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Modeling the flow of compressible media in a vessel by means of Simulink S-functions

Abstract. The paper deals with modeling the flow of compressible media in a vessel, while elasticity of the vessel can change. It presents mathematical model of a pipeline as system with distributed parameters, consisting of three partial differential equations and two additional equations which describes density and area as functions of pressure and temperature. The solution of the problematic is achieved by means of Matlab&Simulink environment, particularly by use of S-functions and finite difference method (FDM). The paper includes a case study when a set of pressure impulses is brought to the model of a vessel. As a result, media pressure, temperature and velocity are displayed over the time and along the pipeline.

Streszczenie. W pracy opisane jest modelowanie przepływu ściśliwego medium w naczyniu kwionośnym (żyle), gdzie elastyczność naczynia może być zmienna. Przedstawiono model matematyczny rurociągu z rozłożonymi parametrami, który składa się z trzech równań różniczkowo cząstkowych i dwa dodatkowe równania, które opisują gęstości i powierzchnię w funkcji ciśnienia I temperatury. Problem jest rozwiązywany za pomocą środowiska Matlab & Simulink, w szczególności poprzez wykorzystanie S-funkcji i metody różnic skończonych (finite difference method - FDM). Artykuł zawiera analizę przypadku, gdy zestaw impulsów ciśnienia doprowadza się do modelu naczynia. Wynikiem analizy są przebiegi czasowe ciśnienia, temperatury i prędkości wzdłuż naczynia krwionośnego. - Modelowanie przepływu ściśliwego media w naczyniu kwionośnym za pomocą środowiska Matlab &Simulink i S-funkcji.

Keywords: Modeling, Simulation, Simulink, S-function, Partial Differential Equations
Słowa kluczowe: Modelowanie, Symulacja, Simulink, S-funkcja, Równanie Różniczkowe Cząstkowe

Introduction

The paper describes a vessel with flowing liquid with pressure p, temperature T and velocity v. It also includes the following issues affecting the dynamics of the system: thermal volume expansivity and pressure expansivity of the vessel, thermal volume expansivity of media, specific media density changes doe to pressure and temperature. The construction of the model is based on presumption that media has Newton character, the flow is one-directional and there are no phase changes of the media. The model supposes that the dynamics of the vessel can be represented as dynamics of the elastic pipeline with given parameters described in [1].

Mathematical model of flowing media (blood) in an elastic pipeline (vessel) is implemented by means of modern high-performance S-functions. This approach makes it possible to create interconnection circuits and thus representing large distributed-parameters systems. Besides other parameters, each block can be customized by geometric dimension and expansivity modulus. Compared to other ways of flow modeling introduced in [9],[10],[11], this one offers advantages regarding simplicity of creation resulting complex model.

S-functions

S-functions make it possible to add customized algorithms to Simulink models, either written in MATLAB or C. By following a set of simple rules it possible to implement the algorithms in an S-function. After S-function has been written and placed its name in an S-Function block (available in the User-defined Functions sub library), it's time to customize the user interface by using masking. An S-function is a computer language description of a dynamic system. S-functions can be written using MATLAB or C. C language S-functions are compiled as MEX-files using the *mex* utility described in the Application Program Interface Guide. As with other MEX-files, they are dynamically linked into MATLAB when needed. S-functions use a special calling syntax that enables you to interact with Simulink's equation solvers. This interaction is very similar to the interaction that takes place between the solvers and built-in Simulink blocks. The form of an S-function is very general and can accommodate continuous, discrete, and hybrid systems. As a result, nearly all Simulink models can be described as S-functions. The most common use of S-functions is to create custom Simulink blocks. S-functions can be effectively used for a variety of applications, including:

- Adding new general purpose blocks to Simulink
- Incorporating existing C code into a simulation
- Describing a system as a mathematical set of equations
- Using graphical animations

An advantage of using S-functions is that it is possible to build a general purpose block that can be used many times in a model, varying parameters with each instance of the block.
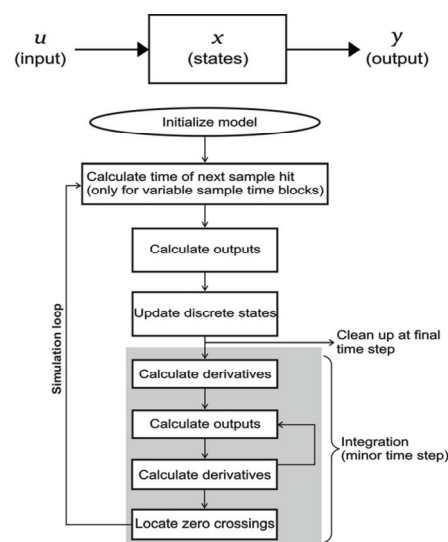

Fig. 1. The description of Simulink loop.

An M-file or a CMEX-file that defines an S-Function block must provide information about the model; Simulink needs this information during simulation. As the simulation proceeds, Simulink, the ODE solver, and the M-file interact to perform specific tasks. These tasks include defining initial conditions and block characteristics, and computing derivatives, discrete states, and outputs. Simulink provides a template M-file S-function that includes statements that

define necessary functions, as well as comments to help with writing the code needed for a particular S-function block.

M-file S-functions work by making a sequence of calls to S-function routines, which are M-code functions that perform tasks required by customized S-function. This table lists the S-function routines available to M-file S-functions. C MEX-file S-functions have the same structure and perform the same functions as M-file S-functions. In addition, C MEX S-functions provides more functionality than M-file S-functions [3].

Mathematical Model

The model below describes flowing of the elastic fluid in the segment of elastic pipeline. The original mathematical model is described by three nonlinear partial differential equations of the first order. The fundamental of the model consists of Newton equation (1), Equation of continuity (2) and Bernoulli equation (3). The model is extended by equations (4) and (5) with respect to the elasticity of pipeline and transferred fluid [5], [6].

- Newton equation

(1) $$\frac{\partial p}{\partial x} + \rho v \frac{\partial v}{\partial x} + \rho \frac{\partial v}{\partial t} + \rho g \frac{\partial z}{\partial x} + \frac{\lambda \rho v |v|}{2 d_n} = 0$$

- Equation of continuity

(2)
$$\rho F \frac{\partial v}{\partial x} + \rho v \left(\frac{\partial F}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial F}{\partial T} \frac{\partial T}{\partial x} + \frac{\partial F}{\partial x} \right) +$$
$$+ F v \left(\frac{\partial \rho}{\partial p} \frac{\partial p}{\partial x} + \frac{\partial \rho}{\partial T} \frac{\partial T}{\partial x} \right) +$$
$$+ p \left(\frac{\partial F}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial F}{\partial T} \frac{\partial T}{\partial t} \right) + F \left(\frac{\partial \rho}{\partial p} \frac{\partial p}{\partial t} + \frac{\partial \rho}{\partial T} \frac{\partial T}{\partial t} \right) = 0$$

- Bernoulli equation

(3)
$$\frac{\partial}{\partial t} \left(\rho \left(c T + \frac{v^2}{2} \right) \right) + \frac{\partial}{\partial x} \left(\rho v \left(c T + \frac{v^2}{2} \right) \right) + \frac{\partial (p v)}{\partial x} +$$
$$+ \frac{\partial (\rho v g z)}{\partial x} - \frac{\gamma (T_0 - T)}{F} = 0$$

Additional equations for density and cross-section:

(4) $$\rho = \frac{\rho_0}{\exp\left(\frac{1}{K} (p_0 - p) \right) \exp\left(\beta (T - T_0) \right)}$$

(5) $$F = F_0 \exp\left(\frac{d_n}{E d} (p - p_0) \right) \exp\left(2\alpha (T - T_0) \right)$$

Where: c - internal energy of media, d - wall thickness of the pipeline, d_n - internal diameter of the pipeline, E - modulus of elasticity, g - acceleration of gravity, K - media elasticity bulk modulus, l - length of pipeline, p - (relative) pressure of media, p_0 - environment pressure, surrounding pressure, F - cross-sectional area of the pipeline, F_0 - initial cross-sectional area of the pipeline, constant, t – time, T - temperature of media, T_0 - initial environment temperature constant, v - flow velocity of media, x - coordinate along pipeline axis, z - elevation of the pipeline, $\alpha -$ bulk thermal expansivity of pipeline, $\beta -$ bulk thermal expansivity of media, $\gamma -$ heat transfer coefficient, $\lambda -$ friction factor of media, $\rho -$ density of media, ρ_0 - initial density of media, constant.

In order to find a unique solution of set of equations (1) – (3), the boundary and initial conditions must be added.

These are as follows:
The boundary conditions:
- The velocity at the end of the vessel, it is prescribed as relation between velocity and pressure $v(L,t) = k \cdot p(L,t)$.
- The pressure at the beginning of the vessel, it is prescribed by a load characteristic of the source (hard source assumed in this case, no internal load, with no relation between pressure and velocity). The input train of pressure impulses can be seen in Fig. 3.

The initial conditions:
- The distribution of the pressure within the whole length of the vessel.
- The distribution of the velocity within the whole length of the vessel.
- The distribution of the elevation within the whole length of the vessel.
- The distribution of the cross-section within the whole length of the vessel. The cross-section in time $t = 0$ is supposed constant, $d_n = 0.02$ cm [2].

Finite difference method

A set of partial differential equations is solved by a finite difference method (FDM) in particular Simulink S-function. It uses approximation of partial derivatives by numerical differences in particular points using the following formulas, omitting the remainders:

(6) $$\left. \frac{\partial T(x,t)}{\partial x} \right|_{x = x_1} \approx \frac{-3T(x_1,t) + 4T(x_2,t) - T(x_3,t)}{2h}$$

(7) $$\left. \frac{\partial T(x,t)}{\partial x} \right|_{x = x_2} \approx \frac{T(x_3,t) - T(x_1,t)}{2h}$$

(8) $$\left. \frac{\partial T(x,t)}{\partial x} \right|_{x = x_3} \approx \frac{T(x_4,t) - T(x_2,t)}{2h}$$

(9) $$\left. \frac{\partial T(x,t)}{\partial x} \right|_{x = x_n} \approx \frac{T(x_{n-2},t) - 4T(x_{n-1},t) + 3T(x_n,t)}{2h}$$

The particular derivatives are derived from partial differential equations and implemented into *mdlDerivatives* routine of S-function [4], [8].

Wave propagation case study

The paper describes simulation of the following experiment described in Fig. 2. A train of successive pressure impulses (diastolic and systolic values of 8/16 kPa) is brought to inlet of a vessel with given parameters, see Fig. 3.

A load is modelled at the outlet – this is done through the prescribed relation between pressure and velocity at the vessel outlet.
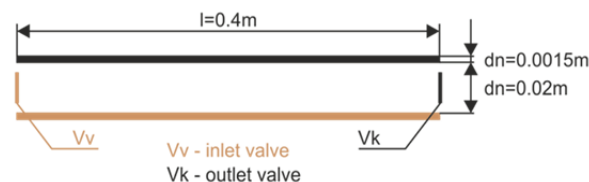

Fig. 2. Representation of a vessel by a pipeline segment [7].

A vessel is represented by an elastic pipeline segment through which the pressure wave propagates. Besides, other state variables are evaluated through x-axis and time (velocity, temperature). These segments can be arranged in

series, thus representing different types of objects in bloodstream, such as aorta, artery, vein and capillary.

This case study can be modified in many ways due to the terms in system equations (for example cross-section distribution along *x*-axis, density changes and cross-section changes due to temperature variations) [2].

Results

Pressure impulses are transformed by the system into the series of the pulses in Fig. 4. or 3D graph Fig. 7. Due to the load at the outlet the pressure waves are propagated back. When a systolic pressure is achieved, the velocity rapidly grows up to 0.32 m/s. The velocity at diastolic pressure works out about 0.15 m/s, see Fig. 5. Due to character of pressure waves and due to its elasticity, the cross-section area depends on time as shown in Fig. 6.

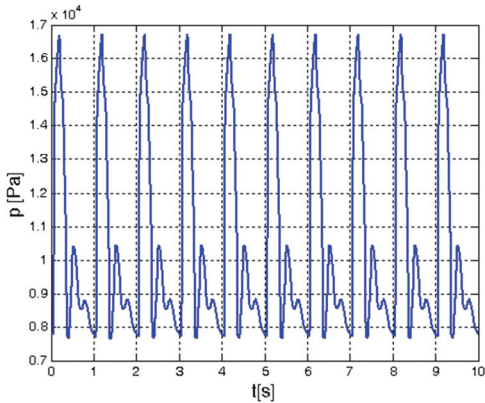Fig. 3. Input pressure impulses to a vessel.

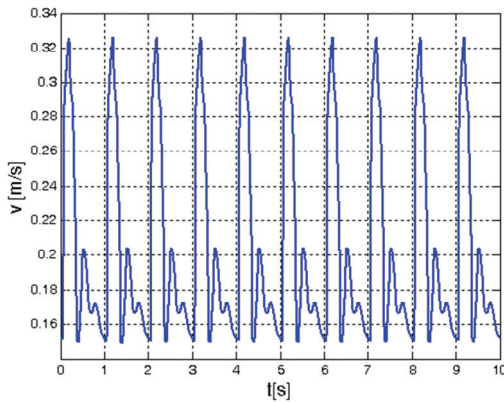Fig. 4. Media pressure at the end of a vessel.

Fig. 5. Media velocity at the end of a vessel.

The first two impulses in Fig. 6 are different from the ones in steady state, due to initial condition for cross-section which is being computed during the simulation.
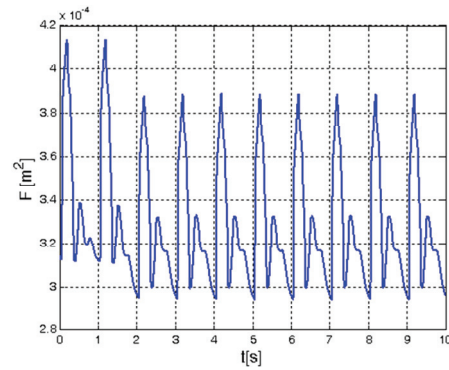

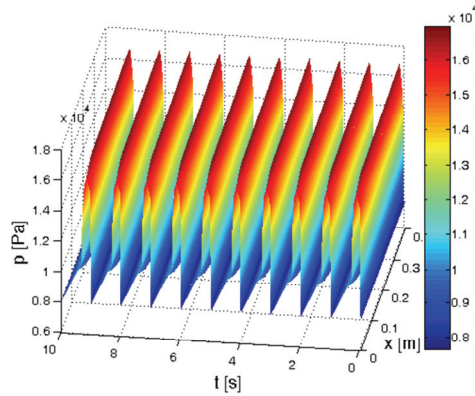Fig. 6. Cross-section area at the end of a vessel.

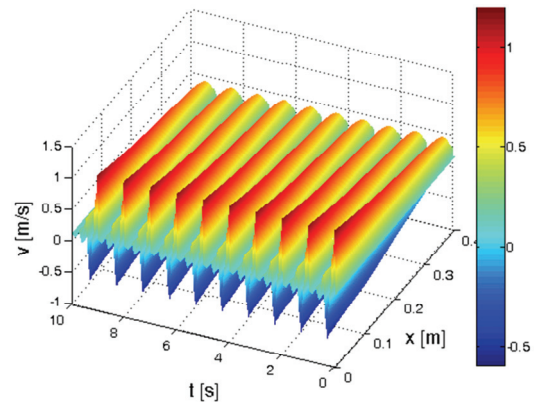Fig. 7. Pressure distribution along a vessel over time and length.

Fig. 8. Velocity distribution along a vessel over time and length.
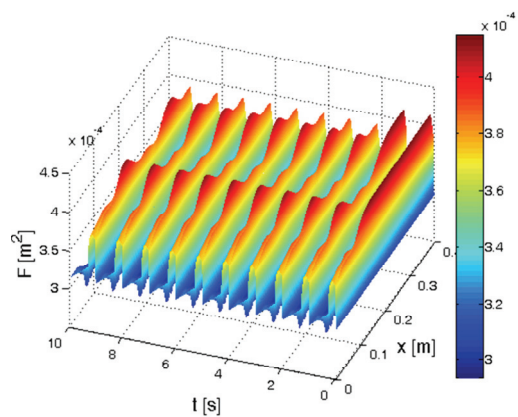

Fig. 9. Cross-section distribution along a vessel over time and length.

Kinetic energy of pressure wave is transformed to the potential energy which is kept due to extension of the cross-section as shown on 3D graphs, see Fig. 8 and Fig. 9. As a result of wave reflection, the original input pressure impulses are in interference with the reflected ones, these interferences depend on parameters of the segment.

Conclusion

The main goal of the paper was not to model the detailed particular medical case with a real specification, this would require a specific assignment of the values.
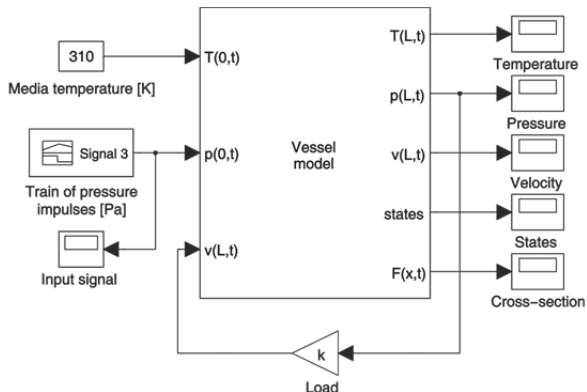

Fig. 10. Simulink model of a vessel.

It was going to show the capabilities and powerful environment if Simulink and especially S-functions for modeling complex problems concerning dynamic systems are used.

However, temperatures, length, diameter arise from approximation of real values, including the pressure impulses that represent systolic and diastolic pressure values. The structure and concept of the modeling scheme is modular – the blocks can be arranged arbitrarily and in a user-defined way, see Fig. 10. Moreover, the specific conditions and relations can be prescribed for inlet and outlet of a model of a vessel. For example, their opening/closing/turning down and up can be triggered by any state conditions such as negative velocity as indication of reflected wave.

The proposed way of modeling in Matlab&Simulink with S-functions uses C syntax for the most crucial parts of the model. This enables user to run any number of instances of the blocks representing different segments with different parameters connected in series.

Simulation of the resulting model has a very high performance. Simulation experiments proved that simulation is approximately 4000 times faster than real time, and 150 times faster than equivalent of this model implemented in Comsol Multiphysics.

To discuss the complexity of the solution, it should be divided into two parts: programming of S-function and implementation of Simulink scheme. The programming part requires certain amount of programming skills and studying the S-function manual to handle the function inputs/outputs in appropriate way. This necessary effort put in this part will pay in the end as the implementation in Simulink is very easy and intuitive. Each of the blocks (connected arbitrary – mostly in series or parallelly) can be customized and have its own parameters.

One of the biggest advantage of this approach is that geometry of the modeled object does not have to be created and meshed to apply FEM (finite element method) in case of Comsol Multiphysics, but it is simply expressed by numbers and functions. It is also very difficult to model such complex models in these tools as they usually offer a predefined set of problems (multiphysic processes, case

studies) or PDES that the modeled problematic would have to match. The approach is thus innovative, as it can handle additional physical relations resulting from the physical nature of the problem. It also models the dynamics considering distributed-parameters system character unlike lumped parameter model introduced in [10]. Similarly to [9], (p. 1002, Figure 1.), it allows to create vascular model construction by segmentation of the computational object.

Correctness of results were verified by comparison with the results from previously published work (solved by different approach) [7] and also in a bigger scale experiment with the measurement hydraulic heads in the pipelines [5].

Beneficial effect of this work may lie in its usage for modeling and design of prosthetic aortic heart valves [11].

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