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A mathematical model of a DC drive on the basis of variational approaches

***Abstract**. In the paper a mathematical model of an electromechanical drive consisting of a DC drive, which drives an active load through an elastic clutch. The system is considered as an electric circuit with fixed parameters. For derivation of differential state equations a variational approach was applied. On the basis of the proposed model the transient states in the drive system have been considered. The results of mathematical simulations are presented in the graphical form.*

***Streszczenie.** W pracy przedstawiono model matematyczny napędu elektromechanicznego składającego z silnika prądu stałego, który przez sprzęgło elastyczne napędza obiekt o aktywny momencie obciążenia. System rozpatrywany jako elektryczny układ o parametrach skupionych. Do wyprowadzenia różniczkowych równań stanu elektromechanicznego wykorzystano podejścia wariacyjne. Na podstawie zaproponowanego modelu przeanalizowano nieustalone procesy dynamiczne w układzie napędowym. Wyniki symulacji matematycznych przedstawiono w postaci graficznej. (**Model matematyczny napędu prądu stałego na podstawie podejść wariacyjnych**).*

Keywords: Hamilton-Ostrogradski principle, Euler-Lagrange' systems, DC drive system.
Słowa kluczowe: zasada Hamiltona-Ostrogradskiego, Euler-Lagrange'a systemy, napęd prądu stałego.

Introduction

In the present paper, starting from a modified Hamilton-Ostrogradski principle, a mathematical model of a DC drive was presented. The drive system consists of a motor, a long elastic shaft and a load mechanism. The application of interdisciplinary approaches in the theory of methods of mathematical modelling of drive systems makes it possible to determine fully all parameters related to movement or design, what is not always possible for classical approaches.

The mathematical model of the system.

In generalized coordinates the electri charges in the stator and rotor windings are denoted as $q_{1,2} = Q_{1,2}$, the rotation angles of the rotor and the supplied load as $q_{3,4} = \gamma_{1,2}$, respectively, the currents in the windings $\dot{q}_{1,2} = i_{1,2}$ and angular velocities of the rotor and the driven mechanism, respectively as $\dot{q}_{3,4} = \omega_{1,2}$, where k – the number of generalized coordinates: two currents i_a, i_f and two rotation velocities $\omega_{1,2}$, i.e. altogether $k = 4$.

For the electromechanical system we can write the Lagrangian components as follows [1]

$$(1)\qquad \tilde{T}^* = \int_0^{i_a} \Psi_a(i_a)di_a + \int_0^{i_a} \Psi'(i_a)di_a + \int_0^{i_a} \Psi''(i_a)di_a +$$

$$+\int_0^{i_f} \Psi_f(i_f)di_f + \frac{J_1\omega_1^2}{2} + \frac{J_2\omega_2^2}{2}, \quad P^* = \frac{c_{1,2}(\gamma_2-\gamma_1)^2}{2};$$

$$(2)\qquad \Phi^* = \int_0^t \frac{1}{2}(r_a + r' + r'')i_a^2 d\tau + \int_0^t \frac{1}{2} r_f i_f^2 d\tau + \frac{\nu_{1,2}(\omega_2-\omega_1)^2}{2};$$

$$(3)\qquad D^* = \int_0^t (u_a i_a + \Delta u i_a + u_f i_f)d\tau + \int_0^t\int_0^{\omega_1} M_{EM} d\omega_1 d\tau -$$

$$-\int_0^t\int_0^{\omega_2} M(\omega_2)d\omega_2 d\tau;$$

where $\tilde{T}^*$ – total kinetic coenergy of the electromechanical system; P^* – total potential energy concentrated in the system; Φ^* – external and internal dissipation of electric and mechanical energy of the system; D^* – energy of active and passive non-potential forces acting on the system from the outside; $\Psi_a, \Psi', \Psi'', \Psi_f$ – total magnetic coupling of the rotor, series excitation winding, additional poles and the independent motor excitation winding, respectively; i_a, i', i'', i_f – currents in the rotor and excitation windings, respectively; r_a, r', r'', r_f – resistances of rotor and excitation windings, respectively; u_a – supply voltage for the rotor; Δu – voltage drop in the brush contact; u_f – supply voltage for the excitation; J_1, J_2 – inertia moments of the rotor and driven mechanism; $c_{1,2}$, $\nu_{1,2}$ – coefficients of stiffness and mechanical energy dissipation; $M_{EM}, M(\omega_2)$ – start-up (electromechanical) moment of the motor and load moment of the load, respectively.

The variation of action functional [1, 4] is given with the relationship

$$(4)\qquad \delta S = \delta\int_0^{t_1} L^* dt = \int_0^{t_1} \delta L^* dt.$$

A variation of the action functional according to Hamilton shall be equal to zero only then, when the dynamic system acts accordingly to the Euler-Lagrange equations: [1]

$$(5)\qquad \frac{d}{dt}\frac{\partial L^*}{\partial \dot{q}_k} - \frac{\partial L^*}{\partial q_k} = 0, \qquad L^* = T^* - P^* + \Phi^* - D^*,$$

where L^* – modified Lagrange function.

On the basis of (1) – (3) the extended Lagrangian is obtained, which is later introduced into Euler-Lagrange equations [1, 4]. As the result, the following formulas are obtained:

$$(6)\qquad \frac{d}{dt}\frac{\partial}{\partial i_a}\left(L^*\right) - \frac{\partial}{\partial Q_a}\left(L^*\right) = \frac{d\Psi_a(i_a)}{dt} + \frac{d\Psi'(i_a)}{dt} +$$

$$+\frac{d\Psi''(i_a)}{dt} + (r_a + r' + r'')i_a^2 - u_a + \Delta u = 0;$$

$$(7)\qquad \frac{d}{dt}\frac{\partial}{\partial i_f}\left(L^*\right) - \frac{\partial}{\partial Q_f}\left(L^*\right) = \frac{d\Psi_f(i_f)}{dt} + r_f i_a - u_f = 0;$$

(8) $$\frac{d}{dt}\frac{\partial}{\partial\omega_1}\left(L^*\right)-\frac{\partial}{\partial\gamma_1}\left(L^*\right)=J_1\frac{d\omega_1}{dt}-c_{1,2}(\gamma_2-\gamma_1)-$$

$$-\nu_{1,2}(\omega_2-\omega_1)-M_{EM}=0\ ;$$

(9) $$\frac{d}{dt}\frac{\partial}{\partial\omega_2}\left(L^*\right)-\frac{\partial}{\partial\gamma_2}\left(L^*\right)=J_2\frac{d\omega_2}{dt}+c_{1,2}(\gamma_2-\gamma_1)+$$

$$+\nu_{1,2}(\omega_2-\omega_1)+M(\omega_2)=0\ ;$$

On the basis of expressions (6) – (9), taking into account relationship (5), the nonlinear differential equations, which describe the DC drive.

(10) $$\frac{d\Psi_j}{dt}=\frac{\partial\Psi_j}{\partial i_k}\frac{di_k}{dt}+\frac{\partial\Psi_j}{\partial\gamma_1}\frac{d\gamma_1}{dt}+\frac{\partial\Psi_j}{\partial t},\quad \frac{\partial\Psi_j}{\partial t}=0,$$

where the derivatives are equal to zero:

(11) $$\frac{\partial\Psi_f}{\partial\gamma_1},\frac{\partial\Psi'}{\partial\gamma_1},\frac{\partial\Psi''}{\partial\gamma_1}\equiv 0\ ,$$

where:

(12) $$\mathbf{\Psi}\equiv\Psi_j=\left[\Psi_a,\Psi',\Psi'',\Psi_f\right]^{\mathrm{T}},\ \mathbf{i}\equiv i_k=\left[i_a,i_f\right]^{\mathrm{T}}.$$

The partial derivatives from the total magnetic couplings are given with the following relationships:

(13) $$\frac{\partial\Psi_j}{\partial i_k}=\mathbf{L}^{\partial}\equiv\begin{bmatrix}L_{aa}^{\partial} & L_a^{\partial\prime} & L_a^{\partial\prime\prime} & L_{fa}^{\partial}\\ L_{af}^{\partial} & L_f^{\partial\prime} & L_f^{\partial\prime\prime} & L_{ff}^{\partial}\end{bmatrix},\ \frac{\partial\Psi_a}{\partial\gamma_1}\frac{d\gamma_1}{dt}\equiv$$

$$\equiv\frac{\partial\Psi_a}{\partial\gamma_1}\omega_1=c_M\omega_1\Phi=E\ ,$$

where Φ – magnetic flux of the motor, c_M – constant coefficient characteristic for the machine [2], L^{∂} – differential inductances of the machine, E – SEM due to rotation.
The mathematical model is written in the following form:

(14) $$\left(L_{aa}^{\partial}+L_a^{\partial\prime}+L_a^{\partial\prime\prime}\right)\frac{di_a}{dt}+\left(L_{af}^{\partial}+L_f^{\partial\prime}+L_f^{\partial\prime\prime}\right)\frac{di_f}{dt}=$$

$$=u_a-(r_a+r'+r'')i_a-c_M\omega\Phi-\Delta u\ ;$$

(15) $$L_{fa}^{\partial}\frac{di_a}{dt}+L_{ff}^{\partial}\frac{di_f}{dt}=u_f-r_f i_f\ .$$

The electromagnetic moment of a DC motor is given with the expression [2].

(16) $$M_{EM}=c_M\Phi i_a\ ,$$

where $\Phi=\Phi(i_a,i_f)$ – magnetic flux of the machine.

After mathematical trasnformations the equations of electric states in the normal Cauchy form for the generalized case:

(17) $$\frac{di_a}{dt}=A_{11}\left(u_a-r_{\Sigma}i_a-c_M\omega\Phi-\Delta u\right)-A_{12}\left(u_f-r_f i_f\right);$$

(18) $$\frac{di_f}{dt}=A_{21}\left(u_a-r_{\Sigma}i_a-c_M\omega\Phi-\Delta u\right)+A_{22}\left(u_f-r_f i_f\right),$$

where

(19) $$r_{\Sigma}=r_a+r'+r''\ ;$$

(20) $$A_{11}=\frac{L_{ff}^{\partial}}{L_{ff}^{\partial}\left(L_{aa}^{\partial}+L_a^{\partial\prime}+L_a^{\partial\prime\prime}\right)-L_{fa}^{\partial}\left(L_{af}^{\partial}+L_f^{\partial\prime}+L_f^{\partial\prime\prime}\right)};$$

(21) $$A_{12}=\frac{L_{af}^{\partial}+L_f^{\partial\prime}+L_f^{\partial\prime\prime}}{L_{ff}^{\partial}\left(L_{aa}^{\partial}+L_a^{\partial\prime}+L_a^{\partial\prime\prime}\right)+L_{fa}^{\partial}\left(L_{af}^{\partial}+L_f^{\partial\prime}+L_f^{\partial\prime\prime}\right)};$$

(22) $$A_{21}=\frac{L_{fa}^{\partial}}{L_{ff}^{\partial}\left(L_{aa}^{\partial}+L_a^{\partial\prime}+L_a^{\partial\prime\prime}\right)+L_{fa}^{\partial}\left(L_{af}^{\partial}+L_f^{\partial\prime}+L_f^{\partial\prime\prime}\right)};$$

(23) $$A_{22}=\frac{1}{L_{ff}^{\partial}}\left(1+\frac{L_{fa}^{\partial}\left(L_{af}^{\partial}+L_f^{\partial\prime}+L_f^{\partial\prime\prime}\right)}{L_{ff}^{\partial}\left(L_{aa}^{\partial}+L_a^{\partial\prime}+L_a^{\partial\prime\prime}\right)-L_{fa}^{\partial}\left(L_{af}^{\partial}+L_f^{\partial\prime}+L_f^{\partial\prime\prime}\right)}\right);$$

For the total mathematical model of the DC drive, the relationships (8), (9) are transformed into the normal Cauchy form

(24) $$\frac{d\omega_1}{dt}=\frac{1}{J_1}\left(M_{EM}+c_{1,2}(\gamma_2-\gamma_1)+\nu_{1,2}(\omega_2-\omega_1)\right);$$

(25) $$\frac{d\omega_2}{dt}=-\frac{1}{J_1}\left(c_{1,2}(\gamma_2-\gamma_1)+\nu_{1,2}(\omega_2-\omega_1)+M(\omega_2)\right),$$

(26) $$\frac{d\gamma_1}{dt}=\omega_1;\ \frac{d\gamma_2}{dt}=\omega_2$$

At a single step, the following set of differential equations is being integrated: (17), (18), (24) – (26), taking into account expressions (12), (13), (19) – (23).

If the DC machine is non-saturated, then the differential inductances are constant.
In the general case the magnetic flux is a function of both machine current $\Phi=\Phi(i_a,i_f)$. Then the calculation of differential inductances becomes very difficult. Therefore in modelling of DC machines the approximate methods are applied. On their basis working magnetic fluxes are calculated. For example, machine saturation may be approximately considered as a function of magnetization current of the machine, i.e. $\Phi=\Phi(i_f)$. Such approach results in a substantial simplification of the mathematical model of themachine, but does not imply any model limitation. Then the coefficient $L_{ff}=L_{ff}(i_f)$ becomes nonlinear.

Results of a computer-aided simulation.

The analysis of transient states was carried out for two cases: in the first one the load moment is constant, in the second one it is given with a function: $M_O=M_N\sin 5t$.

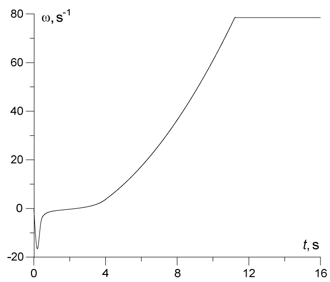

Fig. 1. Transient dependence of the rotation velocity in the first case

In Figs. 1 and 2 the transient dependences of the rotation velocity in the DC drive and of the current in the rotor winding for the first considered case are shown. Because the rotor shaft is loaded with an active moment, at the beginning of movement, right after start-up, the rotor begins to move in the opposite direction being subject to the load moment.

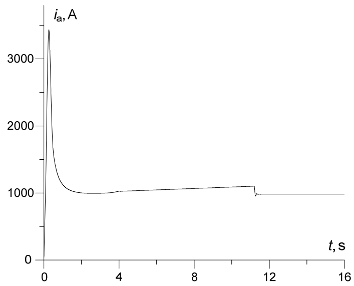

Fig. 2. Transient current in the rotor winding for the first considered case

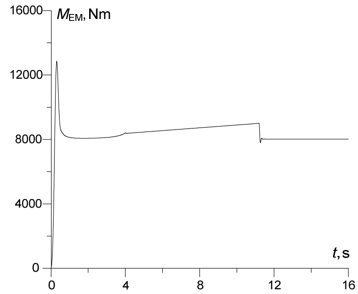

Fig. 3. Transient electromagnetic moment for the first considered case

In Fig. 3 the transient electromagnetic moment of the motor for the first considered case is depicted. Similarly, like in Fig. 2, a sudden impact action of electromagnetic field is visible.

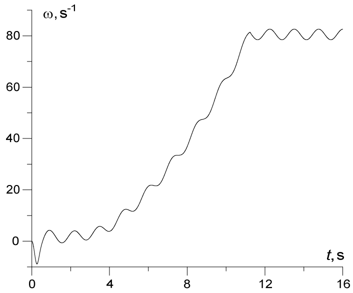

Fig. 4. Transient dependence of the rotation velocity in the second case

In Fig. 4 the transient rotation velocity of the drive system for the second considered case is depicted Because the rotor shaft is loaded with a harmonic moment, the shape of rotation velocity profile should contain additional fluctuations. In the steady state these fluctuations take the harmonic form, own frequency 0,8 Hz.

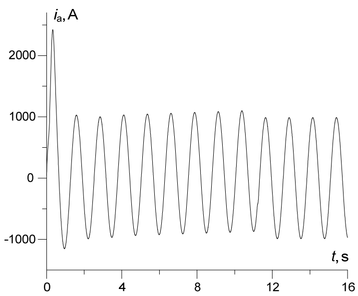

Fig. 5. Transient current in the rotor winding for the second considered case

In Fig. 5 the transient current in the rotor winding for the second considered case is depicted. Here, substantial oscillations are also visible, which are dependent on the type of load. The amplitude of current oscillations is rather high, above 1000 A, it practically reaches nominal values.

Conclusions

1. Mathematical modelling of transient and steady states may be carried with two approaches. The first one is the classic one, on the basis of the law of energy conservation, the second one – variational, on the basis of least action principle. The choice of one of the approaches is made in dependence on the level of system complexity [1,4].

2. Applications of the interdisciplinary method introduced in Ref. [1], which makes it possible to develop a wide class of dynamic objects, with consideration of dissipative processes both for systems with lumped and distributed parameters, makes it possible to develop adequate mathematical models of complicated electrical systems.

3. On the basis of variational approach a mathematical model of a complicated drive system (a DC drive with elastic shaft line) has been developed.

4. Availing of the mathematical model presented in the paper, makes it possible for the reader to analyze transient and steady states in an electromechanical system, followingly to draw conclusions on the use of these systems in reality.

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