I}_4 \\ \hdashline \mathbf{I}_4 & \mathbf{I}_4 \end{array}\right],$$



Rys.1. Illustration of the mechanism to choice of elements of Toeplitz matrix in accordance with the "7"-ordered technique

$$\mathbf{B}^{(3)}_{27\times18} = \mathbf{I}_9 \otimes \mathbf{G}_{3\times2} = \left[\begin{array}{c:c:c:c} \begin{array}{c:c} 1 & 0 \\ \hdashline 0 & 1 \\ \hdashline 1 & 1 \end{array} & \mathbf{0}_{3\times2} & \mathbf{0}_{3\times2} & \mathbf{0}_{3\times2} \\ \hdashline \mathbf{0}_{3\times2} & \begin{array}{c:c} 1 & 0 \\ \hdashline 0 & 1 \\ \hdashline 1 & 1 \end{array} & \mathbf{0}_{3\times2} & \mathbf{0}_{3\times2} \\ \hdashline \vdots & \vdots & \ddots & \vdots \\ \hdashline \mathbf{0}_{3\times2} & \mathbf{0}_{3\times2} & \mathbf{0}_{3\times2} & \begin{array}{c:c} 1 & 0 \\ \hdashline 0 & 1 \\ \hdashline 1 & 1 \end{array} \end{array}\right],$$

$$\mathbf{B}^{(2)}_{18\times12} = \mathbf{I}_3 \otimes \mathbf{G}_{3\times2} \otimes \mathbf{I}_2 = \left[\begin{array}{c:c:c} \begin{array}{c:c} \mathbf{I}_4 & \mathbf{0}_4 \\ \hdashline \mathbf{0}_4 & \mathbf{I}_4 \\ \hdashline \mathbf{I}_4 & \mathbf{I}_4 \end{array} & \mathbf{0}_{6\times4} & \mathbf{0}_{6\times4} \\ \hdashline \mathbf{0}_{6\times4} & \begin{array}{c:c} \mathbf{I}_4 & \mathbf{0}_4 \\ \hdashline \mathbf{0}_4 & \mathbf{I}_4 \\ \hdashline \mathbf{I}_4 & \mathbf{I}_4 \end{array} & \mathbf{0}_{6\times4} \\ \hdashline \mathbf{0}_{6\times4} & \mathbf{0}_{6\times4} & \begin{array}{c:c} \mathbf{I}_4 & \mathbf{0}_4 \\ \hdashline \mathbf{0}_4 & \mathbf{I}_4 \\ \hdashline \mathbf{I}_4 & \mathbf{I}_4 \end{array} \end{array}\right],$$



Rys.2. Illustration of the mechanism to choice of elements of Hankel matrix in accordance with the „mirrored 7"-ordered technique

$$\mathbf{S}_{27\times 1}^{(T)} = \mathbf{C}_{27}^{(3)}\mathbf{C}_{27}^{(2)}\mathbf{C}_{27}^{(1)}\mathbf{X}_{27\times 1}^{(T)},$$

$$\mathbf{S}_{27\times 1}^{(H)} = \mathbf{C}_{3^3}^{(3)}\mathbf{C}_{3^3}^{(2)}\mathbf{C}_{3^3}^{(1)}\mathbf{X}_{27\times 1}^{(H)},$$

$$\mathbf{C}_{27}^{(1)} = \left[\begin{array}{c:c:c} -\mathbf{I}_9 & \mathbf{I}_9 & \mathbf{0}_9 \\ \hdashline -\mathbf{I}_9 & \mathbf{0}_9 & \mathbf{I}_9 \\ \hdashline \mathbf{I}_9 & \mathbf{0}_9 & \mathbf{0}_9 \end{array}\right],$$

$$\mathbf{C}_{27}^{(2)} = \left[\begin{array}{ccc:ccc:ccc} -1 & 1 & 0 & & & & & & \\ -1 & 0 & 1 & & \mathbf{0}_3 & & & \mathbf{0}_3 & \\ 1 & 0 & 0 & & & & & & \\ \hdashline & & & -1 & 1 & 0 & & & \\ & \mathbf{0}_3 & & -1 & 0 & 1 & & \mathbf{0}_3 & \\ & & & 1 & 0 & 0 & & & \\ \hdashline & & & & & & -1 & 1 & 0 \\ & \mathbf{0}_3 & & & \mathbf{0}_3 & & -1 & 0 & 1 \\ & & & & & & 1 & 0 & 0 \end{array}\right],$$

$$\mathbf{C}_{27}^{(3)} = \mathbf{I}_9 \otimes \mathbf{\Gamma}_3 = \left[\begin{array}{ccc:c:c:c}
\begin{matrix}-1 & 1 & 0\\ -1 & 0 & 1\\ 1 & 0 & 0\end{matrix} & \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 \\ \hdashline
\mathbf{0}_3 & \begin{matrix}-1 & 1 & 0\\ -1 & 0 & 1\\ 1 & 0 & 0\end{matrix} & \mathbf{0}_3 & \mathbf{0}_3 \\ \hdashline
\vdots & \vdots & \ddots & \vdots \\ \hdashline
\mathbf{0}_3 & \mathbf{0}_3 & \mathbf{0}_3 & \begin{matrix}-1 & 1 & 0\\ -1 & 0 & 1\\ 1 & 0 & 0\end{matrix}
\end{array}\right]$$

.

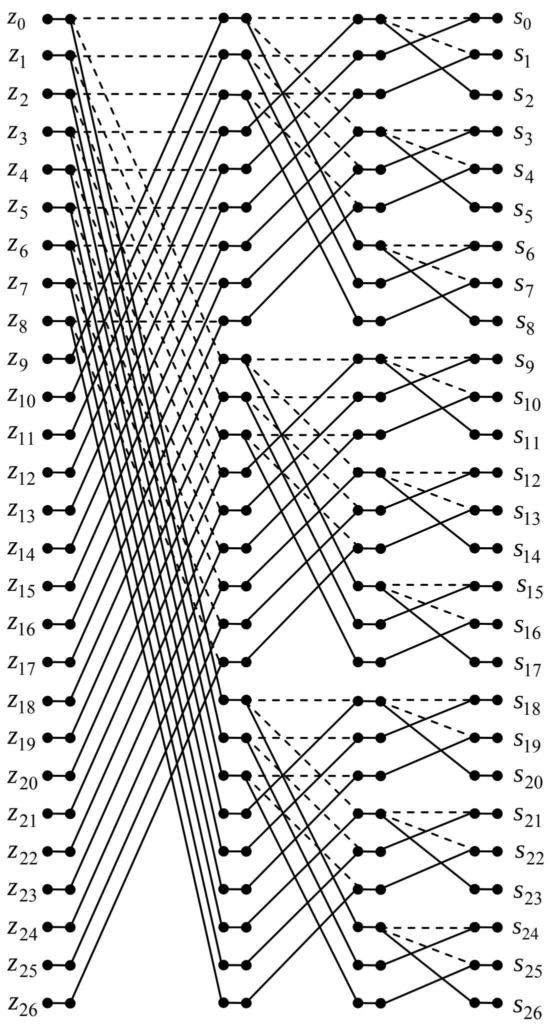
Then the computational procedures that describe fast algorithms for Toeplitz and Hankel matrix-vector multiplication for $N = 8$ are as follows.

(9) $\mathbf{Y}_{8\times1}^{(T)} = \widetilde{\mathbf{A}}_{8\times12}^{(1)} \widetilde{\mathbf{A}}_{12\times18}^{(2)} \widetilde{\mathbf{A}}_{18\times27}^{(3)} \mathbf{S}_{3^m}^{(T)} \mathbf{A}_{27\times18}^{(3)} \mathbf{A}_{18\times12}^{(2)} \mathbf{A}_{12\times8}^{(1)} \mathbf{X}_{8\times1}$

(10) $\mathbf{Y}_{8\times1}^{(H)} = \widetilde{\mathbf{A}}_{8\times12}^{(1)} \widetilde{\mathbf{A}}_{12\times18}^{(2)} \widetilde{\mathbf{A}}_{18\times27}^{(3)} \mathbf{S}_{3^m}^{(H)} \mathbf{B}_{27\times18}^{(3)} \mathbf{B}_{18\times12}^{(2)} \mathbf{B}_{12\times8}^{(1)} \mathbf{X}_{8\times1}$

Figure 3 shows a signal-flow graph representation of the $\mathbf{S}_{27\times1}$ vector elements calculation process for $N = 8$ Toeplitz/Hankel matrix case. It should be noted that these elements are calculated the same way both for the Toeplitz and Hankel matrix. For this reason, the figure 3 used the generalized symbols y_i instead of $\breve{y}_i$ and $\hat{y}_i$ as well as z_i instead of $\breve{z}_i$ and $\hat{z}_i$. We assume that the exact values of the variables will be clear from the context. In this paper, signal-flow graphs are oriented from left to right. Straight lines in the figures here and farther denote the operation of data transfer. At points where lines converge, the data are summarized. (The dashed lines indicate the subtraction operation). We use the common lines, without arrows, not to clutter the picture.

Figure 4 shows a signal flow graph of the fast algorithm of $N = 8$ Toeplitz matrix-vector multiplication. Figure 5 shows a signal flow graph of the fast algorithm of $N = 8$ Hankel matrix-vector multiplication. Note that the circles in these figures show the operation of multiplication by a number (variable) inscribed inside a circle.



Rys.3. The signal graph describing the process of calculating elements of the matrices $\mathbf{S}_{3^m\times1}^{(T)}$ and $\mathbf{S}_{3^m\times1}^{(H)}$ in accordance with the procedures (7) and (8) for the case where $N = 8$

Assessment of computational cost

Both algorithms presented in this paper have the same computational complexity as the Karatsuba's method. The multiplicative complexity of algorithms is $\theta_\times = 3^m$ multiplications and additive complexity of algorithms is $\theta_+ = \sum_{i=1}^{m} (3^m \cdot 2^{m-i})$ additions. The significantly lower than the naïve N^2 complexity provides savings in terms of hardware resources. Additional advantage is possibility of parallelization so let us compares computation times for usual matrix-vector multiplication parallel algorithm and presented solution.

(10) $$\tau^{(norm)} = \tau_\times + \log_2 N \tau_+$$

(11) $$\tau^{(T/H)} = \tau_\times + 2\log_2 N \tau_+$$

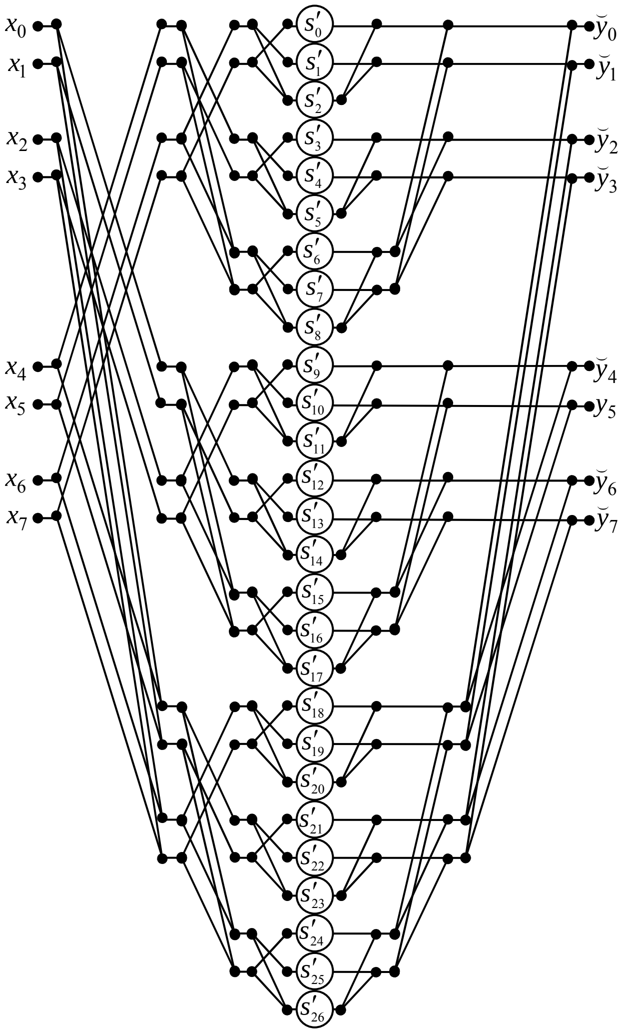
$\tau_\times$ and τ_+ denotes respectively time of one multiplication and one addition.

Even though additions time is two times longer in presented algorithms, the most important multiplication time remains the same. Along with reduced need for multipliers and adders, and limited resources of given hardware platforms we gain great advantage especially for matrices of large order. For example, matrix of order 32 needs 4 times more multiplication than in described method. If only 256 multipliers are available the times of execution are as follows: $\tau^{(norm)} = 4\tau_\times + 5\tau_+$ and $\tau^{(T/H)} = \tau_\times + 10\tau_+$.

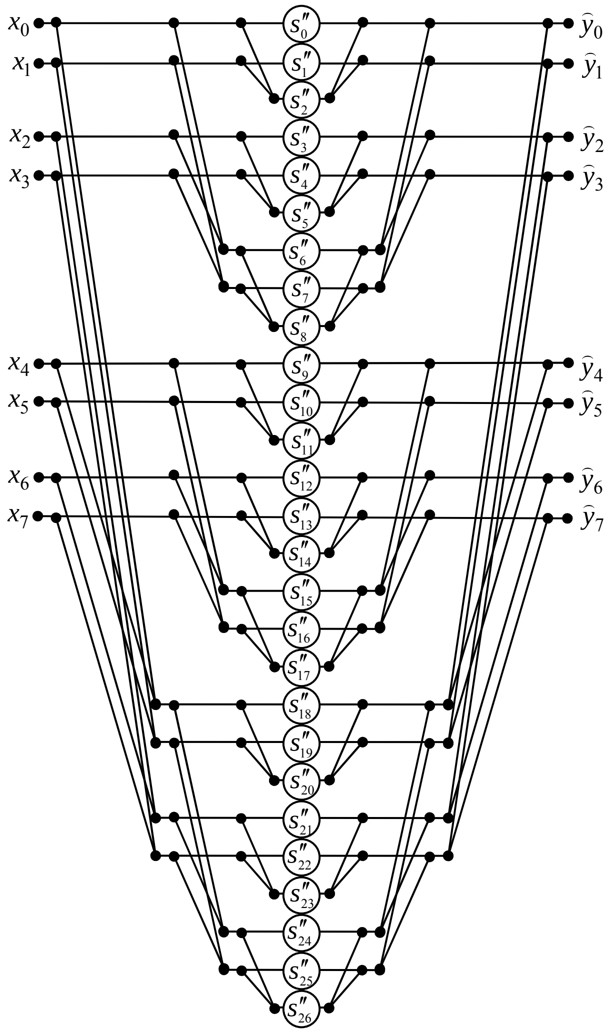
Table 1.Comparison of computational complexity for proposed and naïve algorithms for various dimensions of the original matrices

N	Multiplications		Additions	
	naïve	proposed	naïve	proposed
2	4	3	1	3
4	16	9	9	15
8	64	27	49	57
16	256	81	225	195
32	1024	243	961	633
64	4096	729	3969	1995
128	16384	2187	16129	6177
256	65536	6561	65025	18915
512	262140	19683	261120	57513
1024	1048600	59049	1046500	174075

Table 1 shows the number of arithmetical operations required for execution of the developed algorithms for various dimensions of the original matrices.



Rys.4. Signal-flow graph of the Toeplitz matrix-vector product computation in accordance with the procedures (8)



Rys.5. Signal-flow graph of the Hankel matrix-vector product computation in accordance with the procedures (9)

Conclusion

We present two new parallel algorithms for computing matrix-vector products for Toeplitz and Hankel matrices. These algorithms require fewer multiplications and additions than the naïive way of performing calculations. They require even fewer multiplications than the algorithms based on fast Fourier transforms, as in their implementation there is no need to perform complex operations. Additional advantages are parallelization possibility with no recursion requirement. On specialized hardware platform these algorithms can be significantly faster than any other known approach, however further research is needed in that aspect. Since the calculation of the matrix-vector products with Toeplitz and Hankel matrices have numerous applications, synthesized computational procedures can be used to rationalize many practical problems in various fields of science and technology.

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