Maciej SŁAWIŃSKI

Warsaw University of Technology, Faculty of Electrical Engineering, Institute of Control and Industrial Electronic

Perfect observers for continuous-time nonlinear systems

Abstract. *This paper presents theory of the perfect observers for nonlinear systems. The perfect observers have been proposed for the canonical form of the observable nonlinear systems. The existence conditions have been presented for the perfect observers for nonlinear systems. The perfect observers can be obtained using proposed method. All of the considerations have been supported by the numerical examples.*

Streszczenie. *Artykuł przedstawia teorię obserwatorów doskonałych dla układów nieliniowych. Zostały tu opracowane obserwatory doskonałe dla postaci kanonicznej obserwowalnego układu nieliniowego. Zostały zaprezentowane warunki istnienia obserwatora doskonałego dla układu nieliniowego. Została opracowana metoda pozwalająca na wyznaczenie obserwatora doskonałego dla układu nieliniowego oraz wszystkie rozważania zostały poparte przykładami numerycznymi. (**Obserwatory doskonałe nieliniowych układów ciągłych**)*

Keywords: nonlinear systems, perfect observers, singular systems
Słowa kluczowe: układy nieliniowe, obserwatory doskonałe, układy singularne

Introduction

The models of singular linear systems have been generalized in many papers and books [1], [2], [7]. The existence and design methods of the observers for singular linear systems have been presented in [1], [2], [3], [5], [6], [8], [9], [10], [15]. L. Dai has shown [2] that it is possible to construct a singular observer which reconstruct exactly the state vector $x(k)$ of the singular system $Ex(k+1) = Ax(k) + Bu(k)$ for all $k = 0, 1, ...$. The singular observer which reconstructs exactly the state vector of the system is called perfect. Perfect observer for linear system exists if and only if system is observable [9], [10], [15].

The devices that we meet in a real world may be featured by the differential equations. Those equations are often nonlinear so the perfect observers for nonlinear systems are indispensable for most of them. The main subject of this paper is to extend the concept of the perfect observers for continuous-time nonlinear systems. Necessary and sufficient conditions are established for existence of the perfect observers for nonlinear systems and the procedure for computation of the perfect observer matrices is derived. The procedure is illustrated by an numerical example. Some simulation results are also presented.

Problem formulation

Let $R^{n\times m}$ be the set of real matrices of the dimension $n \times m$ ($R^n := R^{n\times 1}$) and the identity matrix will be denoted by I_n. Lie derivative of the function g in the direction f will be denoted by $L_f g = \sum_{i=1}^{n} f_i \delta g/\delta x_i$. Lie bracket will be denoted by $[f, g] = \frac{dg}{dx} f - \frac{df}{dx} g$ and the adjoint endomorphism will be denoted by $ad_f(g) = [f, g]$.

Let us consider continuous–time nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + g(x, u) \\ y &= h(x) \end{aligned} \tag{1}$$

where $\dot{x} = dx/dt$, $x \in R^n$, $u \in R^m$ and $y \in R$ are the state, input and output vectors, respectively and $f(x_1, x_2, ..., x_n)$, $g(x_1, ..., x_n, u_1, ..., u_m)$ are R^n valued mappings defined on the open set $U : x \in U$ and $h(x_1, x_2, ..., x_n)$ consists of nonlinear functions which are real valued functions defined on U.

The nonlinear system (1) is locally observable in U (including $x = 0$) if and only if [3], [11]

$$rank \begin{bmatrix} dh \\ dL_f h \\ \dots \\ d\left(L_f^{n-1} h\right) \end{bmatrix} = n, \forall x \in U \tag{2}$$

where $L_f^k h$ is k-th Lie derivative of the function h in the direction of f.

Let us consider singular (descriptor) continuous-time nonlinear system

$$\begin{aligned} E\dot{w} &= w + l(u, y) \\ \widetilde{x} &= v(w, u, y) \end{aligned} \tag{3}$$

where $w, \widetilde{x} \in R^n$, $u \in R^m$ and $y \in R$ are real valued vectors and $l(u, y)$, $v(w, u, y)$ are R^n- valued functions.

Definition 1 *The continuous-time nonlinear system (3) is called perfect observer for the continuous-time nonlinear system (1) if and only if*

$$\widetilde{x}(t) = x(t), \forall t > 0 \tag{4}$$

The aim of this paper is to design the perfect observers for nonlinear systems. Necessary and sufficient conditions for existence of a solution to the problem will be established and the procedure for computation matrices of the perfect observers will be proposed. All of the considerations will be supported by numerical example.

Preliminaries

Let us consider standard continuous-time linear system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{5}$$

where $x \in R^n$, $u \in R^m$ and $y \in R^p$ are the state, input and output vectors, respectively and $A \in R^{n\times n}$, $B \in R^{n\times m}$ and $C \in R^{p\times n}$ are known real matrices. If and only if standard linear system (5) is observable

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n \tag{6}$$

then there exists full order perfect observer [9], [10], [15]

$$E\dot{\widetilde{x}} = Fx + Gu - Hy \tag{7}$$

where $E, F \in R^{n\times n}$, $G \in R^{n\times m}$, $H \in R^{n\times p}$, $\widetilde{x} \in R^n$, $u \in R^m$, $y \in R^p$ and $\widetilde{x}(t) = x(t)$ for all $t > 0$.

Nonlinear system (1) can be transformed to its equivalent standard observable linear system using output feedback [1], [3], [13], [14].

Lemma 1 *There exists local diffeomorphism $T : U \to R^n$, $z = T(x)$, $T(0) = 0$, $z \in R^n$ defined on the neighbourhood U of $x = 0$ that transforms nonlinear system (1) to its* *canonical form [2], [3], [14]*

(8) $$\dot{z} = \begin{bmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} z + \begin{bmatrix} l_1(u,y) \\ l_2(u,y) \\ \vdots \\ l_n(u,y) \end{bmatrix}$$

(9) $$y = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix} z$$

if and only if $h(0) = 0$, $g(x,0) = 0$, $\forall x \in R^n$ and f, g, h are smooth in their arguments (they are real-valued functions with continuous partial derivative of any order [3]) and
1) system (1) is observable
and
2) $\left[ad_f^i r, ad_f^j r\right] = 0, 0 \le i, j \le n-1$
and
3) $\left[g, ad_f^j r\right] = 0, 0 \le j \le n-2, \forall n \in R^m$
hold where variable r is solution of the equation

(10) $$\begin{bmatrix} \langle dh, r\rangle \\ \vdots \\ \langle d\left(L_f^{n-1}h\right), r\rangle \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

There exists global diffeomorphism $T : U \to R^n$ if the conditions 1)-3) hold on R^n and
4) $ad_f^i r, 0 \le i \le n-1$
are complete vector fields. The vector field is complete if solution of the equation $\dot{x} = f(x)$ is determined $\forall t \in R$.

Problem solution

The perfect observer for nonlinear system (1) can be created if system satisfies the theorem 1.

Theorem 1 *There exists local perfect observer (11), (12) for the nonlinear system (1) if and only if conditions of the lemma 1 are satisfied.*

(11) $$\begin{bmatrix} 0 & 0 \\ I_{n-1} & 0 \end{bmatrix} \dot{w} = w + l(u,y)$$

(12) $$\hat{x} = v(w,u,y)$$

where $\hat{x} \in U \in R^n$, $\hat{x}(t) = x(t)$, $\forall t \in R$ (U is the neighbourhood of $x = 0$).

Proof. If conditions 1)-3) of the lemma1 are satisfied then there exists local diffeomorphism that transforms nonlinear system (1) to its canonical form (8)-(9). This canonical form is linear continuous-time observable system with nonlinear input. For linear observable system we can create the perfect observer [6], [9], [10], [15]. Using (9) and n–1 rows (starting from the second one) of (8) we obtain singular continuous-time linear system

(13) $$\begin{bmatrix} 0 & 0 \\ I_{n-1} & 0 \end{bmatrix} \dot{w} = I_n w + \begin{bmatrix} l_2(u,y) \\ \vdots \\ l_n(u,y) \\ -y \end{bmatrix}$$

where $w(t) = z(t)$, $\forall t > 0$ in the neighbourhood U. The error equation for this observer can be defined as follows [9], [10], [15].

(14) $$\hat{e}(t) = w(t) - z(t) = 0, \quad \forall t > 0$$

Using the diffeomorphism $z = T(x)$, $w = T(\hat{x})$, $T(0) = 0$ we obtain $T(x) = T(\hat{x})$, $\forall t > 0$ in U and then.

(15) $$\hat{x}(t) = x(t), \forall t > 0$$

in the neighbourhood of U. ■

Analogous results we obtain for global perfect observer. The global perfect observer for continuous-time nonlinear system can be created if there exists global diffeomorphism $z = T(x)$ that transforms nonlinear system to its linear observable form.

Local (global) perfect observer for nonlinear system (1) can be obtained by the use of the following procedure.

Procedure 1 ***Step 1.*** *If nonlinear system (1) satisfies conditions of the lemma1 then local (global) perfect observer (11) - (12) for nonlinear system (1) can be created.*
Step 2. *Find $z = T(x)$: $z_i = t_i(x)$, $t_i(0) = 0$, $i = 1, ..., n$ from the equation*

(16) $$\langle dt_i, ad_{(-f)}^j r\rangle = \begin{Bmatrix} 0 & for & i \ne j+1 \\ 1 & for & i = j+1 \end{Bmatrix}$$

where $1 \le i \le n$ and $0 \le j \le n-1$.
Step 3. *Using the derivative of the $z = T(x)$ and the equations of the system (1) we obtain canonical form (8) - (9).*
Step 4. *Perfect observer (11), (12) for nonlinear system (1) can be created from equation (8) excluding its first row and the equation (9) and the inverse of diffeomorphism $z = T(x)$.*

Numerical example

Create perfect observer for nonlinear system described by the following equations

(17) $$\begin{aligned} \dot{x_1} &= -x_2 + (1 + x_1)u \\ \dot{x_2} &= -x_1 x_2 + u \\ y &= x_1 \end{aligned}$$

where $f = \begin{bmatrix} -x_2 \\ -x_1 x_2 \end{bmatrix}$, $g = \begin{bmatrix} 1 + x_1 \\ 1 \end{bmatrix}$, $h = [x_1]$.

Step 1. First we check conditions of the theorem1.

(18) $$d\left(L_f h\right) = \begin{bmatrix} 0 - 1 \end{bmatrix}$$

Condition 1) is satisfied because

(19) $$rank \begin{bmatrix} dh \\ d(L_f h) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 2$$

Next find the vector r that satisfies condition (10).

(20) $$\begin{aligned} \langle dh, r\rangle = r_1 = 0 \\ \langle d(L_f h), r\rangle = -r_2 = 1 \end{aligned} \quad : \quad r = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

and

(21) $$ad_{(f)} r = \frac{dr}{dx} f - \frac{df}{dx} r = \begin{bmatrix} -1 \\ -x_1 \end{bmatrix}$$

and

(22) $$\begin{bmatrix} r, & ad_{(-f)} r \end{bmatrix} = 0$$

so condition 2) is satisfied, too.

(23) $$\begin{bmatrix} g, & ad_f^0 r \end{bmatrix} = 0$$

and the condition 3) is satisfied, too. Condition 4) of the lemma 1 is also satisfied

(24) $$\begin{bmatrix} r, & ad_f r \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & -x_1 \end{bmatrix}$$

There exists global diffeomorphism for the nonlinear system (17) that transforms it to the canonical linear observable form.
Step 2. From the following equations

$$(25)\qquad \begin{array}{r} \langle dt_1, \quad r\rangle = \frac{\partial t_1}{\partial x_1}r_1 + \frac{\partial t_1}{\partial x_2}r_2 = -\frac{\partial t_1}{\partial x_2} = 1 \\ \langle dt_1, \quad ad_{(-f)}r\rangle = \frac{\partial t_1}{\partial x_1} + \frac{\partial t_1}{\partial x_2}x_1 = 0 \\ \langle dt_2, \quad r\rangle = \frac{\partial t_2}{\partial x_1}r_1 + \frac{\partial t_2}{\partial x_2}r_2 = -\frac{\partial t_2}{\partial x_2} = 0 \\ \langle dt_2, \quad ad_{(-f)}r\rangle = \frac{\partial t_2}{\partial x_1} + \frac{\partial t_2}{\partial x_2}x_1 = 1 \end{array}$$

we obtain global diffeomorphism

$$(26)\qquad T(x) = \begin{bmatrix} \frac{1}{2}x_1^2 - x_2 \\ x_1 \end{bmatrix}$$

that transforms nonlinear system (17) to its canonical linear observable form.
Step 3. Using system equations and derivative of the equation (26) and diffeomorphism $z = T(x)$ we obtain

$$(27)\qquad \dot{z}_1 = x_1\dot{x}_1 - \dot{x}_2, \quad \dot{z}_2 = \dot{x}_1$$

The canonical form for nonlinear system (17) is given by

$$(28)\qquad \begin{array}{c} \dot{z}_1 = (y^2 + y - 1)u \\ \dot{z}_2 = z_1 - \frac{1}{2}y^2 + (1+y)u \\ 0 = z_2 - y \end{array}$$

Step 4. Using $z = T(x)$ and (28) we obtain perfect observer described by the equations

$$(29)\qquad \begin{array}{c} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \dot{w} = w + \begin{bmatrix} -\frac{1}{2}y^2 + (1+y)u \\ -y \end{bmatrix} \\ \hat{x} = \begin{bmatrix} w_2 \\ -w_1 + \frac{1}{2}w_2^2 \end{bmatrix} \end{array}$$

The simulation results are presented in Fig. 1 - Fig. 3. State vector of the system (17) and output signals of the perfect observer (29) are presented in Fig. 2. Errors of the observer output signals $e = \hat{x} - x$ are presented in Fig. 3.

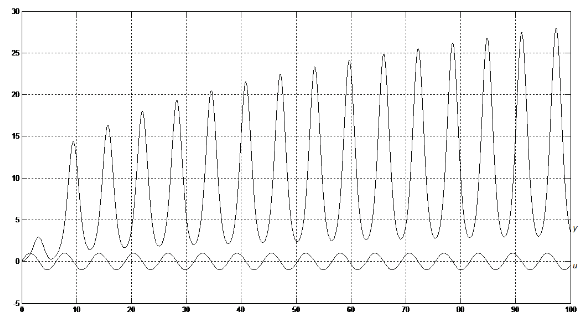

Fig. 1. Input u and output y vector

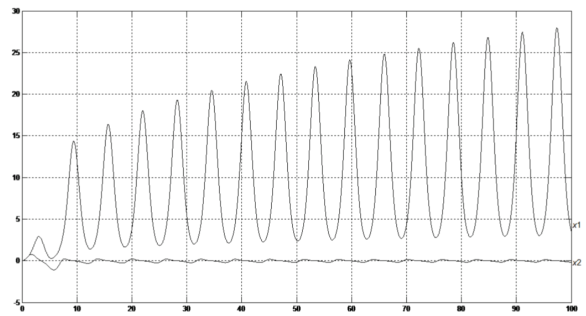

Fig. 2. State vector of the system (17) and observer output signals

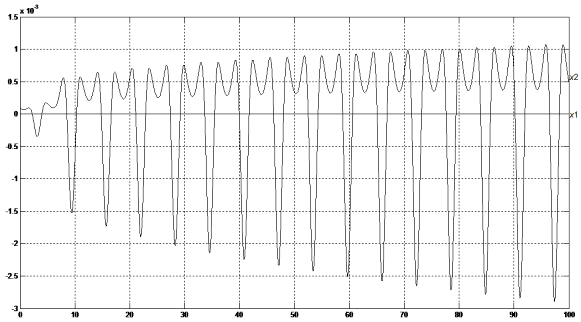

Fig. 3. Observer error

Conclusions

The perfect observers have one drawback that the output signals can depends on the derivative of the input signals. Up to now we don't know the perfect computation method of the signals derivatives. All errors of the observers output signals depends on the simulation parameters (as of the step size).

Perfect observers for special linearisable continuous-time nonlinear systems have been investigated. Procedure for designing of the perfect observers have been derived and illustrated by numerical example. Simulation results for this observer have been also presented. An extension of the considerations for non linearisable nonlinear continuous-time systems is also possible but it will be a subject of the future research.

REFERENCES

[1] Dai L.: Observers for Discrete Singular Systems, IEEE Trans. Autom. Contr., AC-33, No. 2, Febr. 1988, pp. 187-191.
[2] Dai L.: Singular Control Systems, Berlin, Tokyo: SpringerVerlag, 1989.
[3] Isidori A.: Nonlinear Control Systems, Springer, Verlag 1989.
[4] Kaczorek T.: Vectors and Matrices in Automation and Electrotechnics, Warsaw: Polish Scientific Publishers, (in Polish).
[5] Kaczorek T.: Reduced-Order Perfect and Standard Observers for Singular Continuous-Time Linear Systems, Machine Intelligence & Robotic Control, Vol.2, No. 3, pp. 93-98, 2000.
[6] Kaczorek T.: Full-order perfect observers for continuous-time linear systems, PAK 1/2001.
[7] Kaczorek T.: Linear Control Systems, New York: Research Studies, Press and J. Wiley, 1993, vol. 1 and 2.
[8] Kaczorek T.: Perfect Observers for Singular 2-D Fornasini-Marchesini Models, IEEE Trans. on Autom. Contr., Vol. 46, No. 10, October 2001.
[9] Kaczorek T., Sławiński M.: Perfect Observers for Standard Linear Systems, MMAR'2002, 2-5 September 2002, Szczecin, Poland, pp. 399-404.
[10] Kaczorek T., Sławiński M.: Perfect Observers for Standard Linear Systems, Bull. Pol. Acad. Techn. Sci., Vol. 50, No 3,2002.
[11] Marino R., Tomei P.: Nonlinear Control Design, 1995.
[12] Nijmeijer H., van der Schaft A. J.: Nonlinear Dynamical Control Systems, New York: Springer, 1990.
[13] Respondek W.: Global aspects of linearization, equivalence to polynomial forms and decomposition of nonlinear control systems. Proc. Algebraic and Geometric Methods in Nonlinear Control, Dordrecht: Reidel, pp. 257–284.
[14] Röbenack K., Lynch A. F.: Observer Design Using a Partial Nonlinear Observer Canonical Form, Int. J. Appl. Math. Comput. Sci., 2006, Vol. 16, No. 3, pp. 333–343.
[15] Sławiński M., Kaczorek T.: Perfect observers for continuous-time linear systems, PAK 1/2004, pp.39-44.

Maciej SŁAWIŃSKI: Ph.D. Maciej Sławiński, Institute of Control and Industrial Electronics, Faculty of Electrical Engineering, Warsaw University of Technology, ul. Koszykowa 75, 00-662 Warszawa, Poland, email: m.slawinski@ee.pw.edu.pl