Mikołaj BUSŁOWICZ

Białystok University of Technology, Faculty of Electrical Engineering

Stability of continuous-time linear systems described by state equation with fractional commensurate orders of derivatives

Abstract. *The stability problem of continuous-time linear systems described by the state equation with different fractional commensurate orders of derivatives of state variables has been considered. The new method for stability analysis has been given. The method proposed is based on the Mikhailov stability criterion known from the stability theory of natural order systems. The considerations are illustrated by numerical example.*

Streszczenie. *W pracy rozpatrzono problem badania stabilności liniowych ciągłych układów opisanych równaniem stanu o różnych współmiernych niecałkowitych rzędach pochodnych zmiennych stanu. Podano nową metodę badania stabilności. Jest to metoda częstotliwościowa bazująca na kryterium stabilności Michajłowa, znanym z teorii stabilności układów naturalnego rzędu. Rozważania zilustrowano przykładem. (**Stabilność ciągłych układów liniowych opisanych równaniem stanu o współmiernych niecałkowitych rzędach pochodnych**)*

Keywords: linear system, continuous-time, fractional, stability, Mikhailov criterion.
Słowa kluczowe: układ liniowy, ciągły, niecałkowitego rzędu, stabilność, kryterium Michajłowa.

Introduction

The problem of analysis and synthesis of dynamical systems described by fractional order differential (or difference) equations was considered in many papers and books. For review of the previous results see, for example, the monographs [1-5].

Applications of fractional calculus in engineering, automatic control and power electronic can be found in the papers [6-8].

The problems of stability and robust stability of linear fractional order continuous-time and discrete-time linear systems were studied in [9-18] and [19-21], respectively.

The new class of the linear fractional order systems, namely the positive systems of fractional order was considered in [22-27]

The aim of the paper is to give the frequency domain method for stability analysis of fractional continuous-time linear systems described by the state-space model with different commensurate orders of fractional derivatives of state variables. Such models were considered in [18, 25, 9]. The proposed method is based on the Mikhailov stability criterion known from the theory of linear systems [28].

Preliminaries and problem formulation

Consider a continuous-time linear system of fractional orders described by the state equation

(1) $$ {}_0D_t^{\overline{\alpha}}x(t) = Ax(t) + Bu(t), $$

where: $A \in R^{n\times n}$, $B \in R^{n\times n_u}$,

(2) $$ x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} \in R^n, \quad {}_0D_t^{\overline{\alpha}}x(t) = \begin{bmatrix} {}_0D_t^{\alpha_1}x_1(t) \\ \vdots \\ {}_0D_t^{\alpha_n}x_n(t) \end{bmatrix} \in R^n, $$

(3) $$ {}_0D_t^{\alpha_i}x_i(t) = \frac{1}{\Gamma(p_i-\alpha_i)}\int_0^t \frac{x_i^{(p_i)}(\tau)d\tau}{(t-\tau)^{\alpha_i+1-p_i}} $$

is the Caputo definition for fractional α_i -order derivative,

(4) $$ x_i^{(p_i)}(t) = \frac{d^{p_i}x_i(t)}{dt^{p_i}}, \quad p_i - 1 \le \alpha_i \le p_i, $$

p_i is a positive integer and

(5) $$ \Gamma(\alpha_i) = \int_0^\infty e^{-t}t^{\alpha_i-1}dt $$

is the Euler gamma function.
Initial conditions for (1) have the forms [25]

(6) $$ x^{(k)}(0) = x_0^{(k)} \in \Re^n, \quad k = 0,1,\dots,p_i - 1, $$

where: $x^{(k)}(0) = (d^k/dt^k)x(t)_{|t=0}$.

The Laplace transform of the fractional derivative (3) with zero initial conditions has the form

(7) $$ L\{{}_0D_t^{\alpha_i}x_i(t)\} = s^{\alpha_i}X_i(s), \quad X_i(s) = L\{x_i(t)\}. $$

Characteristic matrix of the fractional system (1) has the form

(8) $$ H(s) = \begin{bmatrix} s^{\alpha_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & s^{\alpha_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & s^{\alpha_n} - a_{nn} \end{bmatrix}. $$

The matrix (8) can be computed from the formula

(9) $$ H(s) = I(s) - A, $$

where

(10) $$ I(s) = \begin{bmatrix} s^{\alpha_1} & 0 & \cdots & 0 \\ 0 & s^{\alpha_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & s^{\alpha_n} \end{bmatrix}. $$

From (9), (10) it follows that the characteristic function of the system (1), of the form

(11) $$ w(s) = \det(I(s) - A), $$

is a polynomial of fractional degree

(12) $$ \delta = \alpha_1 + \alpha_2 + \dots + \alpha_n. $$

We will assume that

(13) $$ \alpha_i = v_i / u_i, \quad v_i, u_i \in Z_+ \quad (i = 1,\dots,n), $$

where Z_+ is the set of non-negative integers, v_i and u_i are coprime.

Denote by m the lowest common multiple of the denominators u_i $(i=1,...,n)$ in (13) and we set

(14) $$\alpha = 1/m, \quad \delta_i = m\alpha_i \text{ and } \lambda = s^{\alpha}.$$

If the conditions (13) and (14) hold, then the system (1) is called the system with commensurate orders of fractional derivatives.

In this case from (8) we obtain the natural degree characteristic matrix

(15) $$\widetilde{H}(\lambda) = \begin{bmatrix} \lambda^{\delta_1} - a_{11} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda^{\delta_2} - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda^{\delta_n} - a_{nn} \end{bmatrix}.$$

This matrix can be computed from the formula

(16) $$\widetilde{H}(\lambda) = \widetilde{I}(\lambda) - A,$$

where

(17) $$\widetilde{I}(\lambda) = \begin{bmatrix} \lambda^{\delta_1} & 0 & \cdots & 0 \\ 0 & \lambda^{\delta_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda^{\delta_n} \end{bmatrix}.$$

From the above it follows that the natural degree polynomial associated with the fractional degree polynomial (11) has the form

(18) $$\widetilde{w}(\lambda) = \det \widetilde{H}(\lambda) = a_p \lambda^p + a_{p-1}\lambda^{p-1} + ... + a_0,$$

where a_k ($k = 0,1,..., p$) are constant coefficients and

(19) $$p = \sum_{i=1}^{n} \delta_i = m\delta.$$

The fractional degree polynomial (11) is a multivalued function whose domain is a Riemann surface. In general, this surface has an infinite number of sheets and the fractional polynomial (11) has an infinite number of roots. Only a finite number of which will be in the main sheet of the Riemann surface. For stability reasons only the main sheet defined by $-\pi < \arg s < \pi$ can be considered.

From the theory of stability of linear fractional order systems (see [1, 9, 16], for example) we have the following theorem.

Theorem 1. The fractional order system (1) with commensurate orders of derivatives is stable if and only if the fractional degree characteristic polynomial (11) has no zeros in the closed right-half of the Riemann complex surface, i.e.

(20) $$w(s) = \det(I(s) - A) \neq 0 \text{ for } \operatorname{Re} s \geq 0,$$

or equivalently, the following condition is satisfied

(21) $$|\arg \lambda_i| > \alpha \frac{\pi}{2}, \quad i = 1,2,..., p,$$

where λ_i ($i = 1, 2,..., p$) are the roots of the natural degree polynomial (18) and α is defined in (14).

From (14) it follows that $0 < \alpha < 1$. In this case the stability region described by (21) is shown in Figure 1. Parametric description of the boundary of this region has the form

(22) $$(j\omega)^{\alpha} = |\omega|^{\alpha} e^{j\pi\alpha/2}, \quad \omega \in (-\infty, \infty).$$

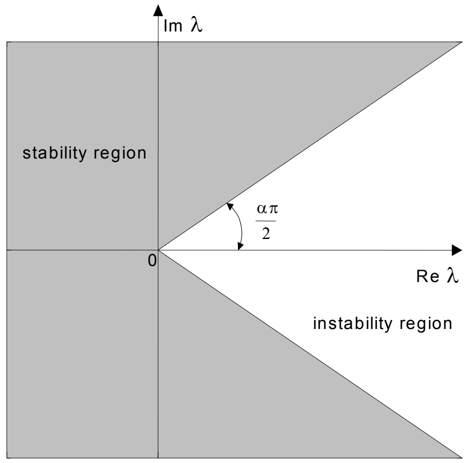

Fig. 1. Stability region of the system (1) with (13), (14)

The aim of this paper is to give the methods for stability analysis of the fractional system (1) with commensurate orders of derivatives.

Problem solution

From Theorem 1 it follows that the fractional system (1) with commensurate orders of derivatives is stable if and only if all roots of the polynomial (18) lie in the stability region shown in Figure 1. Hence, this system may be stable in the case when not all roots of (18) lie in open left half-plane. Moreover, this system may be stable when all roots of (18) are complex conjugate with positive real parts.

From the above we have the following simple sufficient condition for stability.

Lemma 1. The fractional system (1) with commensurate orders of derivatives is stable if all roots of (18) lie in open left half-plane of the complex plane.

From Theorem 1 and Figure 1 we have the following important lemma.

Lemma 2. The fractional system (1) with commensurate orders of derivatives is unstable if the polynomial (18) has at least one non-negative real root. In particular, this holds if

(23) $$\widetilde{w}(0) = \det \widetilde{H}(0) = a_0 = \det(-A) = 0.$$

From Theorem 1 we have the following lemma.

Lemma 3. The fractional order system (1) with commensurate orders of derivatives is stable if and only if

(24) $$\gamma > \alpha \frac{\pi}{2},$$

where

(25) $$\gamma = \min_i |\arg \lambda_i|, \quad i = 1,2,..., p,$$

α is defined in (14) and λ_i denotes the *i*-th root of the polynomial (18).

The method of Theorem 1 requires computation of roots of the polynomial (18). These roots are different from the eigenvalues of the state matrix *A*. Moreover, the degree of polynomial (18) depends on α defined in (14). It is easy to see that investigation of stability of the system (1) by checking the condition (21) (or (24), equivalently) for all roots of natural degree polynomial (18) can be inconvenient with regard on high degree of this polynomial.

To the stability analysis of the fractional system (1) with commensurate orders of derivatives we apply the frequency domain methods. These methods have been proposed in [11-13] (see also Chapter 9 in [27]) for the fractional continuous-time and discrete-time linear systems described by the transfer function. These methods have been applied in [14] to the stability analysis of linear continuous-time systems described by state space models with the same fractional order of derivatives of all state variables.

By generalization of the results of [14] to the case of fractional system (1) with commensurate orders of derivatives we obtain the following theorem.

Theorem 2. The fractional system (1) with commensurate orders of derivatives with characteristic polynomial (11) is stable if and only if

$$\Delta \arg_{0 \le \omega < \infty} w(j\omega) = p\pi/2, \tag{26}$$

where

$$w(j\omega) = \det(I(j\omega) - A) \tag{27}$$

and $I(j\omega)$ is defined by (10) for $s = j\omega$.

Satisfaction of (26) means that the plot of (27) starts for $\omega = 0$ in the point $w(0) = \det(-A)$ and with ω increasing from 0 to ∞ turns strictly counter-clockwise and goes through *p* quadrants of the complex plane.

Plot of the function (27) will be called the generalised (to the class of fractional order systems) Mikhailov plot.

Checking the condition (26) is difficult in general (for large values of *p*), because $w(j\omega)$ quickly tends to infinity as ω grows to ∞.

To remove this difficulty, we consider the rational function

$$\psi(s) = \frac{w(s)}{w_r(s)} = \frac{\det(I(s) - A)}{w_r(s)} \tag{28}$$

instead of the polynomial (11), where $w_r(s)$ is stable the reference fractional polynomial of degree δ (see (12)), i.e.

$$w_r(s) \neq 0 \text{ for } \operatorname{Re} s \geq 0. \tag{29}$$

The reference fractional polynomial can be chosen in the form

$$w_r(s) = (s + c)^{\delta}, \; c > 0. \tag{30}$$

Theorem 3. The fractional system (1) with commensurate orders of derivatives is stable if and only if

$$\Delta \arg_{\omega \in (-\infty, \infty)} \psi(j\omega) = 0, \tag{31}$$

where $\psi(j\omega) = \psi(s)$ for $s = j\omega$ and $\psi(s)$ is defined by (28), i.e. plot of the function $\psi(j\omega)$ does not encircle or cross the origin of the complex plane as ω runs from $-\infty$ to ∞.

Proof. The reference fractional polynomial $w_r(s)$ is stable (by the assumption) and the condition

$$\Delta \arg_{0 \le \omega < \infty} w_r(j\omega) = p\pi/2$$

holds.

From (28) it follows that

$$\Delta \arg \psi(j\omega) = \Delta \arg w(j\omega) - \Delta \arg w_r(j\omega).$$

Hence, (26) holds if and only if (31) is satisfied.

Plot of the function $\psi(j\omega)$, $\omega \in (-\infty, \infty)$, will be called the generalised modified Mikhailov plot.

From (10) – (12), (30) and (28) we have

$$\psi(\infty) = \lim_{\omega \to \pm\infty} \psi(j\omega) = 1 \tag{32}$$

and

$$\psi(0) = \frac{\det(-A)}{c^{\delta}}. \tag{33}$$

From (33) it follows that $\psi(0) \le 0$ if $\det(-A) \le 0$. Hence, from Theorem 3 we have the following lemma.

Lemma 4. If $\det(-A) \le 0$, then the fractional system (1) is unstable.

Illustrative example

Consider the fractional system described by the state equation (1) with $n = 2$, $\alpha_1 = 2/3$, $\alpha_2 = 3/4$ and

$$A = \begin{bmatrix} -1 & 0{,}8 \\ -0{,}8 & -2 \end{bmatrix}. \tag{34}$$

In this case

$$H(s) = \begin{bmatrix} s^{\alpha_1} + 1 & -0{,}8 \\ 0{,}8 & s^{\alpha_2} + 2 \end{bmatrix} = \begin{bmatrix} s^{2/3} + 1 & -0{,}8 \\ 0{,}8 & s^{3/4} + 2 \end{bmatrix}, \tag{35}$$

$\delta = \alpha_1 + \alpha_2 = 17/12$ and characteristic function of the system (1) has the form

$$w(s) = \det H(s) = s^{17/12} + 2s^{2/3} + s^{3/4} + 0{,}64. \tag{36}$$

According to (13) and (14) we have $u_1 = 3$, $u_2 = 4$ and $m = 12$, $\alpha = 1/12$, $\delta_1 = m\alpha_1 = 8$, $\delta_2 = m\alpha_2 = 9$. This means that

$$\widetilde{H}(\lambda) = \begin{bmatrix} \lambda^{\delta_1} + 1 & -0{,}8 \\ 0{,}8 & \lambda^{\delta_2} + 2 \end{bmatrix} = \begin{bmatrix} \lambda^8 + 1 & -0{,}8 \\ 0{,}8 & \lambda^9 + 2 \end{bmatrix} \tag{37}$$

and the natural degree polynomial (18) associated with the fractional degree polynomial (11) has the form

$$\widetilde{w}(\lambda) = \det \widetilde{H}(\lambda) = \lambda^{17} + \lambda^9 + 2\lambda^8 + 0{,}64. \tag{38}$$

The polynomial (38) also follows from (36) for $\lambda = s^{\alpha} = s^{1/12}$.

Plot of the function

$$\psi(j\omega) = \frac{\det H(j\omega)}{(j\omega + 10)^{\delta}}, \quad \omega \in (-\infty, \infty), \quad \delta = \frac{17}{12}, \tag{39}$$

is shown in Figure 2.

According to (32) and (33) we have

$$\psi(\infty)=\lim_{\omega\to\pm\infty}\psi(j\omega)=1,\quad \psi(0)=\det(-A)/10^{\delta}=0.1011.$$

From Figure 2 it follows that the plot of (39) does not encircle the origin of the complex plane. This means, according to Theorem 3, that the system is stable.

Computing roots λ_i ($i=1,2,...,17$) of the polynomial (38) and using (25) we obtain $\gamma=0{,}3159$. Hence

$$\gamma=0{,}3159>\alpha\frac{\pi}{2}=\frac{\pi}{24}=0{,}1309.$$

This means that (24) holds and, according to Lemma 3, the system is stable.

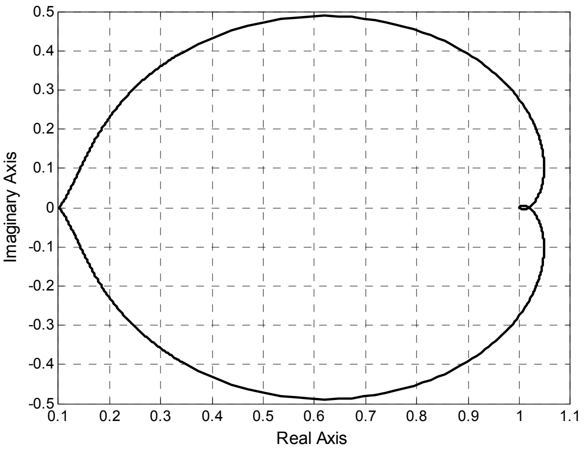

Fig. 2. Plot of the function (39)

Concluding remarks

Generalisation of the classical Mikhailov stability criterion to the class of fractional systems (1) with different fractional commensurate orders of derivatives of the state variables has been given.

In particular it has been shown that the system is stable if and only if the plot of the function (28) for $s=j\omega$ does not encircle or cross the origin of the complex plane as ω runs from $-\infty$ to ∞.

The work was supported by the National Centre of Science in Poland under grant N N514 638940.

REFERENCES

[1] Das S., *Functional Fractional Calculus for System Identification and Controls*. Springer, Berlin 2008.

[2] Kilbas A. A., Srivastava H. M. Trujillo J. J. *Theory and Applications of Fractional Differential Equations,* Elsevier, Amsterdam 2006.

[3] Ostalczyk P., *Epitome of the Fractional Calculus, Theory and its Applications in Automatics,* Publishing Department of Technical University of Łódź, Łódź 2008 (in Polish).

[4] Podlubny I., *Fractional Differential Equations,* Academic Press, San Diego 1999.

[5] Sabatier. J., Agrawal O. P., Machado J. A. T. (Eds), *Advances in Fractional Calculus, Theoretical Developments and Applications in Physics and Engineering*, Springer, London 2007.

[6] Debnath L., Recent applications of fractional calculus to science and engineering, *International Journal of Mathematics and Mathematical Sciences*, 54 (2003), 3413-3442, http://ijmms.hindawi.com.

[7] Calderon A. J., Vinagre B. M., Feliu V., Fractional order control strategies for power electronic buck converters, *Signal Processing,* 86 (2006), 2803-2819.

[8] Pommier V., Sabatier J., Lanuse P., Oustaloup A., Crone control of nonlinear hydraulic actuator, *Control Enginnering Practice,* 10 (2002), 391-402.

[9] Petras I., Stability of fractional-order systems with rational orders: a survey, *Fractional Calculus & Applied Analysis. An International Journal for Theory and Applications*, 12 (2009), 269-298.

[10] Radwan A. G., Soliman A. M., Elwakil A. S., Sedeek A., On the stability of linear systems with fractional-order elements, *Chaos, Solitons and Fractals*, 40 (2009), 2317–2328.

[11] Busłowicz M., Frequency domain method for stability analysis of linear continuous-time fractional systems. In: Malinowski K., Rutkowski L. (Eds.): *Recent Advances in Control and Automation,* Academic Publishing House EXIT, Warsaw 2008, 83-92.

[12] Busłowicz M., Stability analysis of linear continuous-time fractional systems of commensurate order, *Journal of Automation, Mobile Robotics and Intelligent Systems*, 3 (2009), 16-21.

[13] Busłowicz M., Stability of linear continuous-time fractional order systems with delays of the retarded type, *Bull. Pol. Acad. Sci., Tech. Sci.,* 56, (2008), 319-324.

[14] Busłowicz M., Stability of state-space models of linear continuous-time fractional order systems, *Acta Mechanica et Automatica*, 5 (2011), 15-22.

[15] Gałkowski K., Bachelier O., Kummert A., Fractional polynomial and nD systems a continuous case, *Proc. of IEEE Conf. on Decision & Control,* 2006, San Diego, USA.

[16] Sabatier J., Moze M., Farges C., LMI stability conditions for fractional order systems, *Computers and Mathematics with Applications*, 59 (2010), 1594-1609.

[17] Tavazoei M. S., Haeri M., Note on the stability of fractional order systems, *Mathematics and Computers in Simulation*, 79 (2009), 1566-1576.

[18] Deng W, Li C., Lu J., Stability analysis of linear fractional differential systems with multiple time delays, Nonlinear Dynamics, 48 (2007), 409-416.

[19] Busłowicz M., Computer method for stability analysis of linear discrete-time systems of fractional commensurate order, *Przegląd Elektrotechniczny*, 86 (2010), 112-115 (in Polish).

[20] Dzieliński A., Sierociuk D., Stability of discrete fractional state-space systems, *Proc. of 2nd IFAC Workshop on Fractional Differentiation and its Applications (IFAC FDA'06),* 2006, 518-523.

[21] Gałkowski K., Fractional polynomial and nD systems, *Proc. of IEEE Symposium on Circuits and Systems (ISCAS-2005),* 2005, Kobe, Japan.

[22] Kaczorek T., Fractional positive linear systems and electrical circuits, *Przegląd Elektrotechniczny*, 84 (2008), 135-141.

[23] Kaczorek T., Positivity and reachability of fractional electrical circuits, *Acta Mechanica et Automatica*, 5 (2011), 42-51.

[24] Kaczorek T., Necessary and sufficient stability conditions of fractional positive continuous-time linear systems, *Acta Mechanica et Automatica*, 5 (2011), 52-54.

[25] Kaczorek T., Positive linear systems consisting of n subsystems with different fractional orders, IEEE Trans. Circuits and Systems – I: Regular papers, vol. 58, No. 7, 2011.

[26] Kaczorek T., *Selected Problems of Fractional Systems Theory*, Publishing Department of Białystok University of Technology, Białystok 2009 (in Polish).

[27] Kaczorek T., *Selected Problems of Fractional Systems Theory*, Springer, Berlin 2011.

[28] Keel L. H. and Bhattacharyya S. P., A generalization of Mikhailov's criterion with applications, *Proc. of the American Control Conference*, Chicago, USA, 6 (2000), 4311-4315.

Autors: prof. dr hab. inż. Mikołaj Busłowicz, Politechnika Białostocka, Wydział Elektryczny, ul. Wiejska 45D, 15-351 Białystok, E-mail: busmiko@pb.edu.pl ;