Volodymyr EMETS, Jan ROGOWSKI

Institute of Computer Science Technical University of Lodz

Iterative regularization method applied to reconstruction of 3D scattering geometry

Abstract. *The problem of the shape reconstruction of a scatterer from the scattered field measured outside under the illumination of an incident plane wave is considered. Theoretically, the inversion algorithm is derived using integral equation formulations together with the Newton - Kantorovich iterative regularization technique. Numerical results show that the good reconstruction is obtained with the only one wave number of the incident wave.*

Streszczenie. *W artykule zostało przedstawione zagadnienie rekonstrukcji kształtu przeszkody przy użyciu płaskich fal. Został przedstawiony algorytm wykorzystujący reprezentację w postaci równania całkowego oraz metody regularyzacji iterowanej Newtona-Kantorowicza. Wyniki numeryczne pokazują, że odtworzenie kształtu przeszkody jest możliwe nawet dla pojedynczej liczby falowej użytej fali.(***Regularyzacja iterowana w zastosowaniu do rekonstrukcji kształtu obiektów rozpraszających***)*

Keywords: iterative regularisation, 3-D scatterer
Słowa kluczowe: regularyzacja iterowana, przeszkoda 3-D

Introduction

Inverse scattering problems are concerned with the reconstruction of the objects from the knowledge of the scattered waves and can be studied nowadays using powerful numerical techniques. These problems have attracted the increasing attention because of interest in noninvasive measurement and remote sensing. However, there are two major theoretical difficulties in solving these problems: uniqueness and ill-posedness. As the measured data are contaminated by noise or not sufficient, the reconstruction is difficult. Reviews of studies of inverse acoustic and electromagnetic scattering problems can be found in [1-5]. Today, basically three different categories of numerical methods for the treatment of the full nonlinear scattering problem are known: iterative methods, decomposition methods and sampling/probe methods. Iterative reconstruction techniques can use all the knowledge which is available about the problem. They use only minimal data that is necessary to ensure at least local uniqueness of the solution. For inverse obstacle scattering, the methods work when a scatterer is illuminated by one time-harmonic wave coming from different directions. This study proposes an iterative regularisation method for reconstruction of the 3D scatterer geometry from the far-field measurement. Scattering of the incident field by dielectric body with general geometry may be calculated using the Lippmann-Schwinger type integral equation. By applying the discrete dipole approximation method to the solution of this equation, the investigation area is divided into cells small enough that the field value and the dielectric constant in each cell can be taken as constants. Thus, the scattered fields obtained in the direct problem can be regarded as good simulated values of the measured data. Assuming that scattered data are given for several backscattering directions for the only one wave number of the incident plane wave, the solution of the inverse problem is given as the limit of the Newton-Kantorovich iterative process.

Problem formulation

Scattering of the incident field by the dielectric body D with the general geometry may be calculated using the integral equation of the form [6-7]:

(1) $$T \circ \mathbf{E} = \mathbf{E}^i(\mathbf{x}),\ \mathbf{x} \in D,$$

$$T \circ \mathbf{E}(\mathbf{x}) = \mathbf{E}(\mathbf{x}) - k_0^2 \int\limits_D \mathbf{G}(\mathbf{x},\mathbf{y})\,\delta(\mathbf{y})\mathbf{E}(\mathbf{y})\,\mathrm{d}\mathbf{y},$$

$$\mathbf{G}(\mathbf{x},\mathbf{y}) = \left(\frac{1}{\mathrm{k}^2}\nabla\nabla + \mathbf{I}\right)\mathrm{g}(\mathbf{x},\mathbf{y}),$$

$$g(\mathbf{x},\mathbf{y}) = \frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|},\ \delta(\mathbf{y}) = \varepsilon(\mathbf{y}) - \mathrm{n}^2,$$

where $\mathbf{E}$ is the total electric field (incident field $\mathbf{E}^{\mathrm{i}}(\mathbf{x})$ plus scattered field $\mathbf{E}^{\mathrm{s}}(\mathbf{x})$), $k_0 = \omega/c$ is the vacuum wave number, ε is the relative dielectric constant, $n = kk_0^{-1}$ is the refractive index, and $\mathbf{I}$ is the unit tensor. Throughout the paper we assume non-magnetic materials and an $\exp(-i\omega t)$ time dependence for the fields. The scattered field components must satisfy the radiation conditions from which it follows that

(2) $$\mathbf{E}^{\mathrm{s}}(\mathbf{x}) = \frac{1}{4\pi|\mathbf{x}|}\mathrm{e}^{\mathrm{i}k|\mathbf{x}|}\mathbf{f}(\mathrm{k};\boldsymbol{\nu}) + \mathrm{O}(|\mathbf{x}|),\ |\mathbf{x}| \to \infty,$$

$$\mathbf{f}(\mathrm{k};\boldsymbol{\nu}) = \mathrm{k}^2\int_{\mathrm{D}} \mathrm{e}^{-\mathrm{i}k(\boldsymbol{\nu},\mathbf{y})}\delta(\mathbf{y})\mathbf{E}(\mathbf{y})(\mathbf{I} - \boldsymbol{\nu}\boldsymbol{\nu})\mathrm{d}\mathbf{y},$$

$$\boldsymbol{\nu} = \mathbf{x}/|\mathbf{x}|.$$

We shall consider a monostatic scattering data $f^0(k;\mathbf{l})$ as experimentally observed data:

(3)
$$f^0(k;\mathbf{l}) = (\mathbf{f}(\mathbf{k},-\mathbf{l}),\mathbf{d}) = \mathrm{k}^2\int_{\mathrm{D}} \mathrm{e}^{\mathrm{i}k(\mathbf{l},\mathbf{y})}\delta(\mathbf{y})(\mathbf{d},\mathbf{E}(\mathbf{y}))\mathrm{d}\mathbf{y},$$

$\boldsymbol{\nu} = -\mathbf{l}$, where $\mathbf{l}$ is the direction of the incident plane wave E^i, $(\mathbf{d},\mathbf{l}) = 0$, (.,.) is a scalar product.

Let D is a star-like body of revolution, and its surface is parameterized in the form $\mathbf{y} = \rho(\hat{\mathbf{y}})\hat{\mathbf{y}}$, $\hat{\mathbf{y}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, where (θ,ϕ) are the spherical coordinates. Calculating the Frechet derivation of the scattering data (3) at the point ρ_0, we get

(4) $$f^0(k;\mathbf{l}) = \mathrm{A}(\mathrm{k};\mathbf{l}) \circ \rho,$$

$$A'_F \circ = k^2 \int_{S_0} \circ\rho_0^2(\hat{\mathbf{y}})\mathbf{exp}\left[\mathrm{i}k\rho_0(\mathbf{l},\hat{\mathbf{y}})\right]\left(\mathbf{E}(\rho_0\hat{\mathbf{y}},\mathbf{d})\right)\mathrm{dS}_0(\hat{\mathbf{y}})$$

where S_0 is a surface of the unity sphere, $dS_0(\widetilde{\mathbf{y}}) = \sin\theta\mathrm{d}\theta\mathrm{d}\phi$. Thus, a linearization of the equality (3) leads to the relation

(5)
$$D_F \circ \delta\rho = \int_{S_0} \delta\rho(\widetilde{\mathbf{y}})\mathrm{D_F}(\mathrm{k};\mathbf{l},\widetilde{\mathbf{y}})\mathrm{dS}_0(\widetilde{\mathbf{y}}) = \mathrm{f}[\rho_0;\mathbf{l}],\ \mathbf{l} \in \mathrm{S}_0,$$

$$D_F(k;\mathbf{l},\hat{\mathbf{y}}) = \mathrm{x}_0^2(\hat{\mathbf{y}})\mathbf{exp}\left[\mathrm{ix}_0(\hat{\mathbf{y}})(\mathbf{l},\hat{\mathbf{y}})\right]\mathrm{T}^{-1}(\rho_0\hat{\mathbf{y}})\circ\mathbf{E}^{\mathrm{i}}\left[\mathrm{x}_0(\hat{\mathbf{y}})\right],$$

$$\delta\rho = \rho(\widetilde{\mathbf{y}}) - \rho_0(\widetilde{\mathbf{y}}),\ \mathrm{x}_0 = \mathrm{k}\rho_0(\widetilde{\mathbf{y}}),$$

$$f\left[\rho_0;\mathbf{l}\right] = A(k;\mathbf{l})\circ\rho_0 - \mathrm{f}^0(\mathrm{k};\mathbf{l})$$

where $T^{-1}(\rho_0\hat{\mathbf{y}})$ is an inverse operator to the operator (1) for $\mathbf{y} = \rho\hat{\mathbf{y}}$.

Suppose, that $\rho(\hat{\mathbf{y}}) \in \mathrm{L}_2(\mathrm{S}_0)$. Consider a conjugate operator $D_F^* : L_2(S_0) \rightarrow L_2(S_0)$ in the sense of a scalar product in $L_2(S_0)$:

$$D_F^*\circ = \int_{S_0} \circ D_F^*(k;\mathbf{l},\widetilde{\mathbf{y}})\mathrm{dS}_0(\mathbf{l}).$$

Representing an unknown function $\rho(\widetilde{\mathbf{y}})$ as a limit of recurrent sequence, we get

(6)
$$\rho^{(n+1)} = \rho^{(n)} + \left(D_F^* D_F + \gamma_n E\right)^{-1} \times$$

$$\times\left[D_F^*\circ f\left[\rho^{(n)},\mathbf{l}\right] - \gamma_n\rho^{(n)}\right],$$

$$\gamma_n \rightarrow 0,\ n\rightarrow\infty;\ \rho^{(0)} = \rho_0,\ n = 0,1,2,...$$

where E is a unity operator. Supplementing the procedure (6) with a stop rule $n = n(\delta)$, where δ is an error level of $f^0(k;\mathbf{l})$, we obtain a regularizing algorithm. At that approximations (6) exist and are unique relatively to selected $\gamma_n \neq 0$, because the operator $(D_F^* D_F + \gamma_n E)^{-1}$ exists and is limited. Moreover, the function $f[\rho;\mathbf{l}]$ continuously depends on small variations $\delta\rho$, what immediately follows from expressions (1) and (5) with taking into account, that $\mathbf{E}(\rho\hat{\mathbf{y}})$ is a solution of the Fredholm integral equation of the second kind. Suppose, that $0 < a \leq \rho(\hat{\mathbf{y}}) \leq \mathrm{b} < \infty$. Having information about constants a and b, we extract a compact in $L_2(S_0)$, for which belongs $\rho(\hat{\mathbf{y}})$. Hence, by correspondent choice of γ_n one can achieve, that the sequence $\left\{\rho^{(n)}\right\}$ belongs to a compact set, from which one can extract a weakly convergent subsequence.

Numerical model

Turn now to an algebraization of the procedure (6), assuming that D is a solid of revolution and the vector $\mathbf{l}$ lies in the plane (x_1, x_3) and takes discrete values:

$$\mathbf{l} = (\sin\theta_i, 0, \cos\theta_i),\ \theta_i = \arccos\left(\frac{2i}{N_l+1} - 1\right),$$

$$i = 0,1,...,N_l+1.$$

We shall represent the function $\rho(\hat{\mathbf{y}})$ in the form of an expansion via the Legendre polynomials

(7)
$$\rho(\theta) = \sum_{n=0}^{N} a_n P_n(\cos\theta).$$

The algebraization of the equation (1) consists of discretization and transition to a system of linear algebraic equations, the number of which is determined by the wave dimensions of the scattering domain.

One approach to solve Eq. (1) is to discretize the scatterer into N volume elements where the field and the refractive index are assumed constant. This leads to a linear system of equations of the form

(8)
$$T\circ\mathbf{E}_{\mathrm{j}} = \mathbf{E}_{\mathrm{j}}^{\mathrm{i}}(\mathbf{x}_{\mathrm{j}}),\ \mathbf{x}_{\mathrm{j}} \in \mathrm{D},$$

$$\mathbf{E}_{\mathrm{j}}^{\mathrm{i}} = \mathbf{E}_{\mathrm{j}}(\mathbf{x}) - \sum_{\mathrm{m}}\mathbf{G}_{\mathrm{jm}}(\mathbf{x}_{\mathrm{j}},\mathbf{y})\delta(\mathbf{y}_{\mathrm{m}})\mathbf{E}_{\mathrm{j}}(\mathbf{y}_{\mathrm{m}}),$$

(9)
$$\mathbf{G}_{\mathrm{jm}} = \mathrm{k}^2\int_{\mathrm{V_m}}\mathbf{G}(\mathbf{x}_{\mathrm{j}},\mathbf{y})\mathrm{d}\mathbf{y},\ \mathrm{m} = 1...\mathrm{N}.$$

The latter integral over the Green's tensor is rather difficult in the case where $m = j$ due to the singularity of the Green's tensor. In particular, if an exclusion volume is placed around the singularity when carrying out the integral (9), and the limit of the exclusion volume going to zero is observed, the result will depend on the shape of the exclusion volume. For very small volume elements and $m = j$ the expression (9) only depends on the shape of the volume element V_m and not the size. Using expressions [6-7]

$$\mathbf{G}_{\mathrm{jm}} = \mathrm{k}^2\mathbf{G}(\mathbf{x}_{\mathrm{j}},\mathbf{x}_{\mathrm{m}})\mathrm{V_m},\ \mathbf{G}_{\mathrm{jj}} = -\frac{\mathbf{I}}{\mathrm{n}^2}\left(\frac{1}{3} + \mathrm{ik}^2\frac{\mathrm{V_j}}{6\pi}\right)$$

we can obtain a good approximation to Eqs. (8).

The solution of the system (8) gives the values $E_m(y_{1i}, y_{3j})$ inside D, substitution of which, in (5), leads to the values $E_m(\hat{\mathbf{y}}\rho(\theta)) \equiv E_m(\theta) \equiv E_m[\rho(\theta), \theta_i]$. From (5) and (8), we obtain

(10)
$$D_F\circ\delta\rho = 2\pi\int_0^{\pi}\sin\theta\,\delta\rho(\theta)\,D_F(k;\theta,\theta_i)\,d\theta = f[\rho_0;\theta_i],$$

$$D_F(k;\theta,\theta_i) = x_0^2(\theta)\,e^{ix_0(\theta)\cos(\theta-\theta_i)}E_m[\rho_0(\theta),\theta_i],$$

$$\delta\rho(\theta) = \sum_{n=0}^{N}\delta a_n P_n(\cos\theta),\ i = 0,1,...,N_l+1.$$

Discretization of the equation (10) is done by a standard method with use, for example, the simplest template of the quadratic formula - the rectangle method. This reduces the relation (10) to the matrix equation

$$A\circ\delta\mathbf{a} = \mathbf{f},\ \delta\mathbf{a} = (\delta a_0, ..., \delta a_N),$$

$$\mathbf{f} = (f[\rho_0;\theta_1], ..., f[\rho_0;\theta_{N_l+1}]),$$

$$\{A\}_{nj} = 2\pi\Delta\theta\sum_{j=2}^{N_\theta}P_n(\cos\theta_j)\,D_F(k;\theta_j,\theta_i)\sin\theta_j,$$

$$\Delta\theta = \frac{\pi}{N_\theta},\ \theta_j = (j-1)\,\Delta\theta.$$

Taking into account the relation obtained, the sequence (6) is transformed to the form

(11)
$$\mathbf{a}^{(n+1)} = \mathbf{a}^{(n)} + \left(A^T\circ A + \gamma_n E\right)^{-1}\left[A^T\circ\mathbf{f} - \gamma_n\mathbf{a}^{(n)}\right],$$

$$\left\{A^T\right\}_{nj} = \{A\}_{jn}^*,\ n = 0,1,...,N(\delta),$$

where index T denotes transpose of the matrix and $*$ denotes complex conjugate.

Thus, the relations (10) and (11) determine an algorithmization of the procedure reconstruction of body of revolution.

Numerical example

Figures 1 and 2 demonstrate some examples of reconstruction of the revolution surface in the case when the backscattering amplitude is known for $N_l = 3$ and $N_l = 9$ irradiation directions uniformly distributed in the range of angles $\theta_i \in (0, \pi)$. For all that $k = 0.5m^{-1}$, (m is a unity of length) and number n of iteration steps for the incident directions, $\delta(\mathbf{x}) = \mathbf{1}$, $x_0 = 0.2$ and an error level of scattering data $\delta = 10^{-2}$.

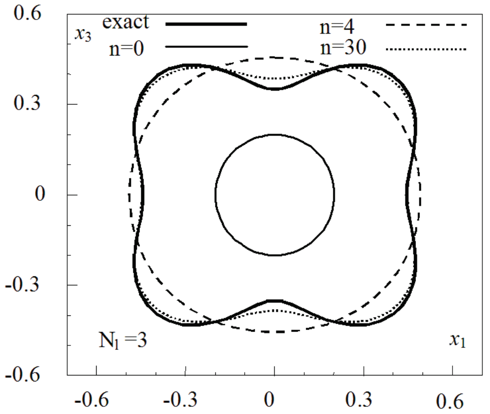

Fig. 1. Reconstruction of the surface $\rho(\theta) = 1 - 0.3P_4(\cos\theta) - 10^{-4}[P_2(\cos\theta) - P_6(\cos\theta)]$ for the incident direction and $N_l = 3$.

On Fig. 1 the true curve is described by the formula $x = k\rho(\theta)$, $m^{-1}\rho(\theta) = 1 - 0.3P_4(\cos\theta) - 10^{-4}[P_2(\cos\theta) - P_6(\cos\theta)]$ where $P_n(\cos\theta)$ are the Legendre polynomials and (ρ, θ) are the cylindrical coordinates. The number of the incident directions is $N_l = 3$.

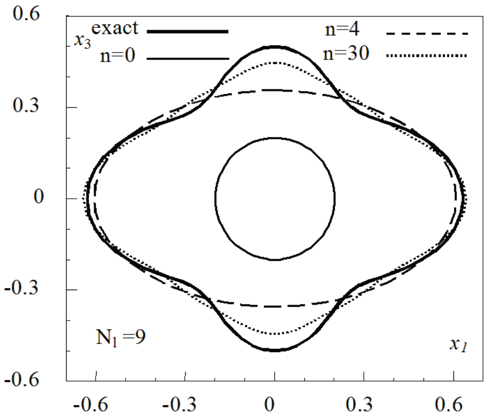

Fig. 2. Reconstruction of the surface $\rho(\theta) = 1 - 0.3[P_2(\cos\theta) - P_4(\cos\theta)]$ for the incident direction and $N_l = 9$.

The result of the reconstruction of the surface $m^{-1}\rho(\theta) = 1 - 0.3[P_2(\cos\theta) - P_4(\cos\theta)]$ is shown on Fig. 2 for $N_l = 9$ and for the same number of iterations as on Fig.1.

The numerical computations show that an augmentation of the number of the incident directions leads to an augmentation of exactness of the scatterer surface reconstruction within the limitations of the measuring errors that immediately ensues from the uniqueness theorem [5].

Conclusions

An iterative algorithm of the form reconstruction for the body of revolution is proposed. Making use of the far field data, the inverse problem can be solved in the angular spectral domain with the use of regularization. In simulation, we use several incident waves and only one frequency. Besides, in solving the direct problem more division cells are used to get a better simulated field measurement. For numerical implementation a step-by-step method is used where at each step a linearized system is being solved. These linear systems, similarly to the full problem, are also ill-posed, and so are studied with the help of iterative regularization. Such an approach is very fast in the sense of computation time in compare with known methods.

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Authors: prof. dr hab. Włodzimierz Jemec, dr inż. Jan Rogowski, Politechnika Łódzka, Instytut Informatyki, ul. Wólczańska 215, 93-005 Łódź