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Reduced Prony method – advanced properties

Streszczenie. *Artykuł opisuje rozszerzone własności metody Prony'ego na podstawie analiz jej zredukowanej wersji. Przedstawione badania opisują wymagania co do szerokości okna analizy oraz częstotliwości próbkowania dla przetwarzanego sygnału, złożonego ze składowych o losowych częstotliwościach interharmonicznych. Artykuł jest rozszerzeniem badań znanych własności metody Prony'ego i może być przydatny podczas jej implementacji w różnych systemach wykorzystujących zaawansowaną analizę sygnałów.*

Abstract. *The article describes successive properties of Prony method based on analysis of its reduced version. The studies describe requirements for width of the analysis window and sampling rate for processed signal, consisting of components with random interharmonic frequencies. The article is an extension of the known properties of Prony method and can be useful for Prony method implementation in a wide variety of advanced signal analysis systems. (**Zredukowana metoda Prony'ego – rozszerzone własności**).*

Słowa kluczowe: jakość energii elektrycznej, metoda Prony'ego, harmoniczne, pomiary.
Keywords: Power quality, Prony's method, harmonics, measurements.

Introduction

Prony method, due to its special properties [1] - [8], i.e. precise estimation of frequency components, possibility of calculating damping factors, shorter analysis window compared to Fourier transformation [9] - [12], or lack of spectrum leakage effects, is used in more and more applications in many areas of science.

Still there is a problem in Prony method application – significant computational complexity and numerical stability of solutions. This article also picks up these threads, indicating parameters for Prony method that enable avoiding solution instability and simplification of complex calculations when frequency components of tested signal are known.

Prony method occurs in several versions [13]. In the article, due to better properties, Prony method of least squares (LS) is considered.

The Prony method represents signal as a combination of exponential functions

$$\hat{x}_n = \sum_{k=1}^{p} h_k z_k^{n-1} \tag{1}$$

where $h_k = a_k \exp(j\theta_k)$ represents time-independent parameter and $z_k = \exp((\alpha_k + j2\pi f_k)T)$ represents time-dependent parameter, for $n = 1,2,...,N$, where N - the length of signal, p - the number of exponentials, T - the sample interval in seconds, a_k - the amplitude of the complex exponentials, α_k – the damping factor in seconds^{-1}, f_k - the sinusoid frequency in Hz and α_k - the initial phase of the sinusoid in radians.

In the first stage of LS Prony method, Toeplitz matrix is calculated [13]. This matrix is made up of successive signal samples. The first line represents samples from p to the first element. The second matrix line represents samples from $p+1$ to second vector element x. The rest of the matrix is constructed by analogy, up to the line p, where the first element has index $2p-1$ and the last one p of signal vector x. Based on Toeplitz matrix SVD (*Singular Value Decompsition*) distribution is calculated, and then the roots of a polynomial, which, after simple transformations, give information about the frequencies and damping factors of the components. Polynomial roots referred to, are formed from parts of matrix U column, with a number equal to the number of the smallest element in the matrix S diagonal. Assuming that the SVD matrix distribution $X_{top}^T X_{top}$, matrix S is a diagonal matrix, matrices U and W are unitary matrices fulfilling relationship $X_{top}^T X_{top} = USW^T$. From the calculated polynomial roots, Vandermonde matrix V is formed, which together with the original signal is used to calculate amplitudes and initial phases of individual components. This matrix consists of successive calculated complex roots, arranged successively in rows of Vandermonde matrix. Wherein the first row of the matrix are roots in zero power (one), the second row are roots in first power, the third row are calculated roots in second power, and similarly up to the row $N-1$, where individual roots are raised to power $N-1$. In the reduced version of Prony method - when signal component frequencies are known and assuming that components are not suppressed exponentially - Prony method can be simplified [1].

Method Description

A. Summary of Known Properties

In the article [1] the authors studied and presented basic properties of the reduced Prony method - the requirement for determining ratio of the maximum analyzed component frequency in the signal to the signal sampling frequency

$$f_s \geq 4 f_{\max} \tag{2}$$

and determined the requirements for the analysis window width

$$T_o \geq 1/\Delta f_{\min} \tag{3}$$

where f_s is the sampling frequency, f_{max} is the maximum component frequency occurring in the analysed signal, T_o and Δf_{min} are the analysis window length and the minimum frequency difference in the vector of the signal component frequencies f_k respectively. At the same time diagram in Fig. 2 [1] was used for analysis to test quality of the signal component determination. To do this, error analysis of signal reconstruction was used from designated signal component parameters according to the formula

$$Absolute\ \ error = \begin{cases} \max|x-\hat{x}| & \text{for} \quad \max|x-\hat{x}| < 1 \\ 1 & \text{for} \quad \max|x-\hat{x}| \geq 1 \end{cases} \tag{4}$$

where x and $\hat{x}$ are respectively time samples of the original signal timing and its reconstructed form.

B. Proof of the Required Sampling Frequency

Equation (2) was determined on the basis of simulation [1]. Correctness of the shown deduction can be verified using the following relations. The maximum frequency of analyzed signal can be determined from the relation

$$f_k = \arctan\left(\frac{\mathrm{Im}\{z_k\}}{\mathrm{Re}\{z_k\}}\right) / 2\pi T \tag{5}$$

Range of solutions for arcustangens is $(-\pi/2;\ \pi/2)$, hence considering only components with non-negative frequencies f_k we can write

$$f_k < \frac{\pi}{2} / 2\pi T \tag{6}$$

or after simplification and substitution: $T=1/f_s$

$$f_k < \frac{f_s}{4} \tag{7}$$

Relation (7) suggests absence of equal sign in equation (2). The simulations carried out show, however, that the component f_k exactly equal ¼ of sampling frequency f_s there is still no deterioration in the observed accuracy of estimated component parameters. Fig. 1 shows characteristics of actual and imaginary parts of vector variability for different frequencies f_k for equations (8) and (9)

$$\mathrm{Re}\{z_k\} = \frac{1}{\sqrt{1+\left(\tan\left(2\pi T f_k\right)\right)^2}} \tag{8}$$

$$\mathrm{Im}\{z_k\} = \frac{\tan\left(2\pi T f_k\right)}{\sqrt{1+\left(\tan\left(2\pi T f_k\right)\right)^2}} \tag{9}$$

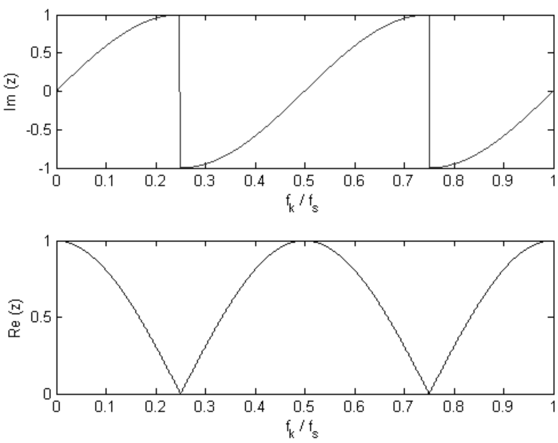

Fig. 1. Variability of actual and imaginary parts of vector z_k depending on the ratio between analyzed component with frequency f_k to sampling frequency f_s

Waveforms show relation on the basis of equation (2). Full range of non-negative frequency values according to equation (5) is obtained as shown in Fig. 1 for the range f_k/f_s equal to $[0;0.25)$.

C. Examination Methodology for Interharmonic Components

The authors carried out studies demonstrating possibility to use shorter analysis windows than would be apparent from equation (2) for the interharmonic components. A series of complex signals with harmonic and interharmonic components with a normalized amplitude equal to 1 and random initial phase were studied. The following research methodology was adopted:

Method I – analysis of signals consisting of 2 harmonic vectors shifted relative to each other on frequency axis (Fig. 2). Components defined by frequency vectors $[h_1\ldots h_n]$ and $[ih_1\ldots ih_n]$, shifted relative to each other by values $[d_1\ldots d_n]$, for $d_1=d_2=d_n$.

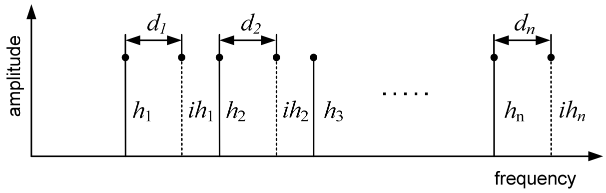

Fig. 2. Signal consisting of harmonic components defined by frequency vectors $[h_1\ldots h_n]$ and $[ih_1\ldots ih_n]$, shifted relative to each other by values $[d_1\ldots d_n]$, $d_1=d_2=d_n$

Method II – analysis of signals consisting of harmonic component frequencies randomly shifted in domain. Shift values make uniform distribution with expected value $\mu=0$ and range of randomized numbers w defined for the needs of the article (Fig. 3).

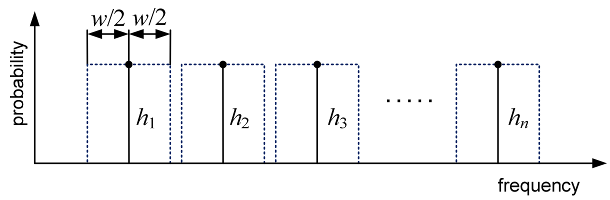

Fig. 3. Signal consisting of components $f_k=[h_1\ldots h_n]$, whose frequencies are random variables with uniform distribution, where: w – range of randomized numbers

Method III – analysis of signals consisting of harmonic component frequencies randomly shifted in domain. Frequency shift values constitute normal distribution with expected value equal $\mu=0$ and variable variance σ^2 (Fig. 4).

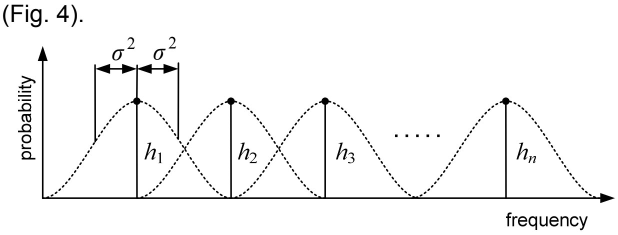

Fig. 4. Signal consisting of harmonic components $f_k=[h_1\ldots h_n]$, whose frequencies are random variables with normal distribution

It should be noted that in each case even despite randomness of component frequencies we assume that they are known. The I part of Prony method is responsible for designation of frequency components. In this article the second part of the method is analyzed (Section 2, Fig. 1, [1]), responsible for amplitudes and initial phases calculations.

The study revealed that in a wide frequency scattering of analyzed components the accuracy of signal reconstruction, according to the formula (4) using set parameters (amplitudes and phases), does not change. This gives a possibility to maintain the stability of solutions for the signals of a more complex nature.

Simulations

Accuracy of modeled signal reconstruction was tested, thereby indicating global accuracy of signal parameter estimation (amplitudes and phases). Signal reconstruction error introduced in relation (4) is a useful criterion for the selection of Prony method parameters and testing its stability in the simulations presented. For example, if the test method does not detect presence of a single or more components in the analyzed signal, absolute error will be 1. If, however, the amplitude determination error for any component is 0.5 and other parameters (amplitude and phase of other components) are determined correctly, then the error defined in relation (4) is also 0.5.

In the first phase of analysis of the reduced Prony method for interharmonic signals Method I was used. The results are shown in Figs. 5-8. For the simulation in Figs. 5-6, two vectors of signal component frequencies $[h_1 \ldots h_n]$ and $[ih_1 \ldots ih_n]$ according to Fig. 2 are uniformly shifted relative to each other with values $[d_1 \ldots d_n]$, for $d_1 = d_2 = d_n$.

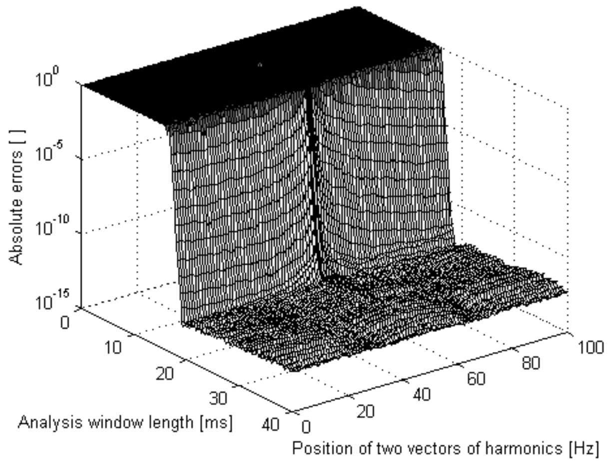

Fig. 5. Uniform shifting of two harmonic component vectors relative to each other (Method I)

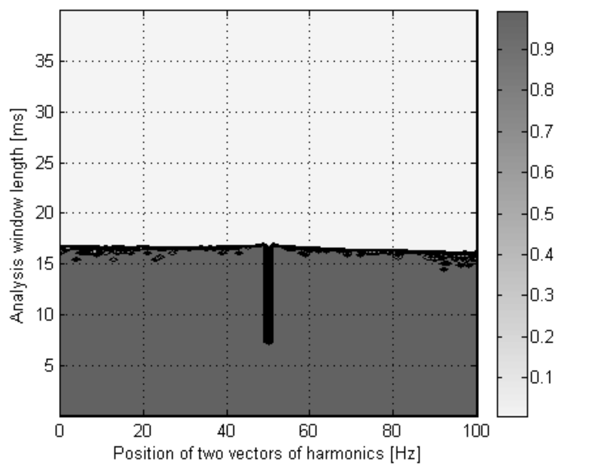

Fig. 6. Uniform shifting of two harmonic component vectors to each other (top view for Fig. 5)

The point for which the component amplitudes are summarized when two frequency vectors overlap, is distinctive. Effect of requirement reduction for analysis window width is visible, which is consistent with previous simulations [1] and is the result of summing components of the same frequencies, therefore twofold reduction of component number.

For simulation from Figs. 7-8 the shift value for vectors h relative to ih is randomized according to uniform distribution with expected value $\mu=0$ and range $w=100$ Hz, but still the relation $d_1=d_2=d_n$ is satisfied. Results obtained are similar to the previous ones. At randomization, however, summary point of vector frequencies h and ih was not achieved, what can be seen in figures.

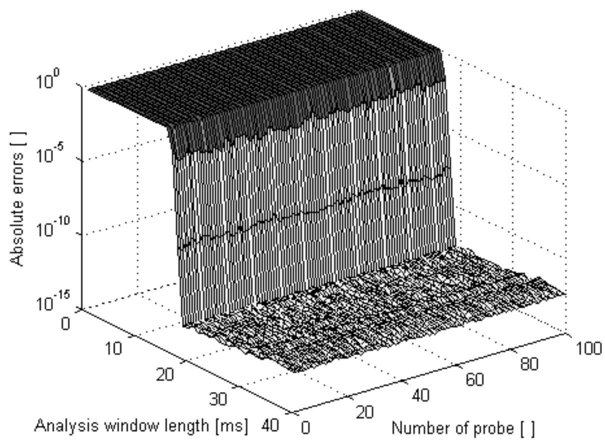

Fig. 7. Shifting of two harmonic component vectors relative to each other (Method I) with random value $d_1=d_2=d_n$

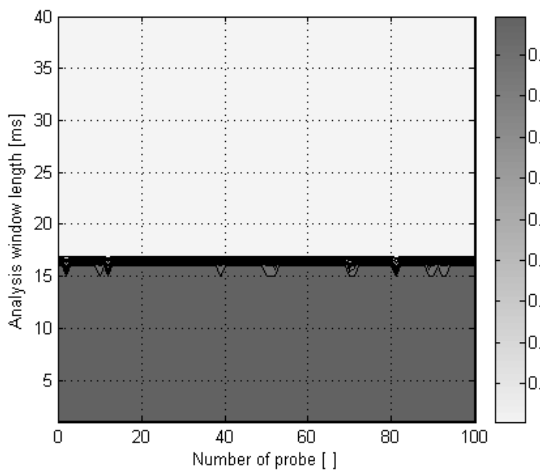

Fig. 8. Shifting of two harmonic component vectors relative to each other (Method I) with random value $d_1=d_2=d_n$ (top view for Fig. 7)

In the next simulation Method II was used, for which each deviation value for vector h component frequencies are randomized. This is a modification of Method I, for which the relation $d_1=d_2=d_n$ is no longer satisfied. Vector ih becomes an integral part of vector h. Examples of frequency values for individual components in subsequent tests are shown in Fig. 9.

Range of randomized frequency deviations for harmonics was set to $w=50$ Hz (Fig. 3) and they are randomized according to a uniform distribution with expected value $\mu=0$. Fig. 15 shows a top view of determined reconstruction errors for simulation parameters defined this way. From the simulation it can be concluded that values for the minimum required analysis window, despite the changing differences between frequencies of individual components, do not change. In Fig. 10, area designated as a Test Field was marked, where maximum reconstruction error will be searched for in subsequent tests. Test area for individual simulations is determined according to equation (3).

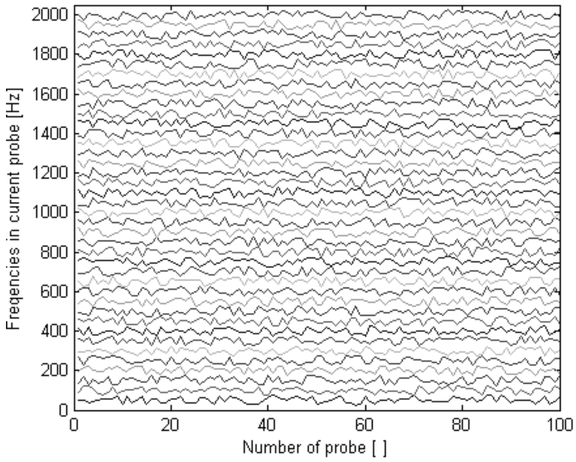

Fig. 9. Values randomized frequencies for subsequent test when determining reconstruction errors for Prony method. Deviation value was randomized according to uniform distribution; range of random numbers: w=50 Hz. Points are combined to increase readability of the diagram

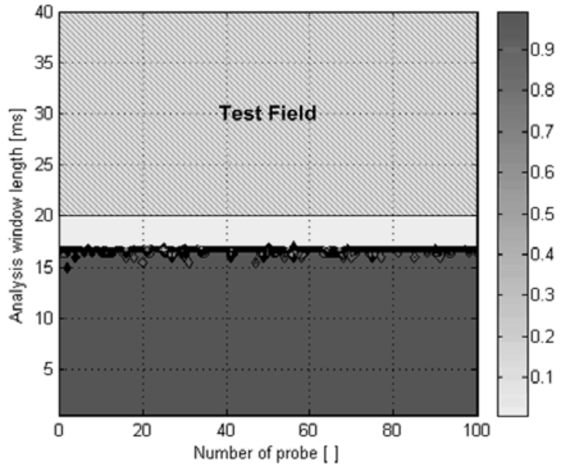

Fig. 10. Test results in reduced Prony method. In figure, area designated as a Test Field was marked, where maximum reconstruction error will be searched for in subsequent tests. The dark area means greater signal reconstruction errors

Reconstruction error to be determined is shown in subsequent simulations as a function of modified variance σ^2 or frequency deviation range w for individual harmonics (Figs. 11-16).

Variability of described reconstruction error was examined depending on adopted deviation values from the harmonic frequencies. For distributions: uniform (Method II) there was defined deviation range w, for normal distribution (Method III) there was defined deviation variance σ^2. For the tests, the following values of frequency range or variance were adopted: 50, 25 & 10 Hz. For each point 1000 randomizations were made, from which the largest reconstruction error was selected.

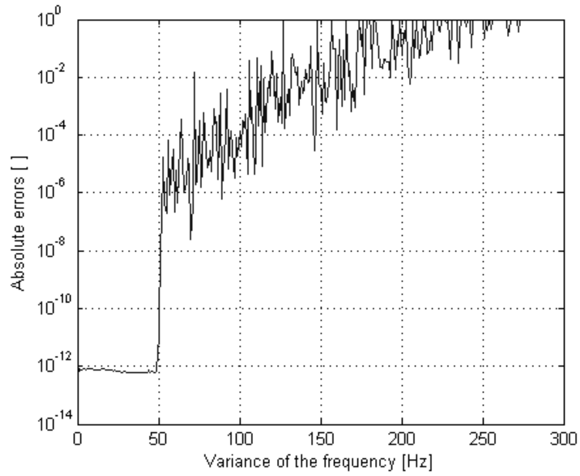

Fig. 11. Signal consisting of harmonic components f_k=[50,100,…,2000] Hz, for f_s=10 kHz, whose frequencies are modified by deviation value, which is randomized for each harmonic based on uniform distribution with the range w shown on OX axis in the diagram – Method II

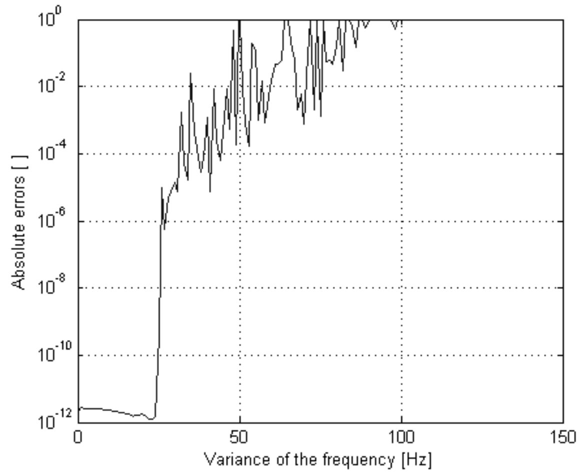

Fig. 12. Signal consisting of harmonic components f_k=[25,50,…,2000] Hz, for f_s=10 kHz, whose frequencies are modified by deviation value, which is randomized for each harmonic based on uniform distribution with the range w shown on OX axis in the diagram – Method II

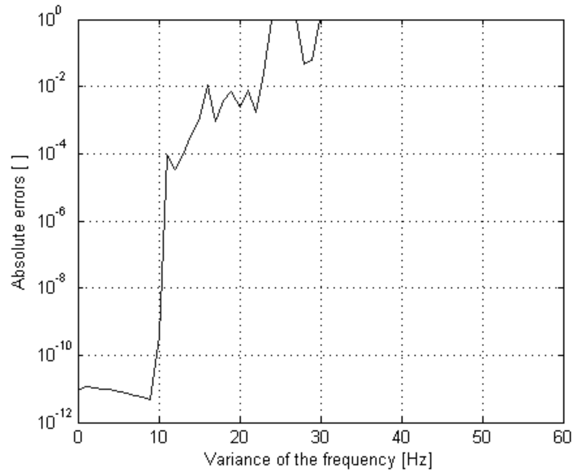

Fig. 13. Signal consisting of harmonic components f_k=[10,20,…,2000] Hz, for f_s=10 kHz, whose frequencies are modified by deviation value, which is randomized for each harmonic based on uniform distribution with the range w shown on OX axis in the diagram – Method II

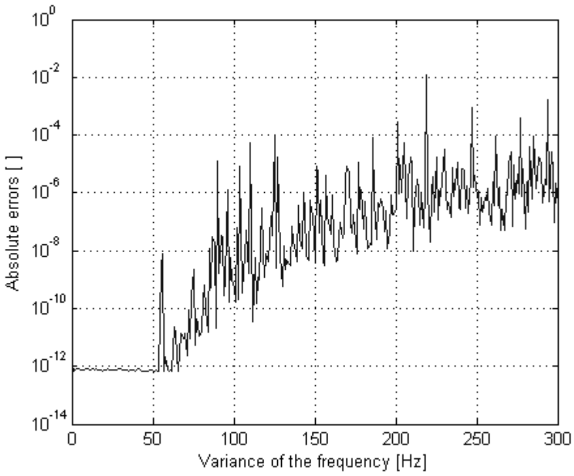

Fig. 14. Signal consisting of harmonic components f_k=[50,100,…,2000] Hz, for f_s=10 kHz, whose frequencies are modified by deviation value, which is randomized for each harmonic based on normal distribution with the variance σ^2 shown on OX axis in the diagram – Method III

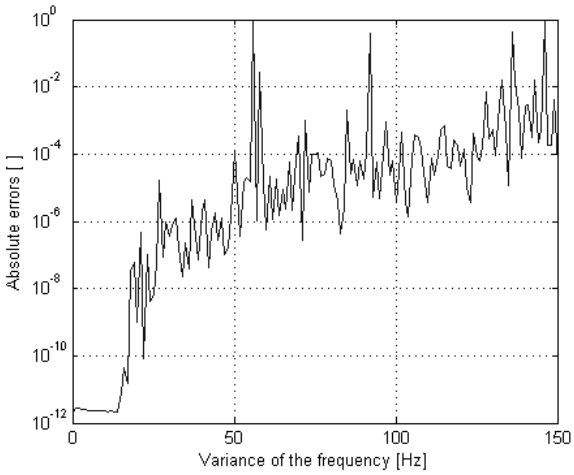

Fig. 15. Signal consisting of harmonic components f_k=[25,50,…,2000] Hz, for f_s=10 kHz, whose frequencies are modified by deviation value, which is randomized for each harmonic based on normal distribution with the variance σ^2 shown on OX axis in the diagram – Method III

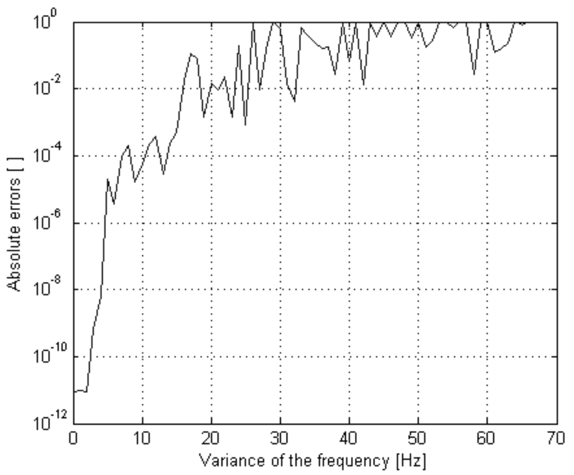

Fig. 16. Signal consisting of harmonic components f_k=[10,20,…,2000] Hz, for f_s=10 kHz, whose frequencies are modified by deviation value, which is randomized for each harmonic based on normal distribution with the variance σ^2 shown on OX axis in the diagram – Method III

Summary

For a variable number of components in uniform distribution it is possible to use all relations included in the publication [1].

For uniform distribution, which by definition limits the value of randomized harmonic frequency deviations to the closed interval, regularity defining minimum requirements for the analysis window can be observed. This can be seen in Figs. 11-13, where the observed accuracy transition for component parameter estimation is observed at points corresponding to the differential frequency between individual components Δf_{min} in vector f_k; for Fig. 11 it is 50 Hz, for Fig. 12 it is 25 Hz and for Fig. 13 we get 10 Hz.

Based on the simulation, this relation can be defined as follows: there is possibility of applying analysis window with length which is the inverse of the minimum difference of expected frequencies between components occurring in the signal, without degrading accuracy of amplitude and phase estimation for analysed signal components, in which for any observation window in the frequency domain with a width f=$2/T_o$ the number of estimated components is less than or equal to 2.

In the case of a normal distribution with a sufficiently large number of samples, a significant concentration of components is observed around random frequency values, which leads to deterioration of the requirements as to the width of the analysis window. As values of randomized frequencies are not limited, there is no way to clearly define minimum requirements as to the width of the analysis window. This is confirmed by simulations from Figs. 14-16. Points of accuracy transition for component parameter estimation do not coincide with the differential frequency values between successive components in the vector f_k. This is particularly visible for the simulation of Fig. 15, for which Δf_{min}=25 Hz, where the decline in the precision of parameter estimation appears already at 15 Hz and for the simulation of Fig. 16, for which Δf_{min}=10 Hz and observed transition appears already at about 2 Hz. Additionally, these values depend on the number of randomizations made.

Conclusion

The article shows possibilities to shorten the required analysis window when applying Prony method. The article [1] describes possibilities of reducing length of the required analysis window for signals with a reduced number of estimated components. This article sets out the conditions for which there is additionally the possibility of using short analysis window without affecting the accuracy of component parameter estimation even with a large number of estimated components. The article presents a group of analysed signals, for which the analysis window may be significantly smaller than the inverse of the smallest difference in component frequencies of analysed signal. These are signals with components whose frequencies deviate in a random way from the consecutive harmonic frequencies, provided however, that as a result of random shifts of individual components they do not change their order on the frequency axis. Therefore there is no local accumulation of more than two components on the frequency axis.

Generally, this conclusion can be written as: there is possibility of applying analysis window with length which is the inverse of the minimum difference of expected frequencies between components occurring in the signal, without degrading accuracy of amplitude and phase estimation for analysed signal components, in which for any observation window in the frequency domain with a width equal to doubled inverse of the analysis window, the number of estimated components is less than or equal 2. Additionally, this article demonstrated the relation defining minimum signal sampling frequency for Prony method, derived based on simulations in the publication [1]. The minimum sampling frequency of a signal to be analysed in Prony method should be at least four times higher than the maximum frequency of any component present in the analysed signal.

Presented properties may be used in Prony method applied for the analysis of actual harmonics, interharmonics or subharmonics, which are part of signal model presented and required to reduce the analysis window.

The presented properties of reduced Prony method can be generalized to the full version of the method, constituting increasingly attractive alternative to Fourier methods, especially in an era of rapid progress in the market development for inexpensive and efficient signal processors. Fourier analysis has now become not precise enough, with too long analysis windows, to keep track of rapidly changing phenomena, imposing many restrictions and simplifying assumptions.

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