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Analysis of the electromagnetic field generated by electric charges moving with variable acceleration and general Lorentz transformations

Abstract: *The problem of determining electromagnetic fields generated by electric charges travelling with a variable acceleration are considered in the paper. By applying Liénarda-Wiechert's formulas the resulting fields have been explicitly defined by a charge in a linear motion exerted by a constant force, i.e. within the relativistic approach, and a distant field generated by a charge in a harmonic motion. On such basis the space-time coordinates relationships between inertial and non-inertial reference frames have been derived, which provide a certain generalization to Lorentz transformations.*

Streszczenie: *W pracy zaprezentowane jest zagadnienie obliczania rozkładów pól elektromagnetycznych generowanych przez ładunki poruszające się ze zmiennym przyśpieszeniem w oparciu o wzory Liénarda-Wiecherta. Na tej podstawie wyprowadzono relatywistyczne transformacje pomiędzy współrzędnymi czasoprzestrzennymi przy przechodzeniu od układu inercjalnego do nieinercjalnego na przypadki ruchu z przyśpieszeniem pod wpływem stałej siły oraz ruchu drgającego.* ***(Analiza pola elektromagnetycznego w otoczeniu ładunków poruszających się ze zmiennym przyśpieszeniem).***

Keywords: electromagnetic field, non-inertial reference frames, Lorentz's transformations, Liénard-Wiechert potentials
Słowa kluczowe: pole elektromagnetyczne, układy nieinercjalne, transformacje Lorentza, potencjały Liénarda –Wiecherta

Introduction

An intense search aimed at providing a mathematical formulation for the electrodynamics laws in non-inertial reference frames has driven us to complete this paper. A concept which served it is basically finding a distribution for the electromagnetic field generated by sources in accelerated motion. Liénard–Wiechert retarded potentials [1, 2] seem perfect to perform this task. If field functions can be put in an explicit form, which is correct only for particular cases of moving sources, then transformation relationships can be found, for both space-time coordinates and field components in both types of the considered reference frames, i.e. inertial and non-inertial ones, which provides general Lorentz transformations. The relations obtained that way shall allow certain differential operators to be used to derive electrodynamics equations in the considered non-inertial reference frame.

The paper deals with fields generated by a point electric charge in a linear motion

a) exerted by a constant force as in the relativistic approach,
b) in a harmonic motion.

For a) case both electromagnetic field and transformations of space-time coordinates, including the classical limit have been fully provided, whereas for b) case electromagnetic field for the distant zone has been found.

General description of the field generated by a point charge in a linear accelerated motion

An electromagnetic field at position $\boldsymbol{r}$ and at time t generated by a point electrical charge Q travelling along arbitrary trajectory (Fig. 1) can be determined with Liénard-Wiechert forms for retarded potentials V, $\boldsymbol{A}$ [1, 2], namely

$$V(\boldsymbol{r},t)=\frac{1}{4\pi\varepsilon_0}\frac{Q}{R-\dfrac{\boldsymbol{R}\cdot\boldsymbol{v}(\tau)}{\mathrm{c}}}$$
(1)
$$\boldsymbol{A}(\boldsymbol{r},t)=\frac{\mu_0}{4\pi}\frac{Q\boldsymbol{v}(\tau)}{R-\dfrac{\boldsymbol{R}\cdot\boldsymbol{v}(\tau)}{\mathrm{c}}}$$

where: $\boldsymbol{R}=\boldsymbol{r}-\boldsymbol{r}_0(\tau)$, $\tau=t-R/\mathrm{c}$, $\boldsymbol{v}(\tau)$ – velocity of the charge, c – the speed of light.

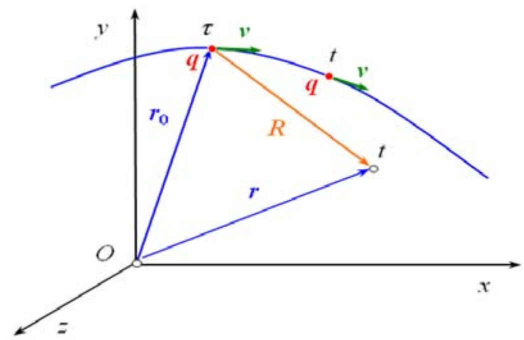

Fig.1. A graphic illustration to Liénard–Wiechert forms

Knowing that

$$\boldsymbol{E}=-\frac{\partial\boldsymbol{A}}{\partial t}-\mathrm{grad}\,V,\ \ \boldsymbol{B}=\mathrm{rot}\,\boldsymbol{A}$$

the electromagnetic field vectors can be determined as:

(2)
$$\boldsymbol{E}(\boldsymbol{r},t)=\frac{Q}{4\pi\varepsilon_0 R_*^3}\left\{\gamma^2\left(\boldsymbol{R}-\frac{R\boldsymbol{v}}{\mathrm{c}}\right)+\frac{1}{\mathrm{c}^2}\left[\boldsymbol{R}\times\left(\left(\boldsymbol{R}-\frac{R\boldsymbol{v}}{\mathrm{c}}\right)\times\boldsymbol{a}\right)\right]\right\}$$
$$\boldsymbol{B}(\boldsymbol{r},t)=\frac{1}{\mathrm{c}}\frac{\boldsymbol{R}}{R}\times\boldsymbol{E}$$

where: $R_*=R-\boldsymbol{R}\cdot\boldsymbol{v}/\mathrm{c}$, $\gamma=\sqrt{1-\beta^2}$, $\beta=v/\mathrm{c}$

The case under our consideration is a charged particle in a linear motion, here along the OX axis (Fig. 2), thus position, velocity and acceleration vectors take their respective forms of:

(3) $\boldsymbol{r}_0=[x_0(\tau),0,0]$, $\quad\boldsymbol{v}=[v(\tau),0,0]$ $\quad\boldsymbol{a}=[a(\tau),0,0]$

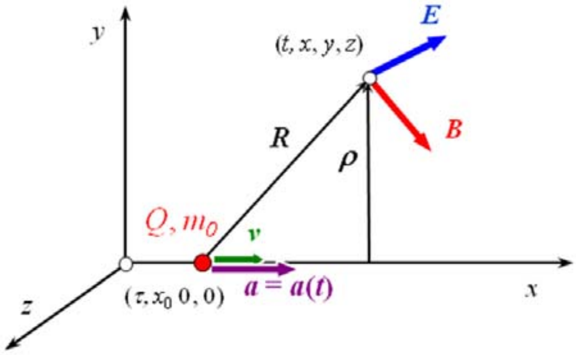

Fig.2. The setup for the considered case

Moreover,

(4) $$R=\sqrt{(x-x_0(\tau))^2+y^2+z^2}$$

and hence

(5) $$\sqrt{(x-x_0(\tau))^2+y^2+z^2}=\mathrm{c}(t-\tau)$$

Taking (2) and (3) transformed into a radial cylindrical coordinate $\rho=\sqrt{y^2+z^2}$ (see Fig. 2) we obtain

(6) $$\begin{aligned}E_x(\mathbf{r},\tau)&=\frac{Q}{4\pi\varepsilon_0R_*^3}\left[\gamma^2\left(x-x_0-\frac{Rv}{\mathrm{c}}\right)-\frac{a}{\mathrm{c}^2}\rho^2\right]\\E_\rho(\mathbf{r},\tau)&=\frac{Q\rho}{4\pi\varepsilon_0R_*^3}\left[\gamma^2+\frac{a}{\mathrm{c}^2}(x-x_0)\right]\\B_\theta(\mathbf{r},\tau)&=\frac{Q}{4\pi}\sqrt{\frac{\mu_0}{\varepsilon_0}}\frac{\rho^2}{R_*^3}\left[\gamma^2+R\frac{a}{\mathrm{c}}\right]\end{aligned}$$

To have these components expressed as a function of "observation" time t a relation between τ time and t needs to be found from equation (5). In general the equation is solved numerically, and in its explicit form only for specific relations $x_0(\tau)$. Two of such cases are to be developed further in the paper.

Description of the field generated by a charge in a linear accelerated motion exerted by a constant force

Shall the point charge be assumed to be moving under constant force $F=a_0m_0$ with an initial velocity $v(0)=0$, then by solving the relativistic motion equations we obtain

(7) $$x_0(\tau)=\frac{\mathrm{c}^2}{a_0}(p(\tau)-1),\ v(\tau)=\frac{a_0\tau}{p(\tau)},\ a(\tau)=\frac{a_0}{p^3(\tau)}$$

where: $p(\tau)=\sqrt{1+(a_0\tau/\mathrm{c})^2}$

For such a case the solution to equation (5) takes the form

(8) $$\tau=\frac{\mathrm{c}\,t\left(A-2c^2t^2\right)+D\sqrt{\Delta}}{2\mathrm{c}\left(D^2-\mathrm{c}^2t^2\right)}$$

where:

$$A=\rho^2+D^2+G,\quad D=x+\frac{\mathrm{c}^2}{a},\ G=\frac{c^4}{a_0^2}+\mathrm{c}^2t^2,$$

$$\Delta=A^2-4\left(D^2G+\mathrm{c}^2t^2\rho^2\right)$$

Formula (8) provides the relationship between τ and x, ρ as well as t. By substituting (8) into (7) and further to (6) an explicit form of the field distribution is obtained, though the form derived is quite complex. Exemplary distributions obtained this way are presented in Figures 3, 4, 5.

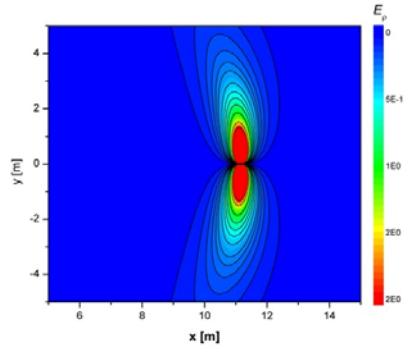

Fig. 3 Components of electric field strength E_ρ

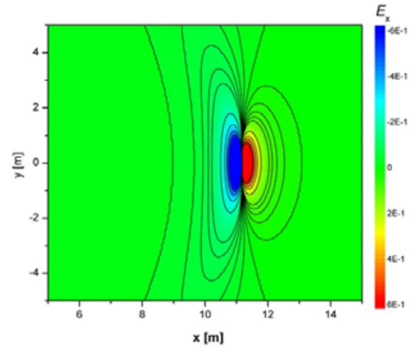

Fig. 4 Component of electric field strength E_x

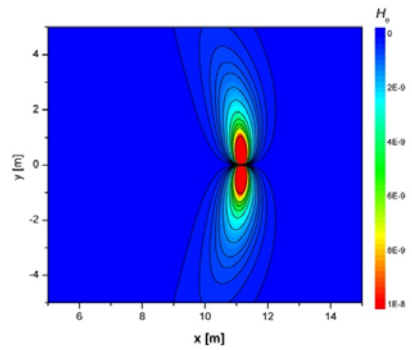

Fig. 5 Component of magnetic field strength H_θ.

Description of the field generated by a charge in an oscillating motion

For such a case the movement of the charge is expressed as:

(8) $$\begin{aligned}&x_0(\tau)=R_0\sin\omega\tau,\quad v(\tau)=\omega R_0\cos\omega\tau\\&a(\tau)=-\omega^2R_0\sin\omega\tau\end{aligned}$$

Equation (5) takes the following form

(9) $$\sqrt{(x-R_0\sin\omega\tau))^2+\rho^2}=\mathrm{c}(t-\tau)$$

Though the exact solution can be obtained only numerically, still within the distant zone (R>>R_0) $R=r=\sqrt{x^2+y^2+z^2}$ and thus

(10) $$\tau = t - \frac{r}{\mathrm{c}}$$

The field components are expressed by

(11)
$$E_x(\mathbf{r},\tau)=\frac{Q}{4\pi\varepsilon_0 R_*^3}\left[\left(1-\frac{R_0^2\omega^2\cos^2\omega\tau}{\mathrm{c}^2}\right)\cdot\left(x_0-R_0\sin\omega\tau-\frac{rR_0\omega}{\mathrm{c}}\cos\omega\tau\right)+\frac{\rho^2R_0\omega^2}{\mathrm{c}^2}\sin\omega\tau\right]$$
$$E_\rho(\mathbf{r},\tau)=\frac{Q\rho}{4\pi\varepsilon_0 R_*^3}\left[\left(1-\frac{R_0^2\omega^2\cos^2\omega\tau}{\mathrm{c}^2}\right)+-\frac{x_0-R_0\sin\omega\tau}{\mathrm{c}^2}R_0\omega^2\sin\omega\tau\right]$$
$$B_\theta(\mathbf{r},\tau)=\frac{Q\rho\, R_0\omega}{4\pi c^2RR_*^3}\left[\left(1-\frac{R_0^2\omega^2\cos^2\omega\tau}{\mathrm{c}^2}\right)r\cos\omega\tau++\frac{\rho^2+(x-R_0\sin\omega\tau)^2}{\mathrm{c}}\omega\sin\omega\tau\right]$$

Transformation of the space-time coordinates

Taking the solution presented in p.3 transformation relationships between O X Y Z inertial reference frame, which is the one where we define the charge movement, and O'X'Y'Z', a non-inertial one set at the charge travelling due to constant force (Fig. 6).

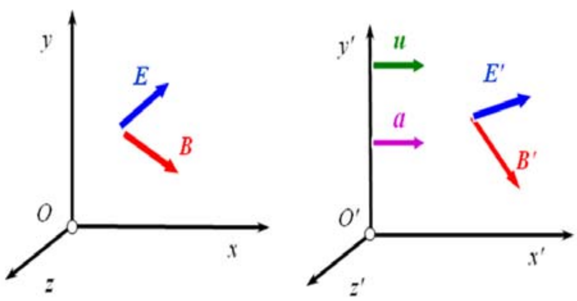

Fig. 6 Reference frames under consideration

For $\rho = 0$ the E_x component from eq. (6) takes the form of

(12)
$$E_x(\mathbf{r},\tau)=\frac{Q}{4\pi\varepsilon_0}\frac{\gamma^2(x-x_0)(1-\beta)}{[(x-x_0)(1-\beta)]^3}=$$
$$=\frac{Q}{4\pi\varepsilon_0}\frac{1}{[(x-x_0)g(\tau)]^2}$$

thus it is a Coulomb field. If observed that similarly to transformation between two inertial reference frames, the field component parallel to the velocity is not transformed, it shall be taken

(13) $$x' = [x - x_0(\tau)]g(\tau),\ \ g(\tau)=\sqrt{\frac{1-\beta}{1+\beta}}$$

Moreover, with an axiomatic assumption that O point moves with reference to O' with $-\boldsymbol{v}$ velocity, an inverse relationship can be derived

(14) $$x = [x' + x_0(\tau')]g_*(\tau'),\quad g_*(\tau')=\sqrt{\frac{1+\beta}{1-\beta}}$$

as well as: $\tau = t - \dfrac{x - x_0(\tau)}{\mathrm{c}}, \quad \tau' = t' - \dfrac{x' + x_0(\tau')}{\mathrm{c}}$

Theoretically equations (13) and (14) can be solved for t and t' which provides the following pairs of operations for space-time coordinates

(15)
$$x' = f(x,t) \qquad x = f'(x',t')$$
$$t' = g(x,t) \qquad t = g'(x',t')$$

The explicit forms for the operations are presented for two cases, namely for $a = a_0 = \mathrm{const}$ (classical boundary case i.e. for $v << c$), and for a determined in eq. (7). Complex operations result in

for $x_0 = \frac{1}{2}a_0\tau^2$

(16)
$$x' = \left(x - \frac{1}{2}a_0\tau^2\right)g(\tau),\ \ x = \left(x' + \frac{1}{2}a_0\tau'^2\right)g_*(\tau')$$
$$t' = \frac{x}{\mathrm{c}}g(\tau) + \frac{\mathrm{c}}{a_0}\left(1-\sqrt{\Delta}\right)\sqrt{g(\tau)}$$
$$t = \frac{x'}{\mathrm{c}}g_*(\tau') - \frac{\mathrm{c}}{a_0}\left(1-\sqrt{\Delta_*}\right)\sqrt{g_*(\tau')}$$

where: $\tau = \dfrac{\mathrm{c}}{a_0}\left(1-\sqrt{\Delta}\right), \qquad \Delta = 1 + \dfrac{2a_0}{\mathrm{c}^2}(x - \mathrm{c}t)$,

$$\tau' = \frac{\mathrm{c}}{a_0}\left(1-\sqrt{\Delta_*}\right), \qquad \Delta_* = 1 - \frac{2a_0}{\mathrm{c}^2}(x' - \mathrm{c}t')$$
$$g(\tau) = \sqrt{\frac{1 - a_0\tau/\mathrm{c}}{1 + a_0\tau/\mathrm{c}}},\ g_*(\tau') = \sqrt{\frac{1 - a_0\tau'/\mathrm{c}}{1 + a_0\tau'/\mathrm{c}}}$$

Since $\Delta \geq 0$, hence:

(17) $$t \leq \frac{\mathrm{c}}{2a} + \frac{x}{\mathrm{c}}$$

while for (7):

(18)
$$x' = (x - x_0(\tau))g(\tau),\ \ x = (x' + x_0(\tau'))g_*(\tau')$$
$$t = \frac{1}{\mathrm{c}}\left[x' + \frac{\mathrm{c}^2}{a}(p_* - 1)\right]g_* + S + \frac{\mathrm{c}}{a} - \frac{\mathrm{c}}{a}p_*$$
$$t' = \frac{1}{\mathrm{c}}\left[x - \frac{\mathrm{c}^2}{a}(p - 1)\right]g + T - \frac{\mathrm{c}}{a} + \frac{\mathrm{c}}{a}p$$

where

$$S = \tau = \frac{\mathrm{c}^2}{a}\sqrt{(p_* - 1)^2 g_*^2 + 2(p_* - 1)g_*}$$
$$T = \tau' = \frac{\mathrm{c}^2}{a}\sqrt{(p - 1)^2 g^2 + 2(p - 1)g}$$
$$\tau = \frac{\left(ct - D + \frac{\mathrm{c}^2}{a}\right)\left(\mathrm{c}t - D - \frac{\mathrm{c}^2}{a}\right)}{2\mathrm{c}(\mathrm{c}t - D)}$$
$$\tau' = \frac{\left(\mathrm{c}t' - D_* - \frac{\mathrm{c}^2}{a}\right)\left(\mathrm{c}t' - D_* + \frac{\mathrm{c}^2}{a}\right)}{2\mathrm{c}(\mathrm{c}t' - D_*)}$$
$$D = x + \frac{\mathrm{c}^2}{a}, \qquad D_* = x' - \frac{\mathrm{c}^2}{a}$$

Comments

For the presented approach to prove proper and correct two postulates need to be met, namely

a) the electric field component parallel to the velocity vector in the reference frame of the travelling charge is of a Coulomb like field, and

b) $v(\tau) = -v(\tau')$,

Considering common Lorentz transformation the postulates set above seem natural, nevertheless it might be worthwhile to present another three arguments proving the derived formulas to be correct.

a) for $v = \mathrm{const}$ leads to the classic Lorentz transformation for inertial reference frames,

b) for $v << c$ in case with $c \to \infty$ it yields Galileo transformations

$$\begin{aligned} &\tau = t \qquad \tau' = t' = t \\ &x' = x - x_0(t) \\ &x = x' + x_0(t) \end{aligned} \tag{19}$$

Moreover,

$$\boldsymbol{E}' = \boldsymbol{E} \qquad \boldsymbol{B}' = \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}$$

c) widely accepted in the literature “the clock postulate” (see p. [3] or else) is also satisfied here, e.i. the derived transformation forms provide for space-time coordinates to depend on the velocity of the systems, but not on the acceleration.

Conclusions

Lienard-Wiechert retarded potentials can be applied to obtain an explicit description of the electromagnetic field originating from a point electrical charge moving with a variable acceleration exerted by a constant force.

For the charge in a harmonic motion, the explicit form for the electromagnetic field description in the distant zone can be derived.

Resulting field distributions allow to provide general Lorentz transformations for the space-time coordinates in the inertial and non-inertial reference frames.

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