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Electromagnetic Wave Diffraction on a perfectly conductive wedge

Abstract: *The analytic approach to solving the problem of plane electromagnetic wave diffraction on a perfectly conductive wedge is presented. The Kontorovich - Lebedev transform and Dirac delta function of a complex argument have been applied. The solution has been numerically tested and proved correct. The solution presented in this paper has a definitely simpler form, which makes it more suitable for and facilitates numerical implementation in comparison to other solutions known from literature.*

Streszczenie: *W pracy zaprezentowano analityczną metodę rozwiązywania zagadnienia rozpraszania fali elektromagnetycznej na idealnie przewodzącym klinie. Metoda ta opiera się na wykorzystaniu transformaty Lebiediewa – Kontorowicza oraz wykorzystuje funkcję delta Diraca z zespolonym argumentem. W porównaniu z innym znanym z literatury rozwiązaniem tego zagadnienia, prezentowane w niniejszej pracy ma zdecydowanie prostszą formę i jest znacznie łatwiejsze do numerycznej implementacji. (**Rozpraszanie fali elektromagnetycznej na idealnie przewodzącym klinie**)*

Keywords: electromagnetic waves, perfectly conductive wedge, Lebedev - Kontorovich transformations, Dirac delta function of a complex argument
Słowa kluczowe: fale elektromagnetyczne, idealnie przewodzący klin, transformaty Lebiediewa – Kontorowicza, delta Diraca z zespolonym argumentem

Introduction

The paper deals with an analytical approach to solving diffraction problem of an electromagnetic plane wave on a perfectly conductive wedge (Fig. 1). Naturally, this solution is widely known, still it has remained expressed in a complex, intricate form [1]. The solution presented in this paper has a definitely simpler form. It was Kontorovich - Lebedev transformation [2], [3], rarely applied numerically, and Dirac delta distribution with a complex argument, which facilitated obtaining such a form [4]. The electromagnetic field final components in the system under consideration can be written as integrals of elementary functions in a relatively simple form. Though it is not possible to compute the integrals analytically, their numerical computation encounters no particular impediments.

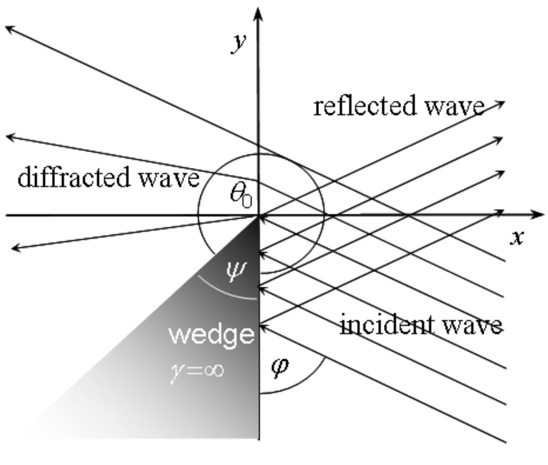

Fig. 1. Electromagnetic wave diffraction on a conductive wedge

The presented paper is a further development of the research reported elsewhere [5]. Here, not only a more general problem, namely a wedge of an arbitrary obtuse angle, is considered, but certain integrals proved to be analytically computable, which in turn allowed the final solution to take a simpler form, significantly easier to implement numerically.

System definition

Geometry of the considered system is shown in Figure 1. A linearly polarised, monochromatic plane electromagnetic wave of pulsation ω and incident angle φ onto one of the walls is the inductive field. Two wave polarisation options are considered, namely $\boldsymbol{E}||\mathrm{OZ}$ and $\boldsymbol{H}||\mathrm{OZ}$, with OZ axis along the wedge edge. This should not limit the generality, as other polarisation options for the incident wave can be treated as a superposition of these two.

It has been assumed that the wedge diffracting the electromagnetic wave is a perfect conductor, and the medium the wave propagates thorough is a non dissipative dielectric of a constant electric and magnetic permeability ε and μ.

Cylindrical coordinates r, θ, z, interrelated with $r=\sqrt{x^2+y^2}$, $\mathrm{tg}\,\theta=-x/y$, i.e. the polar angle θ measured from the wedge wall within $x=0$ plane (see Fig. 1) have been used as field functions.

<u>$\boldsymbol{E}||\mathrm{OZ}$ polarisation</u>

Under the assumptions made and for the wave polarisation option $\boldsymbol{E}||\mathrm{OZ}$, the complex amplitude axial component of the electric field strength vector E_z satisfies the equation

$$\frac{\partial^2 E_z}{\partial r^2}+\frac{1}{r}\frac{\partial E_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 E_z}{\partial \theta^2}=-k^2 E_z \tag{1}$$

where: $k=\omega\sqrt{\varepsilon\mu}=\dfrac{\omega}{c}$

The remaining components of the electric field equal zero, whereas components of the magnetic field can be related to E_z by means of Faraday's law, i.e.

$$H_r=\frac{-1}{j\omega\mu}\frac{\partial E_z}{\partial \theta},\quad H_\theta=\frac{1}{j\omega\mu}\frac{\partial E_z}{\partial r},\quad H_z=0 \tag{2}$$

The assumed perfect conductivity of the wedge results in the following boundary conditions for E_z

$$E_z(r,0)=0\ ,\quad E_z(r,\theta_0)=0 \tag{3}$$

where: $\theta_0=2\pi-\psi$

Thus, basically the solution to the problem is to find Ez function that incorporates disturbance in the form of the incident wave field

(4) $$E_z^{\mathrm{inc}}(r,\theta)=E_0 e^{\mathrm{j}kr\cos(\theta-\varphi)}$$

which satisfies both the equation (1) and the boundary conditions

H||OZ polarisation

In such a case the axial component of the magnetic field intensity satisfies the equation

(5) $$\frac{\partial^2 H_z}{\partial r^2}+\frac{1}{r}\frac{\partial H_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 H_z}{\partial \theta^2}=-k^2 H_z$$

The remaining components of the magnetic field equal zero, whereas electric field components can be related to H_z by Ampère-Maxwell law:

(6) $$E_r=\frac{1}{j\omega\varepsilon}\frac{\partial H_z}{\partial \theta}\,,\qquad E_\theta=\frac{-1}{j\omega\varepsilon}\frac{\partial H_z}{\partial r}\,,\qquad H_z=0$$

Due to the assumed perfect conductivity of the wedge, the component E_r at the wedge walls is zero, hence taking into consideration (6) the following boundary conditions for H_z are obtained

(7) $$\left.\frac{\partial H_z}{\partial \theta}\right|_{\theta=0}=0\ ,\qquad \left.\frac{\partial H_z}{\partial \theta}\right|_{\theta=\theta_0}=0$$

Again then, the solution to the problem is to find H_z function that incorporates the incident wave field

(8) $$H_z^{\mathrm{inc}}(r,\theta)=H_0 e^{\mathrm{j}kr\cos(\theta-\varphi)}$$

which satisfies both equation (5) and the boundary conditions (7).

Solution for *E*||OZ polarisation

A general solution to the problem defined in 2.1. is presented in the following form

(9) $$E_z(r,\theta)=E_0 e^{\mathrm{j}kr\cos(\theta-\varphi)}+\int_0^\infty\left[F_s(\tau)\sinh\tau\theta+F_c(\tau)\cosh\tau\theta\right]\mathrm{H}^{(2)}_{j\tau}(kr)d\tau$$

where: $\mathrm{H}^{(2)}_{j\tau}$ – Hankel function of the second kind,

F_s, F_c – arbitrary integrable functions.

It is not difficult to prove that each function expressed with (9) satisfies the equation (1). Therefore, to find a solution to the problem under consideration functions F_1, F_2 shall be defined to meet the boundary conditions (3). Substituting (9) to the first of the conditions (3) we achieve:

(10) $$\int_0^\infty F_c(\tau)\mathrm{H}^{(2)}_{j\tau}(s)d\tau=-E_0 e^{\mathrm{j}s\cos\varphi}$$

where $s=kr$

The integral in (10) is a Kontorovich - Lebedev transform of F_c function. By applying the formula for the inverse transform [1, 2] we get

(11) $$F_c(\tau)=E_0\frac{\tau\left[\exp(2\pi\tau)-1\right]}{4}\int_0^\infty\frac{e^{\mathrm{j}s\cos\varphi}}{s}\mathrm{H}^{(2)}_{j\tau}(s)ds$$

and hence

(12) $$F_c(\tau)=jE_0\exp\left(\frac{\pi\tau}{2}\right)\cosh\tau(\pi-\varphi)$$

Substituting (12) to (9) and to the second boundary condition (2) and reapplying the formula for inverse Kontorovich - Lebedev transform we obtain:

(13) $$F_s(\tau)=jE_0\exp\left(\frac{\pi\tau}{2}\right)\frac{\cosh\tau(\pi-\theta_0+\varphi)-\cosh\tau(\pi-\varphi)\cosh\tau\theta_0}{\sinh\tau\theta_0}$$

Then, substitution of (12) and (13) into (9) and partial integration yield

(14) $$E_z(r,\theta)=E_0\left\{\frac{1}{2}\left[\exp(\mathrm{j}kr\cos(\theta-\varphi))-\exp(\mathrm{j}kr\cos(\theta+\varphi))\right]+\right.$$
$$\left.+\sum_{n=1}^{6}a_n\int_0^\infty\frac{j}{2}\exp\left(\frac{\pi\tau}{2}\right)\frac{\sinh\theta_n\tau}{\sinh\theta_0\tau}\mathrm{H}^{(2)}_{j\tau}(kr)d\tau\right\}$$

where:

(15) $$\begin{aligned}&a_1=a_2=1, && a_3=a_4=a_5=a_6=\frac{1}{2},\\&\theta_1=\theta-\theta_0+\varphi+\pi, && \theta_2=\theta+\theta_0-\varphi-\pi,\\&\theta_3=\theta+\theta_0-\varphi+\pi, && \theta_4=\theta-\theta_0+\varphi-\pi,\\&\theta_5=\theta-\theta_0-\varphi+\pi, && \theta_6=\theta+\theta_0+\varphi-\pi\end{aligned}$$

Basically, it seems possible to determine E_z from the solution (14) by numerical integration. However, due to the presence of non-elementary Hankel function with an imaginary index within the integral to be computed, some difficulties arise. The values of this function are determined on the base of integral expansion with slightly regular (quickly variable) integrand:

(16) $$\mathrm{H}^{(2)}_{j\tau}(s)=-\frac{2}{j\pi}\exp\left(-\frac{\pi\tau}{2}\right)\int_0^\infty\exp(-js\cosh t)\cos t\tau\,dt$$

It requires appropriately adopted numerical procedure to be applied, which significantly affects both accuracy and time of calculations. Nevertheless, hindrances can be mitigated to great extend. By substituting (16) to (14) the integral in solution (14) takes on the form:

(17) $$-\frac{1}{\pi}\int_0^\infty\exp(-js\cosh t)\left(\int_0^\infty\frac{\sinh\theta_n\tau}{\sinh\theta_0\tau}\cos t\tau\,d\tau\right)dt$$

Let us define:

(18) $$w_n(t)=\int_0^\infty\frac{\sinh\theta_n\tau}{\sinh\theta_0\tau}\cos t\tau\,d\tau$$

For $|\theta_n|<\theta_0$ the integral yields elementary solution [6]

(19) $$w_n(t)=\frac{\pi}{2\theta_0}\frac{\sin(\pi\theta_n/\theta_0)}{\cosh(\pi t/\theta_0)+\cos(\pi\theta_n/\theta_0)},\quad |\theta_n|<\theta_0$$

The condition $|\theta_n|<\theta_0$, however, is not met for all θ_n (see (15)) over the entire considered domain, whereas for $|\theta_n|\geq\theta_0$

the integral (18) proves divergent (in Riemann's terms). For the particular case $\theta_n = \theta_0$ we have

(20) $$w_n(t) = \pi\,\delta(t), \quad \theta_n = \theta_0$$

which allows the integral to be computed further (17), though for $|\theta_n| > \theta_0$ integral (18) no longer can be expressed with the Dirac delta function within its conventional understanding. Still, it turns out that even in such a case opens an opportunity to determine its value in distributive terms. To achieve it, the integral (18) is transformed into the following form:

(21) $$w_n(t) = \int_0^\infty \frac{\sinh(\theta_n - 2\theta_0)\tau}{\sinh\theta_0\tau}\cos t\tau\, d\tau + \\ + 2\int_0^\infty \cosh(\theta_n - \theta_0)\tau \cos t\tau\, d\tau$$

As over the entire domain the condition $\theta_n < 3\theta_0$ is met (see (15)), the first of the integrals in (21) is convergent and can be calculated according to (19) when θ_n is replaced with $\theta_n - 2\theta_0$. The second of the integrals in (21) is divergent, but Dirac delta function with a complex argument helps to keep it meaningful [5]. As the result we have

(22) $$w_n(t) = \pi\delta(t - j(\theta_n - \theta_0)) + \\ + \frac{\pi}{2\theta_0}\frac{\sin(\pi(\theta_n - 2\theta_0)/\theta_0)}{\cosh(\pi t/\theta_0) + \cos(\pi(\theta_n - 2\theta_0)/\theta_0)}, \quad |\theta_n| > \theta_0$$

and making use of the basic property of the function δ

(23) $$\int_0^\infty f(x)\delta(x - jy)dx = \frac{1}{2}f(jy)$$

while calculating the integral (17) we arrive finally at the following solution for the axial component of the electric field

(24) $$E_z(r,\theta) = \frac{E_0}{2}\Big\{\big[\exp(\mathrm{j}kr\cos(\theta - \varphi)) - \exp(\mathrm{j}kr\cos(\theta + \varphi))\big] + \\ - \sum_{n=1}^{6} a_n V_n(r,\theta)\Big\}$$

where for $|\theta_n| < \theta_0$:

(25) $$V_n(r,\theta) = \frac{1}{\theta_0}\int_0^\infty \frac{\sin(\pi\theta_n/\theta_0)}{\cosh(\pi t/\theta_0) + \cos(\pi\theta_n/\theta_0)}\exp(-\,\mathrm{j}kr\cosh t)dt$$

and for $|\theta_n| \geq \theta_0$:

(26) $$V_n(r,\theta) = \exp(-\,\mathrm{j}kr\cos\pi(\theta_n - 2\theta_0)) + \\ + \frac{1}{\theta_0}\int_0^\infty \frac{\sin(\pi(\theta_n - 2\theta_0)/\theta_0)}{\cosh(\pi t/\theta_0) + \cos(\pi(\theta_n - 2\theta_0)/\theta_0)}\exp(-\,\mathrm{j}kr\cosh t)dt$$

The integrals in (25), (26) cannot be determined analytically and have to be computed numerically. The formula for the magnetic field components can be derived on the base of (2).

Solution for $\boldsymbol{H}\|\mathrm{OZ}$ polarisation

From the mathematical point of view the only difference between the problem defined for $\boldsymbol{E}\|\mathrm{OZ}$ and $\boldsymbol{H}\|\mathrm{OZ}$ polarisations consists in different setting the boundary conditions (see (3) and (6)). Nevertheless, such a difference proved immaterial for the reasoning frame presented for $\boldsymbol{H}\|\mathrm{OZ}$, hence – neglecting the derivation details - the final solution can be provided

(27) $$H_z(r,\theta) = \frac{H_0}{2}\Big\{\big[\exp(\mathrm{j}kr\cos(\theta - \varphi)) + \exp(\mathrm{j}kr\cos(\theta + \varphi))\big] + \\ + \sum_{n=1}^{6} b_n V_n(r,\theta)\Big\}$$

where V_n is defined in (24), (25). Moreover,

(28) $$b_1 = 1, \quad b_2 = -1, \quad b_3 = b_5 = \frac{1}{2}, \quad b_4 = b_6 = -\frac{1}{2}$$

Based on the solutions (24) and (27) as well as on (2) and (7) a numerical procedure was set up to determine electromagnetic field distribution in the considered system. Several test runs were performed to examine the presented solutions and prove them correct. As the boundary conditions proved fully satisfied, i.e. tangent components of the electric field at the wedge wall surfaces are null, we believe the solutions to be correct. Results will be described in detail in a separate article.

Summary

The analytic approach to solving the problem of plane electromagnetic wave diffraction on a perfectly conductive wedge is presented. The Kontorovich - Lebedev transform and Dirac delta function of a complex argument have been applied. The provided solution is a sum of six single integrals of elementary functions and is relatively simple in form. The solution has been numerically tested and proved correct. The solution presented in this paper has a definitely simpler form, which makes it more suitable for and facilitates numerical implementation in comparison to other solutions known from literature [1].

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