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Approximation of sensor output data using Chebyshev-Laguerre polynomials

***Abstract**. The integral Laguerre transform is generalized by introduction of scale factor and argument to arbitrary power. The possibility of usage of such transforms for approximation of sensor output data, filtration of the signals in linear filter models and numerical determination of signal derivatives is described.*

***Streszczenie.** Całkowa transformata Laguerre'a może być uogólniona za pomocą wprowadzenia współczynnika skalowania i argumentu w dowolnej potędze. Opisano wykorzystanie takich całkowych transformat do aproksymacji danych wyjściowych sensorów, filtracji sygnałów w liniowych modelach filtrujących i wyznaczenia pochodnych dla sygnałów. (**Aproksymacja danych wyjściowych sensorów za pomocą transformaty Czebyszewa-Laguerre'a**)*

Keywords: generalized Laguerre transform, sensor output data approximation
Słowa kluczowe: uogólniona transformata Laguerre'a, aproksymacja danych wyjściowych sensora

Introduction

The application of sensors for measuring the parameters of oscillating processes requires to develop the appropriate mathematical software for processing the gathered data sets. The basis for the solution of such problems is the spectral representation of the functions in orthogonal bases [1-2]. A comparative analysis of the functions in Fourier, Haar, and Walsh bases has shown that not all criteria imposed on the solutions are satisfied. Hence, the methods in other orthogonal bases such as Jacobi, Chebyshev-Laguerre, and Hermite are developed [3-5].

It seems reasonable to use the Chebyshev-Laguerre polynomials $L_n^{\lambda}(t),\ \lambda > -1$ for research on the oscillating processes in time. The disadvantage of these polynomials is the exponential increase with time at high orders. This disadvantage limits the research area in which the polynomials are used because there are calculation difficulties emerging when the summation of appropriate series for high values of the time variable t is performed. This problem can be solved by inserting a scale factor, but that change in the scale factor entails the repetition of the problem solution and causes to the instability in the calculation of the function which is decomposed into orthogonal series.

The aim of this work is the generalization of Chebyshev-Laguerre basis in order to create a method of its application for determining the parameters of approximation of oscillating functions. That method should allow to restore the values of both the function and its derivatives at given points with reasonable accuracy of approximation, which should be matched with the accuracies of measuring devices, i.e. sensors.

Results of theoretical and computational considerations

1. In the classic case, the Chebyshev-Laguerre transform f_n of the original $f(t)$ is expressed by the formula:

(1) $$f_n = \int_0^{\infty} t^{\lambda} e^{-t} L_n^{\lambda}(t) f(t)dt ,$$

where $\lambda > -1$ is any finite parameter, and L_n are Chebyshev-Laguerre polynomials.

Let us generalize the Chebyshev-Laguerre transform f_n^{ν} of function $f(t)$ in the following way: first introduce the parameters $\mu > 0$, $|\nu| < \infty$, $|\nu| \neq 0$ that are chosen adequately to the task. In this case, the generalized Chebyshev-Laguerre transform could be written as follows:

(2) $$f_n^{\nu} = \int_0^{\infty} t^{\nu\lambda+\nu-1} e^{-\mu t^{\nu}} L_n^{\lambda}(\mu t^{\nu}) f(t)dt .$$

Then the original $f(t)$ of integral transform (2) can be calculated as:

(3) $$f(t) = \sum_{n=0}^{\infty} \frac{n!\, f_n^{\nu}}{\Gamma(n+\lambda+1)} L_n^{\lambda}(\mu t^{\nu}) ,$$

where $\Gamma(x)$ is the Euler's function [2].

If the discrete values of the function which is decomposed into the series (3) are known, then for the calculation of the generalized spectra of f_n^{ν} the optimal in L^2 class quadrature formula is used:

(4) $$f_n^{\nu} = \frac{1}{|\nu|} \mu^{-\lambda-1} \sum_{j=0}^{N} \frac{t_j L_N^{\lambda}(t_j) f\left[(t_j/\mu)^{1/\nu}\right]}{(N+1)^2 \left[L_{N+1}^{\lambda}(t_j)\right]^2} .$$

Such generalization of the Chebyshev-Laguerre series in the basis allows for application of all properties of the Chebyshev-Laguerre polynomials at their investigation. The need to introduce the integral transform (2) is explained especially by the necessity for the processing of fast oscillating signals, for example of the $f(t)=\varphi(t)\sin(a/t+b)$ type, where $\varphi(t)$ is a limited function at $t \to 0$, and both a and b are constants. The approximation of function $f(t)$ by its given classical orthogonal polynomial close to zero is connected with considerable calculation difficulties, i.e. instability of calculation, because the number of oscillations increases if argument tends to zero.

2. The integral transform (2) is used for the information filtration in linear filter models which are described by the convolution type integral equations:

(5)
$$\beta f(x^{-\nu}) + \alpha\beta \int_0^{x^{-\nu}} k'(x^{-\nu} - y^{-\nu}) f(y^{-\nu}) y^{-\nu-1} dy = \varphi(x^{-\nu}) ,$$

where $k(t)$ is the kernel of the equation, $\varphi(t)$ is the output signal, $f(t)$ is the signal to be found, α and β are any finite nonzero parameters. The equations of such type adequately describe fast time-varying processes in some neighbourhood around the origin of the coordinate system. The reasonable choice of the parameters of the integral transform (2) allows to take into account the singularities of behaviour of the solution in the origin of coordinate system. As in the literature the integral transforms of type (1) are well studied, then some identity transforms of introduced integral convolution should be done. Because of the additive property of the integral, the equation (5) can be rewritten as follows:

(6)

$$\beta f(x^{-\nu})+\alpha\beta h_0-\alpha\beta\int_{x^{-\nu}}^{\infty}k'(x^{-\nu}-y^{-\nu})f(y^{-\nu})y^{-\nu-1}dy=\varphi(x^{-\nu})$$

where h_0 is a constant which depends on boundary values of the functions $f(t)$ and $\varphi(t)$. Putting $t=x^{-\nu}$, a shorter formula can be derived:

$$\beta f(t)-\alpha\beta\int_0^t k'(t-\tau)f(\tau)d\tau=\varphi(t)-\alpha\beta h_0 . \quad (7)$$

Let us create an algorithm for solving the following integral equation:

$$\alpha f(t)+\mu\int_0^t k'(t-\tau)f(\tau)d\tau=y(t) \quad (8)$$

in the Chebyshev-Laguerre basis $L_n^{\nu}(t)$, $\lambda>-1$, $y(t)=\varphi(t)-\alpha\beta h_0$. In order to restore the function, the unknown coefficients f_n should be determined.

Integrating the equation (8) by parts, the expression (9) is obtained:

$$(\alpha-\mu k(0))f(t)+\mu\frac{d}{dt}\int_0^t k(t-\tau)f(\tau)d\tau=y(t) \quad (9)$$

For λ=0:

$$\int_0^t L_n(t-\tau)L_m(\tau)d\tau=\int_0^t L_{n+m}(\tau)d\tau ; \quad (10)$$

then substituting the functions in subintegral expression with orthogonal Fourier series one can establish the following relationships:

(11)

$$\frac{d}{dt}\int_0^t k(t-\tau)f(\tau)d\tau=\frac{d}{dt}\int_0^t\sum_{n=0}^{\infty}k_nL_n(t-\tau)\sum_{m=0}^{\infty}f_mL_m(\tau)d\tau$$

$$=\sum_{n=0}^{\infty}k_n\sum_{m=0}^{\infty}f_mL_{n+m}(t)=\sum_{n=0}^{\infty}c_nL_n(t)$$

where

$$c_n=\sum_{m=0}^{\infty}k_mf_{n-m}=\sum_{m=0}^{\infty}k_{n-m}f_m .$$

Here k_m, f_m are the Fourier-Laguerre coefficients of the functions $k(t)$ and $f(t)$ If the Fourier-Laguerre sums are substituted into the above formula for the functions in (9), then equating the coefficients at the same order Chebyshev-Laguerre polynomials, the formula for determination of unknown coefficients can be achieved:

$$f_n=\frac{1}{\alpha+\mu(k_0-k(0))}\left(y_n-\mu\sum_{m=1}^{n}k_mf_{n-m}\right). \quad (12)$$

As the coefficients k_m and y_n are known, the Lagguere spectrum of unknown function $f(t)$ is defined by the formula (12). Hence, the equation (8) is solved.

One of the advantages of the proposed method is that the discretization is eliminated as the integral convolution is transformed into the series convolution. It is known that the discretization is sensitive to input error which takes place if the input information is given in discrete form with not high accuracy. The quadrature formula (4) and recurrence formula (12) allow to calculate a limited number of the first values of Fourier-Laguerre spectrum. In [1] the asymptotic formulae for calculation of the values of Fourier-Laguerre spectrum at large n are given. This allows to create a more precise algorithm for restoring of the searching function $f(t)$ with a reasonable accuracy. The algorithm of the restoring of the searching solution by known values of the generalized spectrum as the follows:

- for small orders of the spectrum the corresponding values are calculated by formula (4);
- on the basis of an a priori information about the searched solution, the asymptotic formulae for calculating the values of higher-order spectra are created;
- the evaluation of the overall error of calculations which consists of the input data error, the method error and the rounding error is obtained;
- on the basis of the received evaluations the parameters for obtaining the solution with required accuracy (the number of terms of orthogonal series, needed accuracy of input information) are found;
- knowing these parameters the solution is restored by the formula (3).

It should be noted that in the case if an a priori information is insufficient for determining the asymptotic distributions of orthogonal spectra, certain numerical methods, based on the analytical form of the dependencies of the spectrum on their orders, are elaborated.

By the a priori information we mean the information about the smoothness of the original and the existence of the first order discontinuities, singular points of the image, information about its behaviour in the infinity, and information about asymptotic behaviour of the original in zero and in infinity. It is possible to calculate the values of the image with guaranteed accuracy and to evaluate the divergence between the accurate and approximate values of a generalized spectrum.

By "calculation with guaranteed accuracy" we mean such organization of the calculation process that at all intermediate stages the computer errors of the arithmetical operations of such order are eliminated so that provides the accuracy of the final result is established.

3. When digital information is processed, not only the approximation signal values but also the signal derivatives values are used. If the input data are given with a fixed pitch and their accuracy is poor, the search for the derivative of a given function is more suitable to be obtained as the solution of the following equation:

$$\int_0^t k(t-\tau)f_l(\tau)d\tau=(l-1)!\,y(t) , \quad (13)$$

where $k(t)=t^{l-1}$, $f_l(t)=y^{(l)}(x)$, $l\geq1$.

If the solution has the form of a series

$$f_l(t)=t^{\lambda_f}\sum_{n=0}^{\infty}\frac{n!\,f_n}{\Gamma(n+\lambda_f+1)}L_n^{\lambda_f}(t) , \quad (14)$$

then the unknown coefficients f_n are calculated as:

$$f_n=\frac{1}{k_0}\left(y_n-\sum_{m=1}^{n-1}f_mk_{n-m}\right), \quad (15)$$

where k_n and y_n are the Fourier-Laguerre coefficients of functions $k(t)$ and $y(t)$, respectively.

If the values of $y(t)$ in the points t_i, where t_i ($i=\overline{1,N}$) are the roots of polynomial $L_{N+1}^{\lambda_f}(t)$, are given, then the coefficients y_n are calculated as:

(16) $$y_n = (l-1)!\sum_{i=0}^{N}\frac{t_i y(t_i)L_n^{\lambda_f}(t_i)}{\left[(N+2)L_{n+2}^{\lambda_f}(t_i)\right]^2}.$$

In this case, the coefficients k_n can be calculated accurately. If the function $k(t)$ is decomposed into the series (14), then

(17) $$k_n = \int_0^\infty e^{-t}k(t)L_n^{\lambda_f}(t)dt.$$

As $k(t)=t^{l-1}$, then

(18) $$k_n = \int_0^\infty t^{l-1}e^{-t}L_n^{\lambda_f}(t)dt = (l-1)!\frac{\Gamma(n+\lambda_k-l+1)}{n!\Gamma(\lambda_k-l+1)}$$

and we obtain the following formula:

(19) $$k_n = \begin{cases}(l-1)!, n=0\\ 0, \quad n\geq 1.\end{cases}$$

For such choice of the parameter λ_k, the coefficients that should be found would be calculated using the following formula:

(20) $$f_n = y_n/(l-1)!.$$

Then

(21) $$f_n = \sum_{i=0}^{N}\frac{t_i y(t_i)L_n^{\lambda_y}(t_i)}{\left[(N+2)L_{N+2}^{\lambda_y}(t_i)\right]^2}.$$

The algorithm of calculation of l-th order derivative of a function $f(t)$ can be described as follows:

1. By given the values $f(t_i)$ ($i=\overline{0,N}$) in the polynomial roots, the coefficients f_n are calculated using the formula (12) at $\lambda = l + \lambda_f$.

2. The values of the l-th order derivatives are calculated substituting the values from (14), whereas the values of the coefficients f_n are calculated from (21).

Let us consider the use of proposed method for approximation of the temperature data in the output of a compression station. The values of the temperature determined at the input of the compression station are shown in Fig. 1.

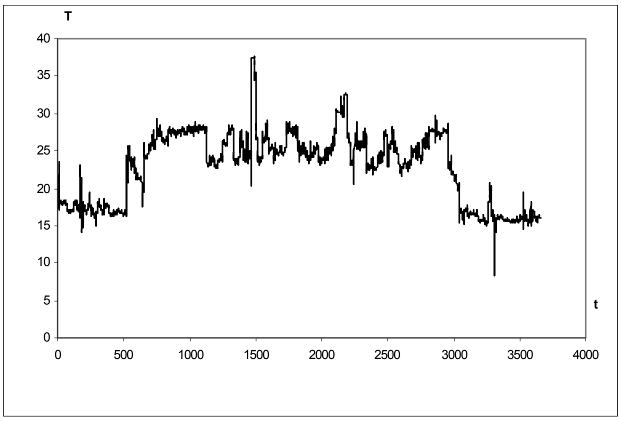

Fig. 1. The dependence of the measured temperature T [°C] on the time t [hours) at the input of compression station

The values of the temperature determined at the output of the compression station and calculated by proposed method are given in Fig. 2.

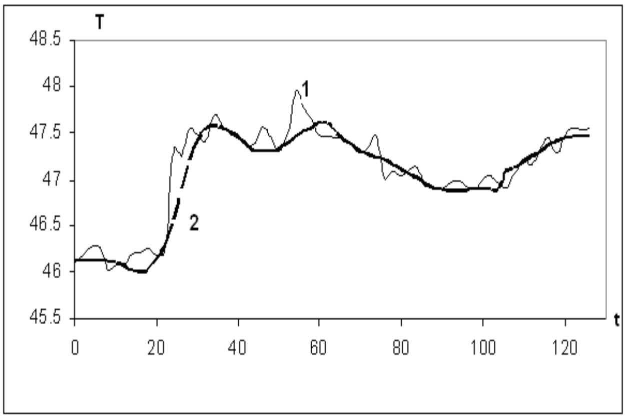

Fig. 2. The dependence of the temperature T (°C) on time t (in hours) at the output of compression station: (1) the measured values, and (2) the values calculated by the proposed method

The observed dependencies confirm the necessity of conducting the abovementioned investigations.

Conclusions

The establishing of the functional relationship with continuous derivative which describes the sensor output data improves the measurement accuracy. The inserting of the argument μt^{ν} into the Chebyshev-Laguerre transform allows to eliminate the calculation difficulties, which otherwise make it impossible to find effective solution of the problem of digital information processing in the case of the oscillating transfer functions of measuring devices.

REFERENCES

[1] П'янило Я.Д. Використання інтегральних перетворень Якобі та Чебишева–Лагерра для розв'язування інтегральних рівнянь, *Доп. НАН України*, 8 (1998), 41-46
[2] Luke Yudell L. Mathematical functions and their approximations, Academic Press Inc., (1975), 568
[3] Xiao-Long Wang, Yao-Lin Jiang. Model order reduction methods for coupled systems in the time domain using Laguerre polynomials, *Computers and Mathematics with Applications,* 62 (2011), 3241–3250
[4] Krasikov I., Zarkh A. Equioscillatory property of the Laguerre polynomials, *Journal of Approximation Theory*, 162 (2010), 2021–2047
[5] Fedele G., Ferrise A. Explicit solution of the finite time L2-norm polynomial approximation problem, *Applied Mathematics and Computation*, 217 (2011), 8354–8359

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