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Output tracking of fractional differential linear control systems

Abstract. *In the article, an elementary method for asymptotic output tracking control of a fractional linear control system is presented. A simulation example is also given.*

Streszczenie. *W artykule przedstawiono elementarną metodę śledzenia pożądanego sygnału przez wyjście ukladu liniowego niecałkowitego rzędu różniczkowego. Rozważania zostały zilustrowane przykładem obliczeniowym wraz z przeprowadzonymi symulacjami komputerowymi.* ***(Śledzenie zadanego wyjścia w układach liniowych niecałkowitego rzędu)***

Keywords: output tracking, fractional linear control systems, fractional calculus
Słowa kluczowe: śledzenie wyjścia, liniowe układy niecałkowitego rzędu, rachunek różniczkowy niecałkowitego rzędu

Introduction

In the last decades fractional integration and differentiation, as generalizations of the notions of integer-order integration and differentiation, have played a very important role in various fields such as electricity, mechanics, economics, chemistry, biology and control theory. It turns out that in many real-life cases, models described by fractional differential equations much better reflect the behavior of a phenomenon than models expressed by means of the classical calculus (see, e.g., [2, 3]). This idea was used successfully in various fields of science and engineering for modeling numerous processes [13]. Mathematical fundamentals of fractional calculus are given in the monographs [4, 6, 7, 9, 10, 12]. An elementary method of steering a fractional linear control system from a given initial state to a given final state was presented in [1]. Some fractional-order controllers were developed in, e.g., [8, 11].

In this article, it will be shown how to follow asymptotically a desired signal by an output of a SISO fractional linear control system starting from an arbitrary initial condition of all state variables. Presented design analysis concerns also the stability problem of the whole dynamics of the system, including the behavior of an unobservable part of a system, which is of great importance from the practical point of view. This method of control assumes all the state variables accessible.

The following notation will be used in the paper: the real and complex field is denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The set of natural numbers containing zero is denoted by $\mathbb{N}_0$. The set of $n \times m$ matrices with real entries will be denoted by $\mathbb{R}^{n\times m}$, and $\mathbb{R}^n := \mathbb{R}^{n\times 1}$. By $C^n(\mathbb{R})$ we mean a set of real-valued functions of one real variable possessing continuous derivatives of order $1, \ldots, n$.

Fractional control systems with Caputo derivative

Let $(\mathrm{I}^{\alpha}_{t_0+}g)(t)$ denote the Riemann-Liouville fractional left-sided integral of order $\alpha \in \mathbb{R}$, on a finite interval of the real line [4, 12]:

$$(\mathrm{I}^{\alpha}_{t_0+}g)(t) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{g(\tau)}{(t-\tau)^{1-\alpha}}\,\mathrm{d}\tau \quad \text{for } \alpha > 0,\ t > t_0,$$

where $\Gamma(\cdot)$ denotes the so-called Euler gamma function.

Let $\alpha \in \mathbb{R}$ and $\alpha \geq 0$. We will use the following definition of Caputo derivative

$$({}^{C}\mathrm{D}^{\alpha}_{t_0+}f)(t) := \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\,\mathrm{d}\tau,$$

where $n = [\alpha] + 1$, and $[\alpha]$ denotes the integer part of α. In particular, if $\alpha = n \in \mathbb{N}_0$, then

$$({}^{C}\mathrm{D}^{n}_{t_0+}f)(t) = f^{(n)}(t),$$

which is the usual n-th order derivative of function $f \in C^n(\mathbb{R})$. We will also use the shorted notations ${}^{C}\mathrm{D}^{n}_{t_0+}f$ or just $f^{(\alpha)}$ for the Caputo derivative introduced above.

We will also use so-called sequential derivative $\mathcal{D}^{k\alpha}_{t_0+}f$ defined by

$$\begin{aligned} &\mathcal{D}^{\alpha}_{t_0+}f = {}^{C}\mathrm{D}^{\alpha}_{t_0+}f, \\ (1)\qquad &\mathcal{D}^{k\alpha}_{t_0+}f = {}^{C}\mathrm{D}^{\alpha}_{t_0+}\mathcal{D}^{(k-1)\alpha}_{t_0+}f, \quad k = 2, 3, \ldots\ . \end{aligned}$$

We introduce the notion of $\mathcal{D}^{\alpha}_{t_0+}f$, because, in general,

$$\underbrace{{}^{C}\mathrm{D}^{\alpha}_{t_0+}\,{}^{C}\mathrm{D}^{\alpha}_{t_0+}\cdots{}^{C}\mathrm{D}^{\alpha}_{t_0+}}_{k\text{-times}} f \neq {}^{C}\mathrm{D}^{k\alpha}_{t_0+}f, \quad k \in \mathbb{N}_0.$$

Output tracking of a linear control system in a state space form

Let us consider a SISO linear fractional control system of the form

$$(2)\qquad \Lambda\colon \quad \begin{aligned} {}^{C}\mathbf{D}^{\alpha}_{t_0+}x &= Ax + bu, \quad 0 < \alpha < 1, \\ y_1 &= Cx, \end{aligned}$$

where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ is a state space vector, $A \in \mathbb{R}^{n\times n}$, $u(t) \in \mathbb{R}$ is a scalar input, $b \in \mathbb{R}^{n\times 1}$, $y_1 \in \mathbb{R}$ is a scalar output, $C \in \mathbb{R}^{1\times n}$, and ${}^{C}\mathbf{D}^{\alpha}_{t_0+}x = ({}^{C}\mathrm{D}^{\alpha}_{t_0+}x_1, \ldots, {}^{C}\mathrm{D}^{\alpha}_{t_0+}x_n)^T$.

The aim of control is to follow asymptotically a desired signal $y_{\mathrm{d}}(t) \in C^n(\mathbb{R})$ by the output $y_1(t)$ of the system Λ. More precisely, we look for a control input u as a function of states variables and suitable time-derivatives of $y_{\mathrm{d}}(t)$ such that $y_1(t)$ follows asymptotically $y_{\mathrm{d}}(t)$, i.e., the error

$$e(t) = (y_1(t) - y_{\mathrm{d}}(t)) \to 0 \quad \text{with} \quad t \to \infty,$$

for any initial conditions

$$x_i(t_0) = x_i^0, \quad 1 \leq i \leq n.$$

In order to simplify the problem, we will transform system Λ to a normal form. To this end, first, define an integer r such that $CA^j b = 0$ for all $j < r-1$, and $CA^{r-1}b \neq 0$. Then, using a transformation

$$y = Tx, \quad T = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^{r-1} \\ t_{r+1} \\ \vdots \\ t_n \end{pmatrix},$$

where the row vectors $t_i \in \mathbb{R}^{1\times n}$ satisfy

$$t_i b = 0, \quad r+1 \le i \le n,$$

and are such that matrix $T \in \mathbb{R}^{n\times n}$ is nonsingular, i.e., $\det T \neq 0$. Therefore, the normal form of the system Λ expressed in the new coordinates $y = (y_1, \ldots, y_n)^T$ is the following

$$\Lambda^{\mathrm{n}}: \begin{array}{l} y_1^{(\alpha)} = y_2 \\ y_2^{(\alpha)} = y_3 \\ \quad \vdots \\ y_{r-1}^{(\alpha)} = y_r \\ y_r^{(\alpha)} = D_1 y + D_2 u \\ y_{r+1}^{(\alpha)} = H_1 y \\ \quad \vdots \\ y_n^{(\alpha)} = H_{n-r} y, \end{array}$$

where $D_1 \in \mathbb{R}^{1\times n}$, $D_2 \in \mathbb{R}$, $H_j \in \mathbb{R}^{1\times n}$ for $1 \le j \le n-r$, and are given by

$$D_1 = CA^r T^{-1}, \quad D_2 = CA^{r-1} b, \quad H_j = t_{r+j} A T^{-1}.$$

Since $D_2 \neq 0$ (by construction of integer number r), we can apply to Λ^{n} a control of the form

$$u = D_2^{-1}(v - D_1 y),$$

with some new single input $v = v(t)$, getting

$$\hat{\Lambda}^{\mathrm{n}}: \begin{array}{l} \bar{y}_1^{(\alpha)} = \hat{A}\bar{y}_1 + \hat{b} v \\ \bar{y}_2^{(\alpha)} = H y, \end{array}$$

where

$$\bar{y}_1 = \begin{pmatrix} y_1 \\ \vdots \\ y_r \end{pmatrix} \in \mathbb{R}^r, \quad \bar{y}_1^{(\alpha)} = \begin{pmatrix} y_1^{(\alpha)} \\ \vdots \\ y_r^{(\alpha)} \end{pmatrix} \in \mathbb{R}^r,$$

$$\bar{y}_2 = \begin{pmatrix} y_{r+1} \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^{n-r}, \quad \bar{y}_2^{(\alpha)} = \begin{pmatrix} y_{r+1}^{(\alpha)} \\ \vdots \\ y_n^{(\alpha)} \end{pmatrix} \in \mathbb{R}^{n-r},$$

and

$$\hat{A} = \begin{pmatrix} 0\,1\,0 \cdots 0 \\ 0\,0\,1 \cdots 0 \\ 0\,0\,0 \ddots 1 \\ 0\,0\,0 \cdots 0 \end{pmatrix} \in \mathbb{R}^{r\times r}, \ \hat{b} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in \mathbb{R}^{r\times 1},$$

$$H = \begin{pmatrix} H_1 \\ \vdots \\ H_{n-r} \end{pmatrix} \in \mathbb{R}^{(n-r)\times n}.$$

We can see that control system in the normal formal $\hat{\Lambda}^{\mathrm{n}}$ possesses a structure consisting of two parts. The first one, i.e., $\bar{y}_1^{(\alpha)} = \hat{A}\bar{y}_1 + \hat{b}v$, is separated from the second one so-called unobservable part, i.e., $\bar{y}_2^{(\alpha)} = Hy$, but not vice versa.

Since, in general, for the system $\hat{\Lambda}^{\mathrm{n}}$

$$y_q^{(\alpha)} \neq y_p^{((q-p+1)\alpha)}, \quad 1 \le p < q \le r,$$

but

$$y_q^{(\alpha)} = \mathcal{D}_{t_0+}^{(q-p+1)\alpha} y_p, \quad 1 \le p \le q \le r,$$

the first part of $\hat{\Lambda}^{\mathrm{n}}$, i.e., $\bar{y}_1^{(\alpha)} = \hat{A}\bar{y}_1 + \hat{b}v$, can be rewritten in an equivalent form given by the following $r\alpha$-order fractional differential equation

$$\mathcal{D}_{t_0+}^{r\alpha} y_1 = v. \tag{3}$$

Applying to $\hat{\Lambda}^{\mathrm{n}}$, or just to (3), a control of the form

$$v = \mathcal{D}_{t_0+}^{r\alpha} y_{\mathrm{d}} - \sum_{i=0}^{r-1} k_i \mathcal{D}_{t_0+}^{i\alpha} e, \tag{4}$$

where $e = y_1 - y_{\mathrm{d}}$, and $k_i \in \mathbb{R}$, $0 \le i \le r-1$, we get

$$\mathcal{D}_{t_0+}^{r\alpha} e + k_{r-1}\mathcal{D}_{t_0+}^{(r-1)\alpha} e + \cdots + k_1 \mathcal{D}_{t_0+}^{\alpha} e + k_0 e = 0, \tag{5}$$

because sequential derivative possesses property of linearity, i.e., $\mathcal{D}_{t_0+}^{q\alpha} e = \mathcal{D}_{t_0+}^{q\alpha} y_1 - \mathcal{D}_{t_0+}^{q\alpha} y_{\mathrm{d}}$, $q \in \mathbb{N}_0$.

Using the already known result (see, e.g., [5]) about the stability of linear fractional differential systems, the following theorem states when a solution of (5) is asymptotically stable, i.e., output $y_1(t)$ follows $y_{\mathrm{d}}(t)$ for $t \to \infty$.

Theorem 1 *Equation* (5) *is asymptotically stable if and only if*

$$|\arg(\sigma_i)| > \alpha\frac{\pi}{2}, \tag{6}$$

where $\sigma_i \in \mathbb{C}$, $1 \le i \le r$, are the zeros of the following polynomial

$$w(\sigma) = \sigma^r + k_{r-1}\sigma^{r-1} + \cdots + k_1\sigma + k_0. \tag{7}$$

Therefore, the problem of output tracking gives rise to the problem of appropriate choice of real coefficients k_i, $1 \le i \le r$, such that Theorem 1 is satisfied.

From the practical point of view, even if the output already follows a given desired signal, we do not still know anything about the state variables represented by the second part of the system $\hat{\Lambda}^{\mathrm{n}}$, i.e., $\bar{y}_2^{(\alpha)} = Hy$. In other words, we have to know whether this part is stable. Notice that the unobservable part of the system $\hat{\Lambda}^{\mathrm{n}}$ is not separated from the first one, indeed, the control v does still indirectly affect state variables $\bar{y}_2$. Assuming the 1-st subsystem can be made stable after applying a suitable feedback (4), all the y_i, $1 \le i \le r$, state variables, as bounded time-varying functions, can be viewed, in some sense, for the 2-nd subsystem as inputs. It can be thus shown that in order to determine the stability of this sub-system it is sufficient to study the stability of this sub-system with all $\bar{y}_1$ variables being zero, i.e.,

$$\bar{y}_2^{(\alpha)} = H\begin{pmatrix} 0 \\ \bar{y}_2 \end{pmatrix}.$$

Therefore, we can rewrite the above system in a more consistent form

$$\bar{y}_2^{(\alpha)} = \bar{H}\bar{y}_2, \tag{8}$$

where

$$\bar{H} = \begin{pmatrix} h_{1,r+1} & \cdots & h_{1,n} \\ \vdots & \cdots & \vdots \\ h_{n-r,r+1} & \cdots & h_{n-r,n} \end{pmatrix} \in \mathbb{R}^{(n-r)\times(n-r)},$$

and $h_{ij} \in \mathbb{R}$, $1 \le i \le n-r$, $r+1 \le j \le n$, are the entries of matrix H.

The stability of system (8) can be now determined by means of Theorem 1, with polynomial (7) replaced by the characteristic polynomial of matrix $\bar{H}$, or using the result given, e.g., in [5].

Example **1** Given a linear control system of fractional order $0 < \alpha < 1$

$$\Lambda: \quad \begin{aligned} {}^{C}\mathbf{D}^{\alpha}_{0+}x &= Ax + bu, \\ y_1 &= Cx, \end{aligned}$$

where

$$A = \begin{pmatrix} -\frac{2}{3} & \frac{4}{3} & -5 & -2 \\ -\frac{10}{3} & \frac{20}{3} & -4 & 2 \\ -1 & 2 & 0 & 1 \\ 5 & -7 & 2 & -3 \end{pmatrix}, \; b = \begin{pmatrix} \frac{4}{3} \\ -\frac{4}{3} \\ 0 \\ 0 \end{pmatrix}, \; C = \begin{pmatrix} 0 \; 0 \; 1 \; 0 \end{pmatrix},$$

find a control $u(t)$ such that the output $y_1(t) = x_3(t)$ will follow a desired signal given by $y_{\rm d}(t) = \sin t$.

First, calculate

$$Cb = 0, \; CAb = -4,$$

and hence $r = 2$. Therefore, matrix of the transformation $y = Tx$ taking system Λ to the normal form $\Lambda^{\rm n}$ is

$$T = \begin{pmatrix} C \\ CA \\ t_3 \\ t_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 2 & 0 & 1 \\ 0 & 0 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}, \; T^{-1} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{4}{3} & \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \\ 1 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 \end{pmatrix}$$

because $t_3 b = 0$ and $t_4 b = 0$. Since

$$TAT^{-1} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 \\ 2 & -2 & 0 & 1 \\ 1 & 0 & -4 & 1 \end{pmatrix}, \; Tb = \begin{pmatrix} 0 \\ -4 \\ 0 \\ 0 \end{pmatrix},$$

we get

$$\Lambda^{\rm n}: \quad \begin{aligned} y_1^{(\alpha)} &= y_2 \\ y_2^{(\alpha)} &= -y_1 + 2y_2 + y_4 - 4u \\ y_3^{(\alpha)} &= 2y_1 - 2y_2 + y_4 \\ y_4^{(\alpha)} &= y_1 - 4y_3 + y_4. \end{aligned}$$

Imposing a control (with some new input v)

(9) $$u = -\frac{1}{4}\left(v + y_1 - 2y_2 - y_4\right)$$

we obtain

$$\hat{\Lambda}^{\rm n}: \quad \begin{aligned} y_1^{(\alpha)} &= y_2 \\ y_2^{(\alpha)} &= v \\ y_3^{(\alpha)} &= 2y_1 - 2y_2 + y_4 \\ y_4^{(\alpha)} &= y_1 - 4y_3 + y_4. \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \bar{y}_1^{(\alpha)} &= \hat{A}\bar{y}_1 + \hat{b}v \\ \bar{y}_2^{(\alpha)} &= Hy, \end{aligned}$$

where

$$\bar{y}_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \; \bar{y}_2 = \begin{pmatrix} y_3 \\ y_4 \end{pmatrix},$$

and

$$\hat{A} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; \hat{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; H = \begin{pmatrix} 2 & -2 & 0 & 1 \\ 1 & 0 & -4 & 1 \end{pmatrix}.$$

The first subsystem of $\hat{\Lambda}^{\rm n}$ can be rewritten in an equivalent form

$$\mathcal{D}^{2\alpha}_{t_0+}y_1 = v,$$

and applying

(10) $$v = \mathcal{D}^{2\alpha}_{0+}y_{\rm d} - k_1\mathcal{D}^{\alpha}_{0+}(y_1 - y_{\rm d}) - k_0(y_1 - y_{\rm d}),$$

we get

$$\mathcal{D}^{2\alpha}_{0+}e + k_1\mathcal{D}^{\alpha}_{0+}e + k_0 e = 0, \quad e = y_1 - y_{\rm d}.$$

Taking, e.g.,

(11) $$k_0 = 1, \; k_1 = \frac{1}{2},$$

obviously satisfying Theorem 1 for each $0 < \alpha < 1$, because $\sigma_{1,2} = -\frac{1}{4} \pm {\rm j}\frac{\sqrt{15}}{4}$, where $|\arg(\sigma_{1,2})| \approx 1.82$ rad, i.e., $|\arg(\sigma_{1,2})| > \alpha\frac{\pi}{2}$ for $0 < \alpha < 1$, we make $\lim_{t\to\infty} e(t) = 0$. Since the first part of dynamics is made stable, we can determine the stability of the 2nd part of system $\hat{\Lambda}^{\rm n}$ by studying the stability of

(12) $$\bar{y}_2^{(\alpha)} = \bar{H}\bar{y}_2, \; \bar{H} = \begin{pmatrix} 0 & 1 \\ -4 & 1 \end{pmatrix},$$

where the eigenvalues of $\bar{H}$ are $\lambda_{1,2} = \frac{1}{2} \pm {\rm j}\frac{\sqrt{15}}{2}$, and $|\arg(\lambda_{1,2})| \approx 1.32$ rad. Thus, in order to (12) be stable, i.e., $|\arg(\lambda_{1,2})| > \alpha\frac{\pi}{2}$, the fractional order α should be less than 0.84.

Finally, using (9) and (10), with k_0 and k_1 given in (11), the control $u(t)$ applied to system Λ, expressed in the original x-coordinates, causing $y_1(t)$ approaching asymptotically $y_{\rm d}(t)$ for $t \to \infty$, is the following:

$$\begin{aligned} u(t) = -\frac{1}{4}(&\mathcal{D}^{2\alpha}_{0+}y_{\rm d} + k_1\mathcal{D}^{\alpha}_{0+}y_{\rm d} + k_0 y_{\rm d} + (1 + k_1)x_1 \\ &- (5 + 2k_1)x_2 + (1 - k_0)x_3 - (3 + k_1)x_4), \end{aligned}$$

where the sequential derivatives of $y_{\rm d}(t) = \sin t$ are

$$\left(\mathcal{D}^{\alpha}_{0+}y_{\rm d}\right)(t) = \sin\left(t + \alpha\frac{\pi}{2}\right),$$

$$\left(\mathcal{D}^{2\alpha}_{0+}y_{\rm d}\right)(t) = \sin\left(t + 2\alpha\frac{\pi}{2}\right) - \frac{\sin\left(\alpha\frac{\pi}{2}\right)}{\Gamma(1-\alpha)}t^{-\alpha}.$$

The simulation plots of system Λ and $\hat{\Lambda}^{\rm n}$, for different cases of fractional orders α, are depicted at Figures 1 - 4.

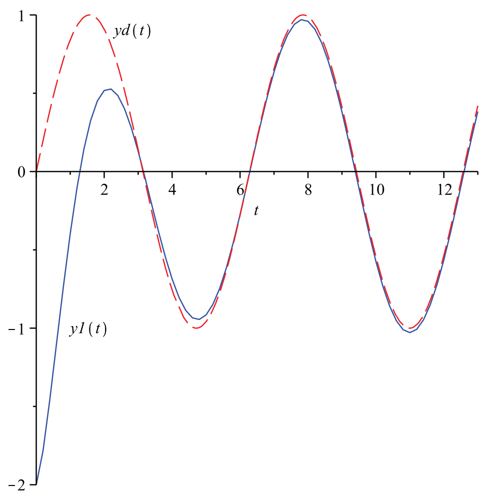

Fig. 1. Simulation results for $\alpha = 0.75$. Plots of $y_1(t) = x_3(t)$ of system Λ with $x_3(0) = -2$, and $y_{\rm d}(t) = \sin t$

In the first case ($\alpha = 0.75$) the output $y_1(t)$ follows asymptotically $y_{\rm d}(t)$, and the unobservable part of $\hat{\Lambda}^{\rm n}$ is stable.

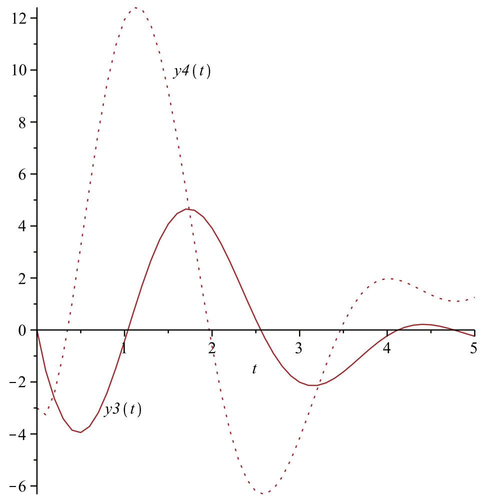

Fig. 2. Simulation results for $\alpha = 0.75$. Plots of stable unobservable dynamics of $\hat{\Lambda}^{\mathrm{n}}$, where $|\arg(\lambda_{1,2})| > \alpha\frac{\pi}{2}$ that is $1.3181 > 1.1781$

In the second case ($\alpha = 0.9$) the output $y_1(t)$ also follows asymptotically $y_{\mathrm{d}}(t)$, however, the unobservable part of $\hat{\Lambda}^{\mathrm{n}}$ is unstable. It is so, because the eigenvalues of $\bar{H}$ do not belong to the allowed stability region given by (6).

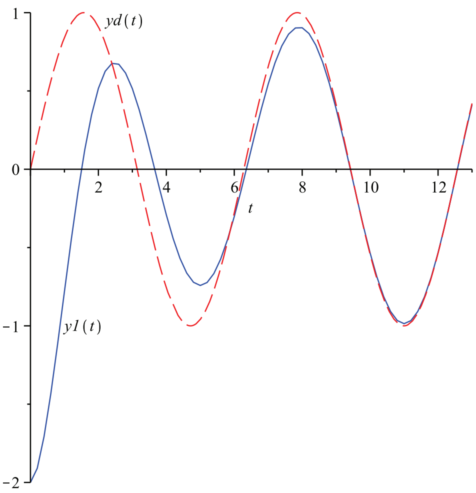

Fig. 3. Simulation results for $\alpha = 0.9$. Plots of $y_1(t) = x_3(t)$ of system Λ with $x_3(0) = -2$, and $y_{\mathrm{d}}(t) = \sin t$

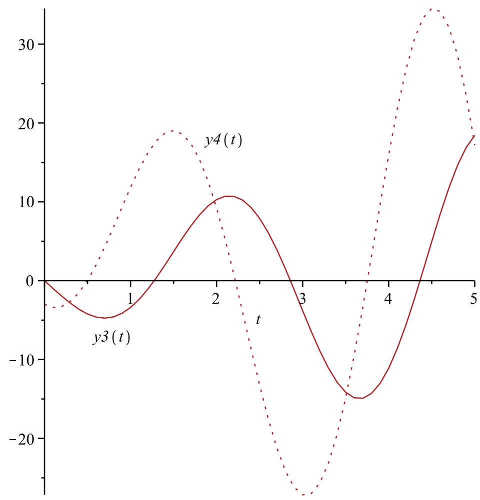

Fig. 4. Simulation results for $\alpha = 0.9$. Plots of unstable unobservable dynamics of $\hat{\Lambda}^{\mathrm{n}}$, where $|\arg(\lambda_{1,2})| < \alpha\frac{\pi}{2}$ that is $1.3181 < 1.4137$

Conclusions

In the article, for a linear fractional SISO control system described by Caputo derivative, an elementary method for asymptotic tracking by an output a desired signal was presented. Since the control design must concern the whole dynamics of the system, the unobservable behavior had been also studied with respect to its stability, which is of great importance. The illustrative simulations given in example emphasized the stability problem of both external part of dynamics and unobservable dynamics of control system. Extending of the presented method of output tracking to the MIMO fractional control systems seems to be quite straightforward.

REFERENCES

[1] A. Dzieliński and W. Malesza. Point to point control of fractional differential linear control systems. *Advances in Difference Equations*, 2011(13):1–17, 2011.

[2] A. Dzieliński, G. Sarwas, and D. Sierociuk. Ultracapacitor parameters identification based on fractional order model. In *Procedings of the European Control Conference, Budapest, Hungary*, 2009.

[3] A. Dzieliński, G. Sarwas, and D. Sierociuk. Time domain validation of ultracapacitor fractional order model. In *Procedings of the 49th IEEE Conference on Decision and Control, Atlanta, CA, USA*, 2010.

[4] A. Kilbas, H. Srivastava, and J. Trujillo. *Theory and Applications of Fractional Differential Equations*. Elsevier, Amsterdam, 2006.

[5] D. Matignon. Stability properties for generalized fractional differential systems. In *ESAIM: Proc. Fractional Differential Systems: Models, Methods and Applications*, pages 145–158, 1998.

[6] K. S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. Wiley, New York, 1993.

[7] K.B. Oldham and Spanier J. *The Fractional Calculus*. Academic Press, New York, 1974.

[8] A. Oustalup. *Commande CRONE*. Hermès, Paris, 1993.

[9] A. Oustalup. *La dérivation non entiére*. Hermès, Paris, 1995.

[10] I. Podlubny. *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[11] I. Podlubny, L. Dorcak, and I. Kostial. On fractional derivatives, fractional order systems and $PI^{\lambda}D^{\mu}$ controllers. In *Proceedings of the 36th IEEE Conference on Decision and Control, San Diego, CA, USA*, 1997.

[12] S. Samko, A. Kilbas, and O. Marichev. *Fractional Integrals and Derivatives*. Gordon and Breach, Amsterdam, 1993.

[13] M. Vinagre, C. Monje, and A. Calderon. Fractional order systems and fractional order control actions. In *Lecture 3 of the IEEE CDC02: Fractional Calculus Applications in Automatic Control and Robotics*, 2002.

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