Dorota MAJORKOWSKA-MECH, Aleksandr CARIOW

West Pomeranian University of Technology in Szczecin

An algorithm for discrete fractional Hadamad transform with reduced arithmetical complexity

Abstract. *This paper presents an algorithm for discrete fractional Hadamard transform computing for the input vector of length 2^n. This algorithm allows for significant reduction in the number of arithmetic operations by taking advantage of the specific structure of discrete fractional Hadamard transformation matrix.*

Streszczenie. *W artykule przedstawiony został algorytm wyznaczania dyskretnej frakcjonalnej transformaty Hadamarda dla wektora danych wejściowych o rozmiarze 2^n. Algorytm ten pozwala na znaczną redukcję liczby operacji arytmetycznych dzięki wykorzystaniu specyficznej struktury macierzy dyskretnej frakcjonalnej transformacji Hadamarda. (**Algorytm wyznaczania dyskretnej frakcjonalnej transformaty Hadamarda ze zmniejszoną złożonością obliczeniową**).*

Keywords: discrete fractional Hadamard transform, eigen decomposition.
Słowa kluczowe: dyskretna frakcjonalna transformata Hadamarda, rozkład w bazie wektorów własnych.

Introduction

Discrete fractional Hadamard transform (DFRHT) is a generalization of the discrete Hadamard transform (DHT) like a discrete fractional Fourier transform is a generalization of the discrete Fourier transform. Since the discrete fractional Fourier transform have been used in the theory and practice of digital signal processing [1-2], it was created also fractional versions of other discrete orthogonal transforms such as discrete fractional sine and cosine transforms [3] and the discrete fractional Hartley transform [4]. In [5] was defined a discrete fractional Hadamard transform for the vector of length $N = 2^n$. Nowadays, a modified version of DFRHT is used in cryptography, especially in images processing [6].

The aim of this paper is to propose an algorithm for calculating the discrete fractional Hadamard transform, for the vector of length $N = 2^n$, with a reduced number of arithmetic operations. Such a reduction is possible due to a special structure of the DFRHT matrix. In [7] were presented the possibilities of reducing the number of arithmetic operations in calculating the matrix-vector products for the some set of matrices with special block-structures. We will show in this paper that the DFRHT matrix has the particular block-structure.

Discrete fractional Hadamard transform

A Hadamard matrix is a symmetric square matrix whose entries are the numbers +1 and -1. The rows (and columns) of this matrix are mutually orthogonal. The normalized Hadamard matrix of order $N = 2^n$ denoted by $\mathbf{H}_N$, can be defined recursively as follows:

$$(1)\qquad \mathbf{H}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \mathbf{H}_N = \frac{1}{\sqrt{2}}\begin{bmatrix} \mathbf{H}_{\frac{N}{2}} & \mathbf{H}_{\frac{N}{2}} \\ \mathbf{H}_{\frac{N}{2}} & -\mathbf{H}_{\frac{N}{2}} \end{bmatrix}$$

for $N = 4, 8, \ldots, 2^n$.

Definition of discrete fractional Hadamard transform is based on eigen decomposition of DHT matrix. Eigen decomposition is also known as matrix diagonalization. Any real nonsingular symmetric matrix (including the Hadamard matrix) can be written as a product

$$(2)\qquad \mathbf{H}_N = \mathbf{Z}_N \mathbf{\Lambda}_N \mathbf{Z}_N^T ,$$

where $\mathbf{\Lambda}_N$ is a diagonal matrix of order $N = 2^n$ whose entries are exactly the eigenvalues of $\mathbf{H}_N$.

$$(3)\qquad \mathbf{\Lambda}_N = \begin{bmatrix} \lambda_0 & & & \\ & \lambda_1 & & \\ & & \ddots & \\ & & & \lambda_{N-1} \end{bmatrix}.$$

$\mathbf{Z}_N = [\mathbf{z}_{N\times 1}^{(0)}, \mathbf{z}_{N\times 1}^{(1)}, \ldots, \mathbf{z}_{N\times 1}^{(N-1)}]$ - is the matrix whose columns are normalized, mutually orthogonal eigenvectors of the matrix $\mathbf{H}_N$. Eigenvector $\mathbf{z}_{N\times 1}^{(k)}$ corresponds to eigenvalue λ_k. The superscript T denotes the matrix transposition operation. To obtain the decomposition of the Hadamard matrix in the form (2) it is necessary to calculate the eigenvectors and eigenvalues of the matrix $\mathbf{H}_N$.

In [8], a method for finding the eigenvalues and eigenvectors of Hadamard matrix of order 2^n has been presented. It has been shown there that, if $\mathbf{v}_{N\times 1}^{(k)}$ ($N = 2^n$ and $k = 0,1,\ldots, N\text{-}1$) is an eigenvector of $\mathbf{H}_N$ associated with eigenvalue λ, then vector

$$(4)\qquad \hat{\mathbf{v}}_{2N\times 1}^{(k)} = \begin{bmatrix} \mathbf{v}_{N\times 1}^{(k)} \\ (\sqrt{2}-1)\,\mathbf{v}_{N\times 1}^{(k)} \end{bmatrix}$$

will be an eigenvector of the matrix $\mathbf{H}_{2N}$ associated with the eigenvalue λ.

In [5] it has been shown that, if $\mathbf{v}_{N\times 1}^{(k)}$ ($N = 2^n$ and $k = 0,1,\ldots, N\text{-}1$) is an eigenvector of $\mathbf{H}_N$ associated with eigenvalue λ, then vector

$$(5)\qquad \tilde{\mathbf{v}}_{2N\times 1}^{(k)} = \begin{bmatrix} (1-\sqrt{2})\,\mathbf{v}_{N\times 1}^{(k)} \\ \mathbf{v}_{N\times 1}^{(k)} \end{bmatrix}$$

will be an eigenvector of the matrix $\mathbf{H}_{2N}$ associated with the eigenvalue $-\lambda$.

Knowing the eigenvalues +1 and -1 of the matrix $\mathbf{H}_2$ (they can be easily calculated by solving the characteristic equation) and associated with them eigenvectors

$$(6)\qquad \mathbf{v}_{2\times 1}^{(0)} = \begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}, \qquad \mathbf{v}_{2\times 1}^{(1)} = \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}$$

we are able to obtain eigenvalues and eigenvectors recursively, for arbitrary order $N = 2^n$ Hadamard matrix using expressions (4) and (5).

From what was written above, we can also deduced that the only eigenvalues of Hadamard matrices $\mathbf{H}_N$ for $N = 2^n$ are numbers +1 and -1. It should be noted that for $N > 2$ one eigenvalue of Hadamard matrix corresponds to more than one eigenvector, so the set of eigenvectors determined by this method is not unique.

In [5] it has been also shown that eigenvectors defined by (4) and (5) are mutually orthogonal, namely for $N = 2^n$ and $0 \le k, l \le N\text{-}1$ the following relationships are true:

1) if eigenvectors $\mathbf{v}_{N\times 1}^{(k)}$ and $\mathbf{v}_{N\times 1}^{(l)}$ are orthogonal ($k\neq l$) then eigenvectors $\hat{\mathbf{v}}_{2N\times 1}^{(k)}$ and $\hat{\mathbf{v}}_{2N\times 1}^{(l)}$ are also orthogonal;

2) if eigenvectors $\mathbf{v}_{N\times 1}^{(k)}$ and $\mathbf{v}_{N\times 1}^{(l)}$ are orthogonal ($k\neq l$) then eigenvectors $\tilde{\mathbf{v}}_{2N\times 1}^{(k)}$ and $\tilde{\mathbf{v}}_{2N\times 1}^{(l)}$ are also orthogonal;

3) eigenvectors $\hat{\mathbf{v}}_{2N\times 1}^{(k)}$ and $\tilde{\mathbf{v}}_{2N\times 1}^{(l)}$ are orthogonal.

The recursive method for determining the eigenvectors of the order 2^n Hadamard matrix based on the formulas (4) and (5) allows to obtain a complete set of mutually orthogonal eigenvectors of this matrix. However, the eigenvectors, being after normalization the columns of matrix $\mathbf{Z}_N$, and the corresponding eigenvalues can be ordered in different ways. In many cases, also in generalization of DHT to DFRHT, the frequency ordering of eigenvectors is important. It means that the k-th eigenvector has k sign-changes. In [5] it has been shown that if the number of sign-changes in the eigenvector $\mathbf{v}_{N\times 1}^{(k)}$ of the order 2^n Hadamard matrix is equal to k then the numbers of sign-changes in the generated from it eigenvectors $\hat{\mathbf{v}}_{2N\times 1}^{(k)}$ and $\tilde{\mathbf{v}}_{2N\times 1}^{(k)}$ of order 2^{n+1} Hadamard matrix are equal to $2k$ and $2k+1$. One is with $2k$ sign-changes, and the other will have $2k+1$ sign-changes.

The number of sign-changes in eigenvectors $\mathbf{v}_{2\times 1}^{(0)}$ and $\mathbf{v}_{2\times 1}^{(1)}$ of matrix $\mathbf{H}_2$, defined by (6), is equal to 0 and 1 respectively. If we introduce indication $a = \sqrt{2} - 1$ ($a > 0$), we are able to obtain, using formulas (4) and (5), eigenvectors of matrix $\mathbf{H}_4$:

$$\hat{\mathbf{v}}_{4\times 1}^{(0)} = \begin{bmatrix} 1 \\ a \\ a \\ a^2 \end{bmatrix}, \ \tilde{\mathbf{v}}_{4\times 1}^{(0)} = \begin{bmatrix} -a \\ -a^2 \\ 1 \\ a \end{bmatrix}, \ \hat{\mathbf{v}}_{4\times 1}^{(1)} = \begin{bmatrix} -a \\ 1 \\ -a^2 \\ a \end{bmatrix}, \ \tilde{\mathbf{v}}_{4\times 1}^{(1)} = \begin{bmatrix} a^2 \\ -a \\ -a \\ 1 \end{bmatrix}.$$

We can note that the number of sign-changes in the above eigenvectors are equal to 0, 1, 3, 2 respectively. Therefore, to obtain the frequency-ordered sequence of eigenvectors, they should be numbered as follows:

$$\mathbf{v}_{4\times 1}^{(0)} = \hat{\mathbf{v}}_{4\times 1}^{(0)}, \ \mathbf{v}_{4\times 1}^{(1)} = \tilde{\mathbf{v}}_{4\times 1}^{(0)}, \ \mathbf{v}_{4\times 1}^{(2)} = \tilde{\mathbf{v}}_{4\times 1}^{(1)}, \ \mathbf{v}_{4\times 1}^{(3)} = \hat{\mathbf{v}}_{4\times 1}^{(1)}.$$

The corresponding eigenvalues of matrix $\mathbf{H}_4$ will be equal to:

$$\lambda_0 = 1, \ \lambda_1 = -1, \ \lambda_2 = 1, \ \lambda_3 = -1.$$

We can easily check that for $N = 8$ the frequency-ordered eigenvectors of matrix $\mathbf{H}_8$ will will be as follows:

$$\mathbf{v}_{8\times 1}^{(0)} = \hat{\mathbf{v}}_{8\times 1}^{(0)}, \ \mathbf{v}_{8\times 1}^{(1)} = \tilde{\mathbf{v}}_{8\times 1}^{(0)}, \ \mathbf{v}_{8\times 1}^{(2)} = \tilde{\mathbf{v}}_{8\times 1}^{(1)}, \ \mathbf{v}_{8\times 1}^{(3)} = \hat{\mathbf{v}}_{8\times 1}^{(1)},$$

$$\mathbf{v}_{8\times 1}^{(4)} = \hat{\mathbf{v}}_{8\times 1}^{(2)}, \ \mathbf{v}_{8\times 1}^{(5)} = \tilde{\mathbf{v}}_{8\times 1}^{(2)}, \ \mathbf{v}_{8\times 1}^{(6)} = \tilde{\mathbf{v}}_{8\times 1}^{(3)}, \ \mathbf{v}_{8\times 1}^{(7)} = \hat{\mathbf{v}}_{8\times 1}^{(3)}$$

and the corresponding eigenvalues will be equal to:

$$\lambda_0 = \lambda_2 = \lambda_4 = \lambda_6 = 1, \ \lambda_1 = \lambda_3 = \lambda_5 = \lambda_7 = -1.$$

The above relations can be easily generalized in following manner:

$$(7) \qquad \begin{cases} \mathbf{v}_{2N\times 1}^{(4l)} = \hat{\mathbf{v}}_{2N\times 1}^{(2l)} \\ \mathbf{v}_{2N\times 1}^{(4l+1)} = \tilde{\mathbf{v}}_{2N\times 1}^{(2l)} \\ \mathbf{v}_{2N\times 1}^{(4l+2)} = \tilde{\mathbf{v}}_{2N\times 1}^{(2l+1)} \\ \mathbf{v}_{2N\times 1}^{(4l+3)} = \hat{\mathbf{v}}_{2N\times 1}^{(2l+1)} \end{cases} \quad \text{for } l = 0,1,\ldots,\frac{N}{2}-1$$

and

$$(8) \qquad \lambda_k = (-1)^k \quad \text{for } k = 0,1,\ldots,2N-1.$$

It shuld be noted that both the eigenvectors of the matrix $\mathbf{H}_2$ and eigenvectors obtained for higher order Hadamard matrices are not normalized. Let the symbol $||\mathbf{v}||$ means the Euclidian norm of vector $\mathbf{v}$.

Lemma: For any $N=2^n$ we have the relationship

$$(9) \qquad \left\| \mathbf{v}_{N\times 1}^{(k)} \right\| = (1+a^2)^n \quad \text{for } k = 0,1,\ldots,N-1,$$

where $a = \sqrt{2} - 1$.

Proof: The proof of this relationship is by induction method. For $n = 1$ we have

$$\left\| \mathbf{v}_{2\times 1}^{(0)} \right\|^2 = \left\| \mathbf{v}_{2\times 1}^{(1)} \right\|^2 = 1 + a^2,$$

then

$$\left\| \mathbf{v}_{2\times 1}^{(k)} \right\|^2 = (1+a^2)^1 \ \text{for } k = 0,1.$$

Let as assume that

$$\left\| \mathbf{v}_{N\times 1}^{(k)} \right\|^2 = (1+a^2)^n.$$

Then

$$\left\| \hat{\mathbf{v}}_{2N\times 1}^{(k)} \right\|^2 = \left\| \tilde{\mathbf{v}}_{2N\times 1}^{(k)} \right\|^2 = \left\| \mathbf{v}_{N\times 1}^{(k)} \right\|^2 (1+a^2) = (1+a^2)^{n+1}$$

for any $k = 0, 1, \ldots, N\text{-}1$. Since each of the vectors $\mathbf{v}_{2N\times 1}^{(l)}$ for $l = 0, 1, \ldots, 2N\text{-}1$ is either $\hat{\mathbf{v}}_{2N\times 1}^{(k)}$ or $\tilde{\mathbf{v}}_{2N\times 1}^{(k)}$ for some $k = 0, 1, \ldots, N\text{-}1$, then

$$\left\| \mathbf{v}_{2N\times 1}^{(l)} \right\|^2 = (1+a^2)^{n+1},$$

which completes the proof.

If we introduce the designation $c = 1+a^2$, then normalized eigenvectors of 2^n order Hadamard matrix will take the form

$$(10) \qquad \mathbf{z}_{N\times 1}^{(k)} = \frac{\mathbf{v}_{N\times 1}^{(k)}}{\left\| \mathbf{v}_{N\times 1}^{(k)} \right\|} = \frac{\mathbf{v}_{N\times 1}^{(k)}}{\sqrt{c^n}} \ \text{for } k = 0,1,\ldots,N-1.$$

Taking into account above relationship, the eigen decomposition (2) of 2^n order Hadamard matrix, can be written as follows:

(11) $$\mathbf{H}_N = \frac{1}{c^n}\mathbf{V}_N \mathbf{\Lambda}_N \mathbf{V}_N^T,$$

where Λ_N is the diagonal matrix whose nonzero elements are

(12) $$\lambda_k = (-1)^k = e^{-jk\pi} \quad \text{for } k = 0,1,...,N-1$$

and $\mathbf{V}_N$ is matrix, whose columns are frequency-ordered eigenvectors of $\mathbf{H}_N$, recursively obtained as it was described above.

The discrete fractional Hadamard transform (DFRHT) matrix of order $N = 2^n$ is defined by [5]

(13) $$\mathbf{H}_N^{(\alpha)} = \frac{1}{c^n}\mathbf{V}_N \mathbf{\Lambda}_N^{(\alpha)} \mathbf{V}_N^T,$$

where

(14) $$\lambda_k^{(\alpha)} = e^{-jk\alpha} \quad \text{for } k = 0,1,...,N-1.$$

The DFRHT is controlled by single angular parameter α which is connected with the angle of rotation in time-frequency space. It is easy to chack that for $\alpha = 0$ the DFRHT matrix become the identity matrix and for $\alpha = \pi$ it is transformed into ordinary Hadamard matrix. Generally the DFRHT matrix is complex-valued. Based on the definition (13) of the DFRHT matrix for $N = 2^n$ it is easy to verify that it has unitary property [5]:

(15) $$(\mathbf{H}_N^{(\alpha)})^{-1} = (\mathbf{H}_N^{(\alpha)})^* = \mathbf{H}_N^{(-\alpha)},$$

where superscript $*$ denotes a complex conjugation of all elements of the matrix.

Specific structure of DFRHT matrix

As it was described above, we are able to obtain recursively eigenvectors of 2^{n+1} order Hadamard matrix from eigenvectors of 2^n order Hadamard matrix. Therefore, it is possible to obtain the matrix of eigenvectors $\mathbf{V}_N$ recursively too. Let us consider the matrix $\mathbf{V}_N$ for $N = 2, 4, 8$ ($a = \sqrt{2} - 1$).

$$\mathbf{V}_2 = \begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix}, \quad \mathbf{V}_4 = \begin{bmatrix} 1 & -a & a^2 & -a \\ a & -a^2 & -a & 1 \\ a & 1 & -a & -a^2 \\ a^2 & a & 1 & a \end{bmatrix}.$$

Let's transpose the second and fourth column of the matrix $\mathbf{V}_4$, and then the third and fourth columns of the resulting matrix. We obtain the following matrix:

$$\overline{\mathbf{V}}_4 = \begin{bmatrix} 1 & -a & -a & a^2 \\ a & 1 & -a^2 & -a \\ a & -a^2 & 1 & -a \\ a^2 & a & a & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{V}_2 & -a\mathbf{V}_2 \\ a\mathbf{V}_2 & \mathbf{V}_2 \end{bmatrix}.$$

The matrix $\mathbf{V}_4$ differs from the matrix $\overline{\mathbf{V}}_4$ only in the order of columns. Therefore, one of these matrices can be obtained from another by multiplying it by the permutation matrix. In this case, we can write:

$$\mathbf{V}_4 = \overline{\mathbf{V}}_4 \mathbf{P}_4,$$

where

$$\mathbf{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For $N = 2$ we can also write

$$\mathbf{V}_2 = \overline{\mathbf{V}}_2 \mathbf{P}_2 = \overline{\mathbf{V}}_2,$$

where $\mathbf{P}_2$ is an identity matrix of order two

$$\mathbf{P}_2 = \mathbf{I}_2.$$

Now let us consider the case $N = 8$

$$\mathbf{V}_8 = \begin{bmatrix} 1 & -a & a^2 & -a & a^2 & -a^3 & a^2 & -a \\ a & -a^2 & a^3 & -a^2 & -a & a^2 & -a & 1 \\ a & -a^2 & -a & 1 & -a & a^2 & a^3 & -a^2 \\ a^2 & -a^3 & -a^2 & a & 1 & -a & -a^2 & a \\ a & 1 & -a & -a^2 & a^3 & a^2 & -a & -a^2 \\ a^2 & a & -a^2 & -a^3 & -a^2 & -a & 1 & a \\ a^2 & a & 1 & a & -a^2 & -a & -a^2 & -a^3 \\ a^3 & a^2 & a & a^2 & a & 1 & a & a^2 \end{bmatrix}.$$

In this case by moving the eighth column of this matrix in place of the second one, the fourth column in place of the third, the fifth column in place of the fourth, the second column in place of the fifth, the seventh column in place of the sixth, and the sixth column in place of the eighth, we obtain the following matrix:

$$\overline{\mathbf{V}}_8 = \begin{bmatrix} 1 & -a & -a & a^2 & -a & a^2 & a^2 & -a^3 \\ a & 1 & -a^2 & -a & -a^2 & -a & a^3 & a^2 \\ a & -a^2 & 1 & -a & -a^2 & a^3 & -a & a^2 \\ a^2 & a & a & 1 & -a^3 & -a^2 & -a^2 & -a \\ a & -a^2 & -a^2 & a^3 & 1 & -a & -a & a^2 \\ a^2 & a & -a^3 & -a^2 & a & 1 & -a^2 & -a \\ a^2 & -a^3 & a & -a^2 & a & -a^2 & 1 & -a \\ a^3 & a^2 & a^2 & a & a^2 & a & a & 1 \end{bmatrix}.$$

This matrix can be written as:

$$\overline{\mathbf{V}}_8 = \begin{bmatrix} \overline{\mathbf{V}}_4 & -a\overline{\mathbf{V}}_4 \\ a\overline{\mathbf{V}}_4 & \overline{\mathbf{V}}_4 \end{bmatrix}.$$

As previously we can write:

$$\mathbf{V}_8 = \overline{\mathbf{V}}_8 \mathbf{P}_8,$$

where

$$\mathbf{P}_8=\begin{bmatrix}1&0&0&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&1&0&0&0&0\\0&0&0&0&1&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&1&0&0\end{bmatrix}.$$

Generalizing the above considerations we can write:

$$\mathbf{V}_N=\overline{\mathbf{V}}_N\mathbf{P}_N \quad \text{for } N=2,4,...,2^n . \tag{16}$$

The order-2^{n+1} permutation matrix $\mathbf{P}_{2N}$ can be obtained from $\mathbf{P}_N$ permutation matrix of order 2^n in the following way. The first part (half) of the row with an even index $2k$ in the matrix $\mathbf{P}_{2N}$ is equal to the row with the index k in the matrix $\mathbf{P}_N$. The second part (half) of the row with an even index $2k$ in the matrix $\mathbf{P}_{2N}$ consists of all zeros. The first part (half) of the row with an odd index $2k$+1 in the matrix $\mathbf{P}_{2N}$ consist of all zeros and the other half is a mirror reflection of row k in the matrix $\mathbf{P}_N$. The rows are indexed starting from zero. Thus the permutation matrix in (16) can be generated recursively according to the following rules:

$$\mathbf{P}_2=\begin{bmatrix}1&0\\0&1\end{bmatrix},\quad \begin{cases}\mathbf{P}_{2N}(2k,l)=\mathbf{P}_N(k,l)\\ \mathbf{P}_{2N}(2k,N+l)=0\\ \mathbf{P}_{2N}(2k+1,l)=0\\ \mathbf{P}_{2N}(2k+1,N+l)=\mathbf{P}_N(k,N-1-l)\end{cases} \tag{17}$$

for $k, l = 0, 1, \dots, N$-1.

If we write the matrix $\mathbf{V}_N$ as a product $\overline{\mathbf{V}}_N\mathbf{P}_N$, then the expression (13) takes the form

$$\mathbf{H}_N^{(\alpha)}=\frac{1}{c^n}\overline{\mathbf{V}}_N\mathbf{P}_N\mathbf{\Lambda}_N^{(\alpha)}\mathbf{P}_N^T\overline{\mathbf{V}}_N^T . \tag{18}$$

It should be noted that the product $\mathbf{P}_N\mathbf{\Lambda}_N^{(\alpha)}\mathbf{P}_N^T$ is also a diagonal matrix, where the diagonal elements have the same values as the diagonal elements in the matrix $\mathbf{\Lambda}_N^{(\alpha)}$, only their order is different. The product $\mathbf{P}_N\mathbf{\Lambda}_N^{(\alpha)}\mathbf{P}_N^T$ multiplied by the factor $1/c^n$ (responsible for the normalization of eigenvectors) will be denoted by $\overline{\mathbf{\Lambda}}_N^{(\alpha)}$ (it is oboviously a diagonal matrix). Since we can rewrite (18) in the form

$$\mathbf{H}_N^{(\alpha)}=\overline{\mathbf{V}}_N\overline{\mathbf{\Lambda}}_N^{(\alpha)}\overline{\mathbf{V}}_N^T \tag{19}$$

where the matrix $\overline{\mathbf{V}}_N$ can be generated recursively:

$$\overline{\mathbf{V}}_2=\begin{bmatrix}1&-a\\a&1\end{bmatrix},\quad \overline{\mathbf{V}}_{2N}=\begin{bmatrix}\overline{\mathbf{V}}_N&-a\overline{\mathbf{V}}_N\\a\overline{\mathbf{V}}_N&\overline{\mathbf{V}}_N\end{bmatrix} \tag{20}$$

The 2^n order DFRHT matrix has a specific block-structure.

Theorem: Any matrix $\mathbf{S}_N=\overline{\mathbf{V}}_N\mathbf{D}_N\overline{\mathbf{V}}_N^T$ ($N=2^n$), where $\mathbf{D}_N$ is some diagonal matrix and matrix $\overline{\mathbf{V}}_N$ is defined by (20) has a structure

$$\mathbf{S}_N=\begin{bmatrix}\mathbf{F}_{\frac{N}{2}}&\mathbf{B}_{\frac{N}{2}}\\\mathbf{B}_{\frac{N}{2}}&\mathbf{G}_{\frac{N}{2}}\end{bmatrix}, \tag{21}$$

where $\mathbf{F}_{N/2}$, $\mathbf{B}_{N/2}$, $\mathbf{G}_{N/2}$ are some squere matrix and each of them has a block-structure similar to the structure of the matrix $\mathbf{S}_N$.

Proof: The proof will be by induction method. We will start from n=1 (N=2^1):

$$\mathbf{S}_2=\overline{\mathbf{V}}_2\mathbf{D}_2\overline{\mathbf{V}}_2^T=\begin{bmatrix}1&-a\\a&1\end{bmatrix}\begin{bmatrix}d_0&0\\0&d_1\end{bmatrix}\begin{bmatrix}1&a\\-a&1\end{bmatrix}=$$

$$=\begin{bmatrix}d_0+a^2d_1&a(d_0-d)\\a(d_0-d)&a^2d_0+d_1\end{bmatrix}=\begin{bmatrix}\mathbf{F}_1&\mathbf{B}_1\\\mathbf{B}_1&\mathbf{G}_1\end{bmatrix}.$$

Let us assume that the property (21) is true for a fixed $n\ge 1$ ($N=2^n$). We will show that it is true for $n+1$ ($2^{n+1}=2N$). We use the fact, that we can write the diagonal matrix $\mathbf{D}_{2N}$ in the form

$$\mathbf{D}_{2N}=\begin{bmatrix}\widehat{\mathbf{D}}_N&\mathbf{0}_N\\\mathbf{0}_N&\breve{\mathbf{D}}_N\end{bmatrix}.$$

$$\mathbf{S}_{2N}=\overline{\mathbf{V}}_{2N}\mathbf{D}_{2N}\overline{\mathbf{V}}_{2N}^T=$$

$$=\begin{bmatrix}\overline{\mathbf{V}}_N&-a\overline{\mathbf{V}}_N\\a\overline{\mathbf{V}}_N&\overline{\mathbf{V}}_N\end{bmatrix}\begin{bmatrix}\widehat{\mathbf{D}}_N&\mathbf{0}_N\\\mathbf{0}_N&\breve{\mathbf{D}}_N\end{bmatrix}\begin{bmatrix}\overline{\mathbf{V}}_N^T&a\overline{\mathbf{V}}_N^T\\-a\overline{\mathbf{V}}_N^T&\overline{\mathbf{V}}_N^T\end{bmatrix}=$$

$$=\begin{bmatrix}\overline{\mathbf{V}}_N(\widehat{\mathbf{D}}_N+a^2\breve{\mathbf{D}}_N)\overline{\mathbf{V}}_N^T&\overline{\mathbf{V}}_N(a\widehat{\mathbf{D}}_N-a\breve{\mathbf{D}}_N)\overline{\mathbf{V}}_N^T\\\overline{\mathbf{V}}_N(a\widehat{\mathbf{D}}_N-a\breve{\mathbf{D}}_N)\overline{\mathbf{V}}_N^T&\overline{\mathbf{V}}_N(a^2\widehat{\mathbf{D}}_N+\breve{\mathbf{D}}_N)\overline{\mathbf{V}}_N^T\end{bmatrix}=$$

$$=\begin{bmatrix}\mathbf{F}_N&\mathbf{B}_N\\\mathbf{B}_N&\mathbf{G}_N\end{bmatrix}.$$

All blocks $\mathbf{F}_N$, $\mathbf{B}_N$, $\mathbf{G}_N$ have form $\overline{\mathbf{V}}_N\mathbf{D}_N\overline{\mathbf{V}}_N^T$, where $\mathbf{D}_N$ is some diagonal matrix, so they have the same bloc-structure as the matrix $\mathbf{S}_N$. □

Since the matrix $\overline{\mathbf{\Lambda}}_N^{(\alpha)}$ occurring in an expression (19) is a diagonal matrix, so by virtue of the above proof, the DFRHT matrix has a block-structure such as matrix $\mathbf{S}_N$ in (21).

Possibility of reducing the number of arithmetic operations in calculating the matrix-vector product

The calculation of each discrete transform is related to the calculation of the matrix-vector product. Suppose we want to calculate the product

$$\mathbf{y}_{N\times 1}=\mathbf{S}_N\mathbf{x}_{N\times 1}, \tag{22}$$

where $N = 2^n$. We assume that the complex-valued matrix $\mathbf{S}_N$ and real-valued input data vector $\mathbf{x}_{N\times 1}$ are known. If we know nothing about the structure of matrix we have to do $2N^2$ multiplications and $2N(N\text{-}1)$ additions of real number.

If the matrix $\mathbf{S}_N$ has a structure as in expresion (21) then there is posibility of reducing the number of arithmetic operation in calculating the output signal $\mathbf{y}_{N\times 1}$, as it was shown in [7]. For $N = 2$ we can write

$$\mathbf{S}_2 = \mathbf{T}_{2\times 3}\, diag(f-b,\ g-b,\ b)\mathbf{T}_{3\times 2}^T , \tag{23}$$

where $\mathbf{T}_{2\times 3}$ is an auxiliary matrix

$$\mathbf{T}_{2\times 3} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \tag{24}$$

and $\mathbf{S}_2$ has the form (f, b, g - some complex numbers)

$$\mathbf{S}_2 = \begin{bmatrix} f & b \\ b & g \end{bmatrix}. \tag{25}$$

If we want to calculate the matrix-vector product for $N = 2$

$$\mathbf{y}_{2\times 1} = \mathbf{S}_2 \mathbf{x}_{2\times 1} \tag{26}$$

and we use the expanded form (23) of the $\mathbf{S}_2$ matrix as follows

$$\mathbf{y}_{2\times 1} = \mathbf{T}_{2\times 3}\, diag(f-b, g-b, b)\mathbf{T}_{3\times 2}^T \mathbf{x}_{2\times 1} , \tag{27}$$

then we have to perform 1 real addition (when we multiply $\mathbf{T}_{3\times 2}^T$ by $\mathbf{x}_{2\times 1}$), 3 multiplications complex numbers by real numbers (when we multiply $diag(f\text{-}b, g\text{-}b, b)$ by previously obtained real-valued vector) equivalent to 6 multiplications of real numbers and 2 complex additions (when we multiply $\mathbf{T}_{2\times 3}$ by previously obtained complex-valeud vector) equivalent to 4 additions of real numbers. Figure 1 shows a dataflow diagram for this case. The dataflow diagram is oriented from left to right. Straight lines in the figures denote the operation of data transfer. Points where lines converge denote summation. Note that one addition of complex numbers is equivalent to two real additions. The circles in this figure shows the operatons of multiplications real numbers by complex numbers inscribed inside the circles. One such multiplication corresponds to two multiplications of real numbers.

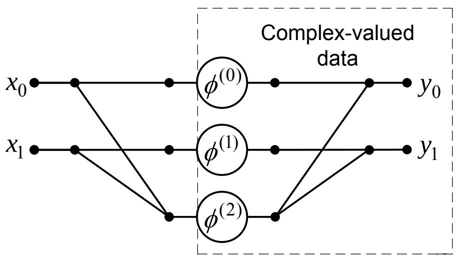

Fig. 1. Dataflow diagram of the matrix-vector product calculation for $N = 2$ according to formula (27); $\varphi^{(0)} = f\text{-}b$, $\varphi^{(1)} = g\text{-}b$, $\varphi^{(2)} = b$

Hence the number of real multiplications and real additions required for computing the matrix-vector product (26) according to (27) are equal to 6 and 5 respectively. If we had not taken into account the special structure of the matrix $\mathbf{S}_2$ and performed the product (26) in the usual way we would need to perform 8 multiplications of real number and 4 real-number additions. Although the calculation gain seems to be in this case small, but for the larger length of input data it will be significant.

Let us analyze the matrix-vector product (22) calculation in the general case for $N = 2^n$, where the matrix $\mathbf{S}_N$ has form (21). Generalizing in this case the relation (23) we can write

$$\mathbf{S}_N = (\mathbf{T}_{2\times 3} \otimes \mathbf{I}_{\frac{N}{2}})(\mathbf{\Phi}_{\frac{N}{2}}^{(0)} \oplus \mathbf{\Phi}_{\frac{N}{2}}^{(1)} \oplus \mathbf{\Phi}_{\frac{N}{2}}^{(2)})(\mathbf{T}_{2\times 3}^T \otimes \mathbf{I}_{\frac{N}{2}}) , \tag{28}$$

where symbols $\otimes$, $\oplus$ denote Kronecker product and direct sum of matrices respectively and $\mathbf{\Phi}_{N/2}^{(0)} = \mathbf{F}_{N/2} - \mathbf{B}_{N/2}$, $\mathbf{\Phi}_{N/2}^{(1)} = \mathbf{G}_{N/2} - \mathbf{B}_{N/2}$, $\mathbf{\Phi}_{N/2}^{(2)} = \mathbf{B}_{N/2}$. It should be noted that the matrices $\mathbf{\Phi}_{N/2}^{(0)}$, $\mathbf{\Phi}_{N/2}^{(1)}$ and obviously $\mathbf{\Phi}_{N/2}^{(2)}$ have also the same block-structure as matrix $\mathbf{S}_N$ but their sizes are $N/2$. Using expression (28) the procedure of matrix-vector product (22) calculation can be written as foolows

$$\mathbf{y}_{N\times 1} = \mathbf{\Omega}_{N\times\frac{3N}{2}} (\mathbf{\Phi}_{\frac{N}{2}}^{(0)} \oplus \mathbf{\Phi}_{\frac{N}{2}}^{(1)} \oplus \mathbf{\Phi}_{\frac{N}{2}}^{(2)}) \mathbf{\Omega}_{\frac{3N}{2}\times N}^T \mathbf{x}_{N\times 1} , \tag{29}$$

where

$$\mathbf{\Omega}_{N\times\frac{3N}{2}} = \mathbf{T}_{2\times 3} \otimes \mathbf{I}_{\frac{N}{2}} .$$

Figure 2 shows a graph-structural model for this case.

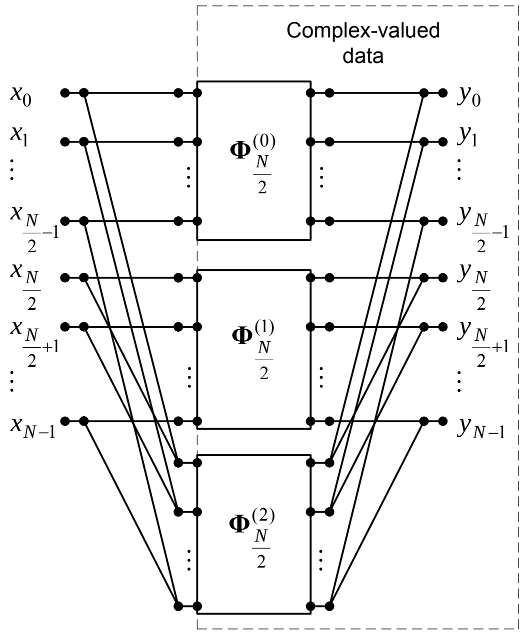

Fig. 2. Graph-structural model of the matrix-vector product calculation for $N = 2^n$ according to formula (29)

Expression (29) describes obviously only one step of reducing the number of arithmetic operations in calculating the matrix-vector product (22). Since the matrices $\mathbf{\Phi}_{N/2}^{(0)}$, $\mathbf{\Phi}_{N/2}^{(1)}$ and $\mathbf{\Phi}_{N/2}^{(2)}$ have the same structure as the matrix $\mathbf{S}_{N/2}$, the same mechanism can be used to reduce the number of arithmetic operations in the next steps, when these matrices will be multiplied by suitable vectors.

Algorithm with a reduced number of arithmetic operations for the DFRHT computing

We want to calculate the DFRHT transform for real-valued input vector $\mathbf{x}_{N\times 1}$

(30) $$\mathbf{y}_{N\times 1} = \mathbf{H}_N^{(\alpha)}\mathbf{x}_{N\times 1},$$

where $N = 2^n$. As was mentioned above the DFRHT matrix has the block-structure as follows:

(31) $$\mathbf{H}_N^{(\alpha,0)} = \begin{bmatrix} \mathbf{F}_{\frac{N}{2}}^{(0)} & \mathbf{B}_{\frac{N}{2}}^{(0)} \\ \mathbf{B}_{\frac{N}{2}}^{(0)} & \mathbf{G}_{\frac{N}{2}}^{(0)} \end{bmatrix}.$$

We assume that this matrix was calculated in advance for suitable parameter α. We have added an upper index 0, to emphasize that there is an initial matrix. In the first step from this matrix we construct the next matrix

(32) $$\mathbf{H}_{\frac{3N}{2}}^{(\alpha,1)} = \mathbf{\Phi}_{\frac{N}{2}}^{(1,0)} \oplus \mathbf{\Phi}_{\frac{N}{2}}^{(1,1)} \oplus \mathbf{\Phi}_{\frac{N}{2}}^{(1,2)},$$

where $\mathbf{\Phi}_{N/2}^{(1,0)} = \mathbf{F}_{N/2}^{(0)} - \mathbf{B}_{N/2}^{(0)}$, $\mathbf{\Phi}_{N/2}^{(1,1)} = \mathbf{G}_{N/2}^{(0)} - \mathbf{B}_{N/2}^{(0)}$ and $\mathbf{\Phi}_{N/2}^{(1,2)} = \mathbf{B}_{N/2}^{(0)}$. Obviously to calculate the product (30) using $\mathbf{H}_{3N/2}^{(\alpha,1)}$ we also need to perform some additions on input and output data, as it was shown in the graph in Figure 2. Each of blocks $\mathbf{\Phi}_{N/2}^{(1,k)}$ occurring on a diagonal of $\mathbf{H}_{3N/2}^{(\alpha,1)}$ in (32) has a block-structure as follows:

(33) $$\mathbf{\Phi}_{\frac{N}{2}}^{(1,k)} = \begin{bmatrix} \mathbf{F}_{\frac{N}{4}}^{(1,k)} & \mathbf{B}_{\frac{N}{4}}^{(1,k)} \\ \mathbf{B}_{\frac{N}{4}}^{(1,k)} & \mathbf{G}_{\frac{N}{4}}^{(1,k)} \end{bmatrix} \quad \text{for } k = 0,1,2$$

so we can repeat the same procedure for each of the blocks $\mathbf{\Phi}_{N/2}^{(1,k)}$. In the second step we obtain the matrix

(34) $$\mathbf{H}_{\frac{9N}{4}}^{(\alpha,2)} = \mathbf{\Phi}_{\frac{N}{4}}^{(2,0)} \oplus \mathbf{\Phi}_{\frac{N}{4}}^{(2,1)} \oplus \ldots \oplus \mathbf{\Phi}_{\frac{N}{4}}^{(2,8)}.$$

Each of blocks $\mathbf{\Phi}_{N/4}^{(2,k)}$ occurring on a diagonal of $\mathbf{H}_{9N/4}^{(\alpha,2)}$ in (34) has a block-structure as follows:

(35) $$\mathbf{\Phi}_{\frac{N}{4}}^{(2,k)} = \begin{bmatrix} \mathbf{F}_{\frac{N}{8}}^{(2,k)} & \mathbf{B}_{\frac{N}{8}}^{(2,k)} \\ \mathbf{B}_{\frac{N}{8}}^{(2,k)} & \mathbf{G}_{\frac{N}{8}}^{(2,k)} \end{bmatrix} \quad \text{for } k = 0,1,...,8.$$

Following this way, in l-th step we can detrmine the matrix blocks, $\mathbf{\Phi}_{N/2^l}^{(l,3k)}$, $\mathbf{\Phi}_{N/2^l}^{(l,3k+1)}$ and $\mathbf{\Phi}_{N/2^l}^{(l,3k+2)}$ occurring in $\mathbf{H}_{3^l N/2^l}^{(\alpha,l)}$ from the blocks $\mathbf{\Phi}_{N/2^{l-1}}^{(l-1,k)}$ obtained in step (l-1) by calculating the matrix $\mathbf{H}_{3^{l-1}N/2^{l-1}}^{(\alpha,l-1)}$ according to the formula

(36) $$\begin{cases} \mathbf{\Phi}_{\frac{N}{2^l}}^{(l,3k)} = \mathbf{F}_{\frac{N}{2^l}}^{(l-1,k)} - \mathbf{B}_{\frac{N}{2^l}}^{(l-1,k)} \\ \mathbf{\Phi}_{\frac{N}{2^l}}^{(l,3k+1)} = \mathbf{G}_{\frac{N}{2^l}}^{(l-1,k)} - \mathbf{B}_{\frac{N}{2^l}}^{(l-1,k)} \\ \mathbf{\Phi}_{\frac{N}{2^l}}^{(l,3k+2)} = \mathbf{B}_{\frac{N}{2^l}}^{(l-1,k)} \end{cases} \quad \text{for } k = 0,1,...,3^{l-1}-1,$$

where each of blocks $\mathbf{\Phi}_{N/2^{l-1}}^{(l-1,k)}$ occurring on the diagonal of $\mathbf{H}_{3N/2^{l-1}}^{(\alpha,l-1)}$ has a block-structure:

(37) $$\mathbf{\Phi}_{\frac{N}{2^{l-1}}}^{(l-1,k)} = \begin{bmatrix} \mathbf{F}_{\frac{N}{2^l}}^{(l-1,k)} & \mathbf{B}_{\frac{N}{2^l}}^{(l-1,k)} \\ \mathbf{B}_{\frac{N}{2^l}}^{(l-1,k)} & \mathbf{G}_{\frac{N}{2^l}}^{(l-1,k)} \end{bmatrix} \quad \text{for } k = 0,1,...,3^{l-1}-1.$$

After n steps we will have matrix

(38) $$\mathbf{H}_{3^n}^{(\alpha,n)} = \mathbf{\Phi}_{\frac{N}{2^n}}^{(n,0)} \oplus \mathbf{\Phi}_{\frac{N}{2^n}}^{(n,1)} \oplus \ldots \oplus \mathbf{\Phi}_{\frac{N}{2^n}}^{(n,3^n-1)}.$$

It should be noted that this matrix is diagonal, because the blocks occuring of its diagonal have size 1×1. This matrix does not depend on the input data and can be calculated in advance based on the knowledge of the input data length and the parameter α. Since the size of this complex-valued matrix is 3^n, so calculating the product of this matrix by a real-valued vector requires $2{\cdot}3^n$ multiplications of real numbers.

The general procedure for calculating the DFRHT transform (30) by using the matrix $\mathbf{H}_{3^n}^{(\alpha,n)}$ will take the form

(39) $$\mathbf{y}_{N\times 1} = \overline{\mathbf{\Omega}}_{N\times 3^n}\mathbf{H}_{3^n}^{(\alpha,n)}\overline{\mathbf{\Omega}}_{3^n\times N}^T\mathbf{x}_{N\times 1},$$

where

$$\overline{\mathbf{\Omega}}_{N\times 3^n} = \mathbf{\Omega}_{N\times\frac{3N}{2}}(\mathbf{I}_3 \otimes \mathbf{\Omega}_{\frac{N}{2}\times\frac{3N}{4}})\ldots(\mathbf{I}_{3^{n-1}} \otimes \mathbf{\Omega}_{\frac{N}{2^{n-1}}\times\frac{3N}{2^n}})$$

and

$$\mathbf{\Omega}_{\frac{N}{2^{k-1}}\times\frac{3N}{2^k}} = \mathbf{T}_{2\times 3} \otimes \mathbf{I}_{\frac{N}{2^k}} \quad \text{for } k = 1,2,...,n.$$

We consider an example of this procedure for $N = 2^3 = 8$

$$\mathbf{y}_{8\times 1} = \overline{\mathbf{\Omega}}_{8\times 27}\mathbf{H}_{27}^{(\alpha,3)}\overline{\mathbf{\Omega}}_{27\times 8}^T\mathbf{x}_{8\times 1},$$

where

$$\overline{\mathbf{\Omega}}_{8\times 27} = \mathbf{\Omega}_{8\times 12}(\mathbf{I}_3 \otimes \mathbf{\Omega}_{4\times 6})(\mathbf{I}_9 \otimes \mathbf{\Omega}_{2\times 3})$$

and

$$\mathbf{\Omega}_{8\times 12} = \mathbf{T}_{2\times 3} \otimes \mathbf{I}_4, \quad \mathbf{\Omega}_{4\times 6} = \mathbf{T}_{2\times 3} \otimes \mathbf{I}_2, \quad \mathbf{\Omega}_{2\times 3} = \mathbf{T}_{2\times 3} \otimes \mathbf{I}_1.$$

Figure 3 shows a dataflow diagram for this case. The diagonal entries of the diagonal matrix $\mathbf{H}_{27}^{(\alpha,3)}$ are denoted h_k for $k = 0,1,...,26$.

Table 1 includes a comparison of the number of multiplications and additions of real numbers, required to calculate the DFRHT transform in the usual way, and using the proposed algorithm (39) for different input vector length N, assuming that the vector of the input data is real-valued. We can note that the proposed algorithm significantly reduces the number of arithmetic operations, especially multiplications, in the calculation DFRHT transform and that the calculation gain increases with the length of the input data.

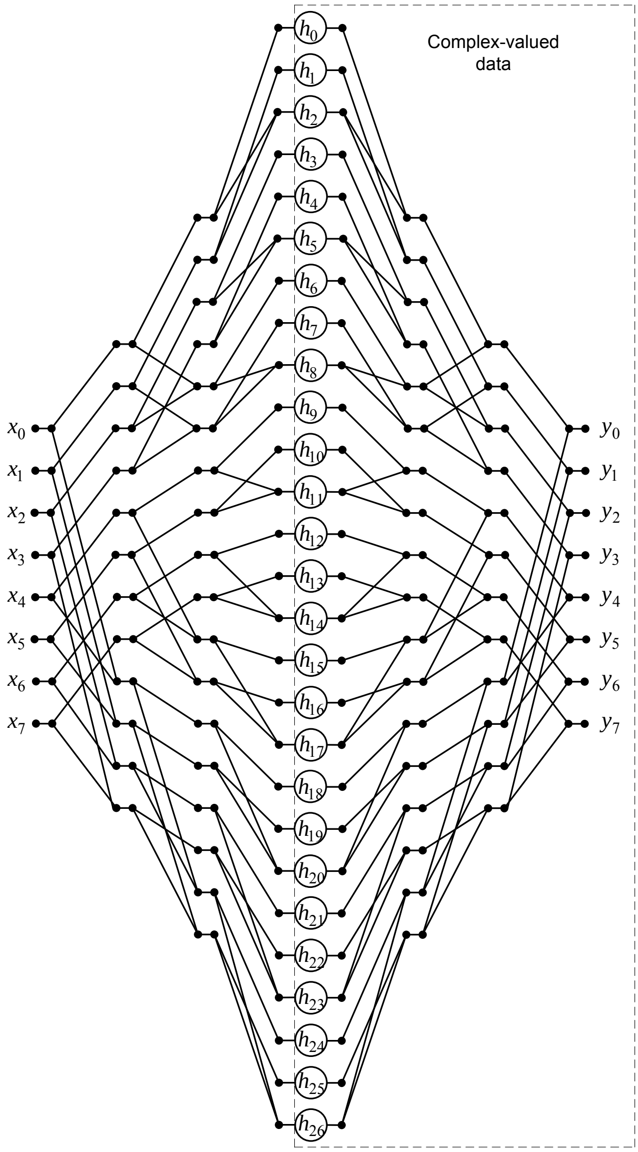

Fig. 3. Dataflow diagram of the DFRHT calculation for $N = 8$ according to formula (39)

Tab. 1. Comparison of the number of multiplications and additions requred to calculate DFRHT transform for input vector of length N

N	Multiplications		Additions	
	Usual way	Proposed algorithm	Usual way	Proposed algorithm
2	8	6	4	5
4	32	18	24	25
8	128	54	112	95
16	512	162	480	325
32	2048	486	1984	1055
64	8192	1458	8064	3325
128	32768	4374	32512	10295
256	131072	13122	130560	31525

Since the matrix DFRHT satisfies property (15), the inverse discrete fractional Hadamard transform (IDFRHT)

(40) $$\mathbf{x}_{N\times1} = (\mathbf{H}_N^{(\alpha)})^{-1}\mathbf{y}_{N\times1}$$

can be calculated in the same manner as DFRHT transform, only the parameter α should be changed to $-\alpha$.

The general procedure for calculating the IDFRHT transform (40) by using the matrix $\mathbf{H}_{3^n}^{(-\alpha,n)}$ will take the form

(41) $$\mathbf{x}_{N\times1} = \overline{\mathbf{\Omega}}_{N\times3^n}\mathbf{H}_{3^n}^{(-\alpha,n)}\overline{\mathbf{\Omega}}_{3^n\times N}^{T}\mathbf{y}_{N\times1}.$$

It is easy to chack, that

(42) $$\mathbf{H}_{3^n}^{(-\alpha,n)} = (\mathbf{H}_{3^n}^{(\alpha,n)})^*,$$

so if the matrix $\mathbf{H}_{3^n}^{(\alpha,n)}$ for calculating DFRHT transform was determined, the matrix $\mathbf{H}_{3^n}^{(-\alpha,n)}$ for calculating IDFRHT transform is also determined. The number of real number multiplications and additions required to calculate IDFRHT transform according to formula (41) will be slightly different from these, which are presented in Table 1, since in this case the input vector $\mathbf{y}_{N\times1}$ is complex-valued and the output vector $\mathbf{x}_{N\times1}$ is real-valued, but they will be also smaller than if we had not taken into account the special structure of DFRHT/IDFRHT matrix.

Summary

The article presents the rationalized DFRHT algorithm with reduced number of arithmetic operations compared to the direct way of the DFRHT implementation. Almost the some algorithm can be used for IDFRHT transform calculation, since DFRHT and IDFRHT matrices have the some block-structures.

For simplicity, as example we considered the synthesis of this algorithm for the DFRHT calculation for $N = 2^3$. However, it is clear that the proposed procedure was developed for the arbitrary case when the length of input data is a power of two.

REFERENCES

[1] Almeida L.B., The fractional Fourier transform and time-frequency representation, *IEEE Trans. Signal Process.*, 42 (1994), No. 11, 3084-3091

[2] Santhanam B., McClellan J.H., The discrete rotational Fourier transform, *IEEE Trans. Signal Process.*, 44 (1996), 994-998

[3] Pei S.C., Yeh M.H., The discrete fractional cosine and sine transforms, *IEEE Trans. Signal Process.*, 49 (2001), No. 6, 1198-1207

[4] Pei S.C., Tseng C.C., Yeh M.H., Shyu J.J., The discrete fractional Hartley and Fourier transforms, *IEEE Trans. Circuits Syst.* II, 45 (1998), 665-675

[5] Pei S.C., Yeh M.H., Shyu J.J., The discrete fractional Hadamard transforms, in *Proceeding of IEEE International Symposium on Circuits and Systems*, 1485-1488, Jul 1999

[6] Tao R., Lang J., Wang Y., The multiple parameter discrete fractional Hadamard transform, *Opt. Commun.*, 282 (2009), No. 8, 1531-1535

[7] Ţariov A., *Algorithmic aspects of the rationalization of the calculations in digital signal processing*, West Pomeranian University Press, Szczecin, 2011 (in polish)

[8] Yarlagadda R., A note on the eigenvectors of Hadamard matrices of order 2^n, *Linear Algebra and Its Applications*, 45 (1982), 43-53

***Authors**: dr Dorota Majorkowska-Mech, E-mail: dmajorkowska@wi.zut.edu.pl; dr hab. inż. Aleksandr Cariow, prof. ZUT, E-mail: atariov@wi.zut.edu.pl, Zachodniopomorski Uniwersytet Technologiczny w Szczecinie, Wydział Informatyki, ul. Żołnierska 49,* 71-210 Szczecin.