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A kinematic approach to the optimal shape synthesis of electric field

***Abstract**. A parameter-free and global-oriented method of optimal field synthesis is proposed. The key idea is to consider the boundary of the field region to synthesize as a moving boundary, whose velocity leading to the optimal solution is unknown. The proposed method does not require to solve the transport equation and the adjoint variable problem. The design optimisation of the guard ring in a high-voltage transformer is considered as the case study. Finite-element method and conformal mapping method are used for field analysis.*

***Streszczenie.** W artykule zaproponowano otymalną syntezę pola przy użyciu metody globalnie zorientowanej ze szswobodnymi parametrami. Kluczową ideą jest rozważenie brzegu obszaru polowego, który ma podlegać syntezie, jako brzegu poruszającego się w kierunku optymalnego rozwiązania z nieznaną prędkością z prędkością. Proponowana metoda nie wymaga rozwiązywania równania transport ani problem sprzężonego. Optymalny projekt pierścienia ochronnego w transformatorze wysokonapięciowym stanowi przykładowe studium przypadku. Do analizy pola wykorzystano metodę elementów skończonych i metodę odwzorowań konforemnych.(**Podejście kinematyczne do syntezy optymalnego kształtu pola elektrycznego**).*

Keywords: Finite-element method, conformal mapping, moving boundary, optimal shape design
Słowa kluczowe: metoda elementów skończonych, odwzorowanie konforemne, poruszający się brzeg, projektowanie optymalnego kształtu.

Introduction

In the traditional approach to optimal shape design, the geometry of the region to synthesize is parametrized by means of a finite set of variables, which are updated by a minimisation algorithm according to the value of the objective function. This way proved to be effective for problems with a low or moderate number of variables; however, difficulties occur when the number increases. In fact, the performance of any minimisation algorithm deteriorates when several variables are to be handled simultaneously, because the ill-conditioning of the associated inverse problem increases. On the other hand, the description of complicated geometries necessarily asks for a large number of parameters.

In this respect, a parameter-free approach can be promising. In topology optimisation, the continuous-valued parametrization of the geometric model virtually enables all feasible shapes of the device under consideration to be explored [1], and this feature is definitely interesting for synthesizing a new device. The methodological background can be found in the level set method [2], where a transport equation of the diffusive kind governs the material distribution. In practice, the material distribution in the field region – which is unknown – is forced to vary gradually from void to solid state, according to an acceptance criterion preventing the occurrence of distorted shapes [3]. In the case of a source synthesis in a magnetic domain, the distribution of current-carrying conductors (or permanently magnetized domains) is unknown. In general, the time evolution of material distribution is governed by a transport equation of the diffusive kind that must be solved numerically. On the other hand, however, topology optimisation methods are not based on a regularization principle; therefore, the evolution of the material distribution might fall into a local minimum, and the relevant solution would depend on the initialisation. Moreover, the adjoint variable problem must be solved at each iteration for updating the sensitivity analysis and so evaluating the objective function gradient [4-5].

In order to go beyond the traditional – or parametric – approach to optimal shape design as a guided search within a finite-dimensional design space, it appears that the key idea is to consider the boundary of the field region to synthesize as a moving boundary, in which the distribution of velocity leading to the optimal solution is unknown. In contrast to various methods of topology optimisation based on the level set theory, the method here proposed does not require to solve the transport equation and the adjoint variable problem. A parameter-free and global-oriented method for optimal field synthesis is presented.

Kinematic approach to the optimal field synthesis

The kinematic approach to the shape design is to consider the boundary of the region to synthesize as a front propagating with a velocity depending on the objective function. So doing, a parameter-free optimisation problem is originated. Actually, by treating the boundary line (in 2D) or surface (in 3D) as a moving boundary, changes in the propagating front are easily handled.

In natural processes, like *e.g.* crystal growth, the velocity of a propagating front might be an arbitrary function of its curvature, and the front is passively moved by an associated flow. Likewise, in a problem of optimal shape design, the velocity is a vector field defined in each node of the moving boundary, and depends on the value of the objective function.

The methodological background can be found in the theory of fronts propagating with curvature-dependent velocity [6].

In principle, a simple algorithm could be defined as follows:

i) given a feasible shape of the boundary, solve the direct problem;
ii) compute the objective function (OF);
iii) compute the OF-dependent velocity components;
iv) solve the transport equation;
v) update the boundary position;
vi) iterate until the velocity of the boundary is zero.

In other words, the boundary of the field region to synthesize is considered as a moving boundary, and the distribution of velocity leading to the optimal solution is unknown. A simplified solution to the moving boundary problem could be obtained by means of a kinematic approach. In contrast to various methods of topology optimisation based on the level set theory (see *e.g.* [7]), the kinematic approach does not require to solve the diffusion equation governing the phenomenon. In order to mitigate the local behaviour of the moving boundary method, it is here proposed to synthesize the kinematic law, leading to the optimal boundary, by means of a global-minimum algorithm. The key idea is defining the velocity vector *e.g.* as follows:

(1) $$v_x(x,y,t) = v_0 \frac{x-x_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}} e^{-\frac{t}{T}}$$

(2) $$v_y(x,y,t) = v_0 \frac{y-y_0}{\sqrt{(x-x_0)^2+(y-y_0)^2}} e^{-\frac{t}{T}}$$

where speed constant v_0 and time constant T are unknown, while (x,y) are the coordinates of n_g moving nodes along boundary γ, and (x_0, y_0) is a reference point. With this choice, the velocity is radially directed towards point (x_0, y_0). Accordingly, the correction vectors are:

(3) $$x_{k+1} = x_k + v_x \Delta t \ , \ x \in \gamma$$

(4) $$y_{k+1} = y_k + v_y \Delta t \ , \ y \in \gamma$$

with k iteration index. Having prescribed Δh as the maximum displacement of γ, the time step Δt is defined according to the following constraint:

(5) $$\left[\sup_{(\gamma,t)} \|v\| \right] \Delta t = \Delta h$$

where the velocity components refer to iteration k, while Δt refers to iteration $k+1$.

At each iteration, the evolution of the boundary velocity is considered in the time interval $t \in [0,5T]$, and the final shape of the boundary is taken. The field analysis is updated at $t=5T$, when the boundary movement is practically expired due to the assumption of exponential time dependence. This way, only one call to the finite-element (FE) solver is required per iteration. Note that definition (1)-(2) allows for negative components of velocity, so enabling both expansion and contraction of the boundary. Because the shape of γ is governed by $2n_g$ parameters, a multi-dimensional search, controlled by only two degrees of freedom (v_0, T) and requiring one field analysis per iteration, is originated. If a derivative-free algorithm of evolutionary computing is used, the search is global-oriented too. The user-defined objective function is implemented as a routine, according to the usual format required by the optimisation algorithm.

A more sophisticated strategy would be defining a curvature-dependent velocity field, in order to fully exploit the flexibility inherent the moving boundary method.

Case study: guard-ring design optimisation

The subject of electric field synthesis has a long history. Rather recently, the boundary-element method was combined with a search technique for the optimal design of an electrode [8-9]. Moreover, the problem of the optimal location of an electrode was solved in [10], using the FE method for the discretisation of the field region and a min-max formulation. Here, the design optimisation of the guard ring of the high-voltage winding (HVW) in a power transformer is considered as the case study. The ring is located near the ends of the HVW, where the highest electric stress occurs [11].

For modelling purpose, a two-dimensional Cartesian model of the oil-filled (ε_r=2.2) field region Ω is developed. The test of applied voltage is simulated: accordingly, low-voltage winding (LVW), core and tank are short-circuited at the ground potential; moreover, a terminal of the HVW is at floating potential, while the other one is connected to an ideal voltage source. In terms of the analysis problem, Ω is a source-free doubly-connected domain; therefore, the Laplace equation of electric potential u in static conditions applies, subject to u = U at the ring and HVW surface, u = 0 at the LVW surface, and $D_n u$ = 0 elsewhere.

In view of the optimal field synthesis, a possible pair of objective functions, both to be minimised with respect to (v_0, T), is the maximum field strength in the oil-filled region (hot spot)

(6) $$f_1(v_o, T) = \sup_{\Omega} \left\| \overline{E}(\gamma(t), r, v_o, T)\big|_{t=5T} \right\|$$

and the maximum field deviation (inhomogeneity)

(7) $$f_2(v_o, T) = \sup_{R} \left\| E_y(\gamma(t), r, v_o, T)\big|_{t=5T} \right\|$$

with $r \in \Omega$ position vector and $\overline{E} = -\overline{\nabla} u$. In turn, $R \subset \Omega$ is a controlled region defined as $[a_1,a_2]$x$[a_3,a_4]$ where a_1 and a_2 are the abscissae of LVW and HVW ends delimiting the oil-channel, a_3 is set to a fixed ordinate value, and a_4 is the highest ordinate value along the moving boundary, with $a_4 > a_3$. The pair (v_0, T) minimising both (6) and (7) at $t=5T$ is searched for, according to Paretian optimality.

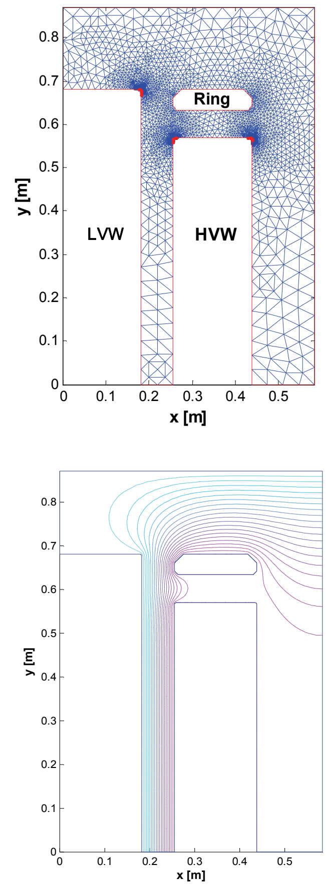

Fig. 1. Transformer model: FE mesh (about 4,500 linear elements), and potential lines.

In Fig. 1 the transformer model is shown. The ring boundary exhibits n_g=17 moving nodes, whereas the maximum displacement of the boundary in a step is constrained to be Δh = 6 cm. As a result of optimisation, in Fig. 2 a set of nine non-dominated solutions is represented in the objective space, so proving the existence of a conflict between objective f_1 and objective f_2 . Results have been obtained by means of a (1+1) evolutionary algorithm [12], applied to an objective weighting formulation with variable weights. Under a prescribed search tolerance equal to 10^{-3}, the typical number of convergence iterations was 80.

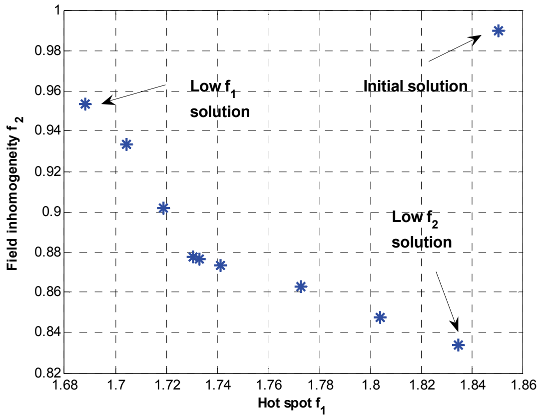

Fig. 2. A set of non-dominated solutions in the objective space (dimensionless values, referred to the uniform field in the oil channel).

Fig. 3 shows the solution corresponding to the left-end point of the non-dominated set represented in Fig. 2, and the hot spot position is shown.

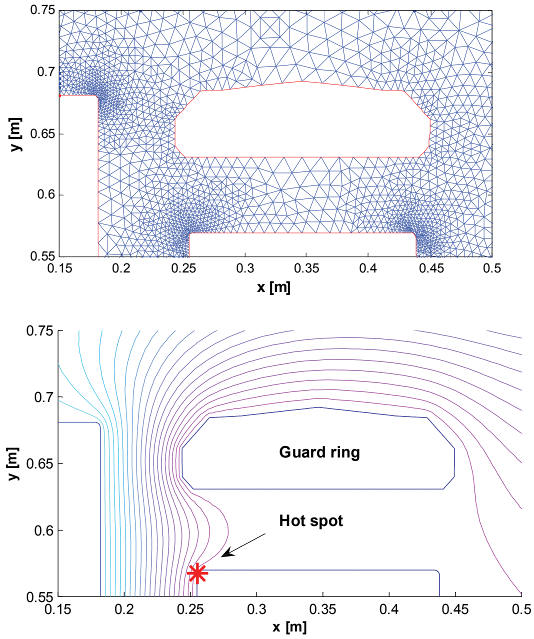

Fig. 3. Solution of low hot spot (FE mesh, up, and potential lines, down): v_0= 58 mms^{-1}, T= 123 ms (marked as "low f_1" in Fig. 2).

It can be noted that an expansion (v_0 > 0) leads to a reduction of the hot spot value.

Methodological aspects

As far as the inverse problem is concerned, a comparison of advantages and drawbacks of the proposed moving-boundary method is summarized in Table I.

Table I - Advantages and drawbacks.

Advantages	Drawbacks
Is able to process any boundary shape	Similarity between initial and final shape of the boundary
Needs two design variables only (parameters of the velocity-time curve)	
Is global-minimum oriented	Optimisation of a given boundary rather than synthesis of a boundary
Fits well with an existing FEA code	
Both outward and inward velocity field are feasible	The modulus of velocity is constant along the boundary

As far as the direct problem is concerned, the numerical simulation of the electric field at the ring boundary is particularly critical when FEA is used. Referring to the computation of objective f_1, the following remarks can be put forward:

i) given a ring boundary, the relevant FE model is based on a polygonal contour, the angles of which give rise to a field singularity;

ii) given a ring boundary, and given a domain mesh composed of first-order nodal elements, the field components are constant by element: under a potential formulation, the transmission conditions at the inter-element boundary in general are not fulfilled, and field components are discontinuous;

iii) given a ring boundary, for prescribed mesh density, different meshes originate different values of field components;

iv) for prescribed mesh density, different ring boundaries originate different distributions of triangular elements close to the ring;

v) considering the layer of triangles adjacent to the ring, their gravity centers in general do not belong to the same potential line; therefore, field components at the centers refer to different values of potential.

Similar remarks hold also for the objective f_2, again relevant to the computation of the highest field component within controlled region R. Due to all these reasons, it has been decided to assess the results of FEA by means of a numerical procedure based on conformal mapping.

Assessing optimisation results by means of conformal map analysis

Accurate computations of flux and potential lines have been performed by means of Schwarz-Christoffel (SC) transformations, and the results have been compared with those derived from FE analyses. For instance, let the geometry of Fig. 4 be considered, corresponding to the solution in the non-dominated set of Fig. 2 for f_1=1.7725 and f_2=0.8629. It was built using the boundary nodes of the FE mesh as polygon vertices and a detailed view, with the evidence of hot spot and highest inhomogeneity points, is shown in Fig. 5. As shown, vertices are close each other when the boundary curvature is large; conversely, their mutual distance increases when the curvature is smaller.

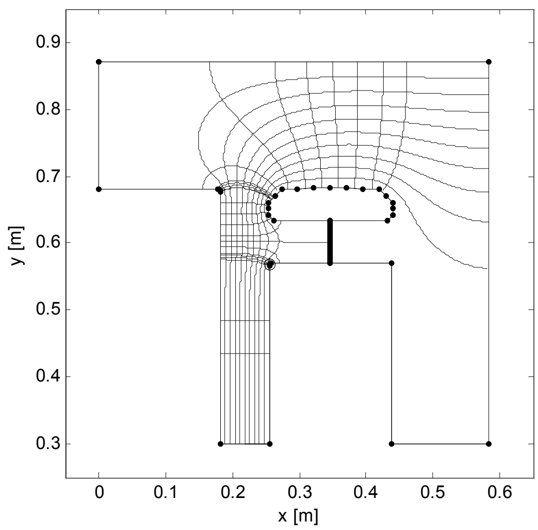

Fig. 4. The geometry analysed by means of SC conformal transformations, after cutting the original doubly-connected domain (pair of thick-lined sides connecting HVW and ring). Some potential lines are shown, with flux lines originating from points lying on boundary sides.

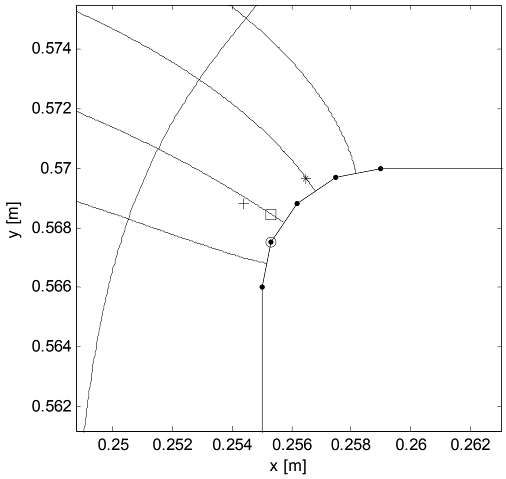

Fig. 5. Position of FE and SC hot spot points (circle and square), and position of FE and SC highest inhomogeneity points (cross and asterisk), relevant to the geometry represented in Fig. 4.

The geometry exhibits elongated regions, and it gives rise to a doubly-connected domain. As known, elongated regions originate severe crowding problems for vertices in the SC map, so modern SC numerical codes [13-15] have been used to cope with them. Very elongated regions have been considered in preliminary calculations, then shortened as shown in Fig. 4 only to speed up the numerical inversion of the SC formula (SCNI), without any loss of accuracy.

To analyse the doubly-connected domain, a simply-connected domain was defined, cutting the geometry by means of a pair of conductive sides, represented by the thick line in Fig. 4, and assessing the accuracy of suitable capacitance values by means of comparison with results obtained from the Hu procedure [15]. As a result, a remarkable speeding up of the whole analysis procedure was obtained.

A rectangular map for the polygonal domain in Fig. 4 is first derived, via SCNI mapping onto the intermediate SC half-plane and then direct SC mapping onto a rectangle. Here, flux and potential lines are immediately obtained, starting both from vertices and from points along the polygon sides, and then backwards mapped onto the original geometry. The singularity effects due to the vertices appear already smoothed along the potential line passing at some 5%, and we considered very significant to investigate field values in a way similar to the FE analysis, mimicking the effect of the mesh. This means neglecting in some way vertex singularities, which appear somewhat artificial when introduced to discretise a curved boundary. Accordingly, a suitable procedure was found considering average fields computed at boundary points far from the vertices and making the ratio of some 2.5% voltage to the distance between the boundary point and the potential line corresponding to the same fractional voltage. Ratios so computed, with fixed distances in voltage instead of fixed distance in geometry, led to field values astonishingly similar to those directly supplied by the FE analysis, and the discrepancies with respect to the FEM results previously presented are not exceeding a few percents. Therefore, they appear to supply a valid model for FE computations.

Conclusion

A simple method is proposed in order to explore shapes of the boundary of an electric field region, in the search for the optimal one satisfying prescribed objective functions. Starting from a given configuration, the boundary is expanded, or contracted, by a kinematic approach, *i.e.* by applying a law of velocity against time to a discrete number of points of the boundary. The resulting shape depends on such a law, and not all possible shapes are explored. Nevertheless, feasible initial shapes produce improved shapes, similar to the initial ones, with little computational effort. The application to the design of the guard ring in a HV transformer results in improvement of the order of 10-15% in the reduction of maximum field strength and field deviation.

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