Mirosław LUFT[1], Radosław CIOĆ[1], Daniel PIETRUSZCZAK[1]

Kazimierz Pułaski Technical University of Radom, Faculty of Transport and Electrical Engineering, Radom, Malczewskiego 29, Poland

Application of fractional calculus in modelling of the transducer and the measurement system

Streszczenie. *W artykule przedstawiono przykład modelowania przetwornika pomiarowego oraz rzeczywistego systemu pomiarowego za pomocą rachunku różniczkowo-całkowego ułamkowych rzędów (rachunku ułamkowego). Przedstawiono algorytm wyznaczania tych modeli zapisem ułamkowym oraz porównano ich z modelami opisanymi klasycznymi równaniami różniczkowymi. Badania symulacyjne wykonano w środowisku programistycznym Matlab-Simulink. (**Zastosowanie rachunku różniczkowo-całkowego ułamkowych rzędów do modelowania przetworników i systemów pomiarowych)***

Abstract*. The paper outlines an example of modelling the measurement transducer and actual measurement system with the use of fractional calculus. The algorithm determining these models is presented in the form of a fractional calculus notation and then the models are compared to the ones described by means of classical differential equations. Tests are executed in the programming environment Matlab-Simulink. (*

Słowa kluczowe: rachunek różniczkowo-całkowy ułamkowych rzędów, przetwornik pomiarowy, system pomiarowy, metoda ARX
Keywords: fractional calculus, measurement transducer, measurement system, ARX method

Introduction

Modelling of an actual measurement system consisting of many devices requires considering the response dynamics of each of them [2], [3], [4]. When one knows the input signal and the response signal, it is possible to obtain a description of the system dynamics in the form of a differential equation. The accuracy of a thus obtained model depends mainly on the applied method of identification. Application of fractional order derivative-integral calculus (in brief: fractional calculus) for identification purposes provides new possibilities of obtaining a model which reflects the dynamics of the examined object in a more accurate way [8], [9], [10].

Fractional calculus is not a new concept. It dates back to the 17th century. References to it can be found in the letters of G. W. Leibnitz to de l'Hopital (1695), in the works of L. Euler (1738) or P. S. Laplace (1812) [12]. Many physical phenomena, such as liquid permeation through porous substances, load transfer through an actual insulator, or heat transfer through a heat barrier are described more accurately by means of derivative-integral equations. The dynamics of physical processes such as acceleration, displacement, liquid flow, electric current power or magnetic field flux are modelled by means of differential equations. The courses of these processes are actually continuous variables, $m+1$ - fold differentiable, where m is determined subject to the order of the examined fractional derivative. Mass cannot be relocated from one place to another in an infinitely short time. Neither is it possible to change temperature or pressure in an actual object infinitely fast.

A classical notation of the measurement transducer dynamics is based on differential equations, which constitute their mathematical model in the time domain. In the case of fractional calculus, operators of the function differentiation and integration are combined into one operator D^n.

For differentiation, the n order assumes positive values of $n = 1,2,3,...$ and for integration – negative values: $n = -1, -2, -3, ...$. The neutral operator for the order of $n = 0$ is also defined:

$$D^n f(t) = \begin{cases} \dfrac{d^n f(t)}{dt^n} & \text{for} \quad n > 0 \\ f(t) & \text{for} \quad n = 0 \\ \int\limits_{t_0}^{t}\left[\int\limits_{t_0}^{\tau_1} ... \left[\int\limits_{t_0}^{\tau_{-n-1}} f(\tau_{-n})d\tau_{-n}\right]...d\tau_2\right]d\tau_1 & \text{for} \quad n < 0 \end{cases} \tag{1}$$

In fractional order derivative-integral calculus the arbitrary order derivative is treated as an interpolation of the sequence of discrete order operators (1) by continuous order operators. A notation introduced by H. D. Davis is applied here where the fractional order derivative of the $f(t)$ function can be presented in the following way:

$$_{t_0}D_t^{\nu} f(t) \tag{2}$$

where: t_0 and t - terminals of fractional differentiation or integration; ν - order of the derivative of the integral.

It must be emphasized that in formula (2) the $[t_0,t]$ differentiation range is defined, identical with that which appears in classical definite integrals. Hence, a derivative and an integral of the fractional order are defined in the $[t_0,t]$ range, which in the case of the derivative narrows down to the $[t_0,t]$ point (range) for the integer order n.

To distinguish between integer and fractional orders, fractional orders are labelled by Greek letters ν and μ. For integer orders the commonly applied letters m and n are reserved. The order of a derivative or an integral satisfies the condition:

$$\nu \in \mathbf{G}_+, n \in \mathbf{Z}_+ \tag{3}$$

where: $\mathbf{G}_+$ - set of real, fractional, positive numbers; $\mathbf{Z}_+$ - set of non-negative integers.

In order to emphasize the difference between a derivative and an integral, in H.D. Davis's notation it is written down that $_{t_0}D^{-\nu}f(t)$, where the integration order fulfills condition (3), and the minus sign next to the D operator informs that it is the integration operation [6], [12].

The recent dynamic development of research into the use of fractional calculus for the analysis of dynamic systems encouraged the authors to make an attempt at using it for modelling of both measurement transducers and actual measurement systems [10].

2. Identification algorithm of measurement transducer using fractional calculus

In the classic notation, dynamics of a measurement transducer design shown in Figure 1 are expressed as a 2nd order differential equation (4):

(4) $$\ddot{w}(t) + 2\zeta\omega_0\dot{w}(t) + \omega_0^{\,2} w(t) = -\ddot{x}(t)$$

where: ω_0 – natural angular frequency, ζ – damping.

Seismic mass transducers, depending on their execution (selection of parameters characterising their dynamic properties), can be used to measure displacement, speed or acceleration. The displacement x(t) is the input quantity in these transducers. In practical vibration measurements, transducers are applied to acceleration measurements. The parameters of speed and displacement are determined by applying elements that integrate accelerometer signals.

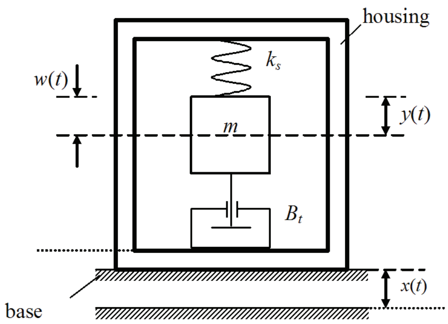

Fig.1. Model of a seismic mass measurement transducer [1], [9]

The transducer comprises: m – transducer seismic mass, k_s – reaction of spring, B – transducer damping, $x(t)$ – displacement of base and housing, $y(t)$ – displacement of mass m in relations to the fixed coordinate system and $w(t)$ – displacement mass m towards base.
Equation (5) can be noted as a difference equation:

(5) $$\begin{aligned} &a_2 w_k + a_1 w_{k-1} + a_0 w_{k-2} = \\ &= b_2 x_k + b_1 x_{k-1} + a_0 x_{k-2} \end{aligned}$$

or a matrix equation (6):

(6) $$\begin{aligned} &\begin{bmatrix} a_2 & a_1 & a_0 \end{bmatrix} \begin{bmatrix} w_k \\ w_{k-1} \\ w_{k-2} \end{bmatrix} = \\ &= \begin{bmatrix} b_2 & b_1 & b_0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \\ x_{k-2} \end{bmatrix} \end{aligned}$$

The derivative-integral expression of (5) becomes (7):

(7) $$\begin{aligned} &A_2\Delta_k^{(2)} w_k + A_1\Delta_{k-1}^{(1)} + A_0 w_{k-2} = \\ &= B_2\Delta_k^{(2)} w_k + B_1\Delta_k^{(1)} x_{k-1} + B_0 w_{k-2} \end{aligned}$$

where $\Delta_k^{(n)}$ is the reverse difference of the discrete function, defined as:

(8) $$\Delta_k^{(n)} f_{(k)} = \sum_{j=0}^{k} a_j^{(n)} f_{(k-j)}$$

Once (8) is taken into consideration, (7) noted as a matrix equation becomes (9):

(9) $$\begin{aligned} &\begin{bmatrix} a_2 & -a_1 - 2a_0 & a_2 + a_1 + a_0 \end{bmatrix} \begin{bmatrix} \Delta_k^{(2)} w_k \\ \Delta_k^{(1)} w_k \\ \Delta_k^{(0)} w_k \end{bmatrix} = \\ &= \begin{bmatrix} b_0 & -b_1 - 2b_0 & b_2 + b_1 + b_0 \end{bmatrix} \begin{bmatrix} \Delta_k^{(2)} x_k \\ \Delta_k^{(1)} x_k \\ \Delta_k^{(0)} x_k \end{bmatrix} \end{aligned}$$

In our examinations we compared responses of the measurement transducer to the sinusoidal input signal. It is described by three models:
A continuous model described by the transfer function:

(10) $$G(s) = \frac{-s^2}{s^2 + 51s + 255}$$

A discrete model obtained from a continuous model (10) described by discrete transfer function:

(11) $$G(z) = \frac{-z^2 + 2z - 1}{z^2 - 1.975z + 0.9748}$$

A discrete model determined by the derivative-integral notation takes the following form:

(12) $$G(z) = \frac{-z^2 + 0.02524z - 6.294e^{-0.005}}{z^2 - 3.161e^{-0.005} z + 1.11e^{-0.016}}$$

Figure 2 depicts a block diagram of the measurement system executed in the Matlab-Simulink package.

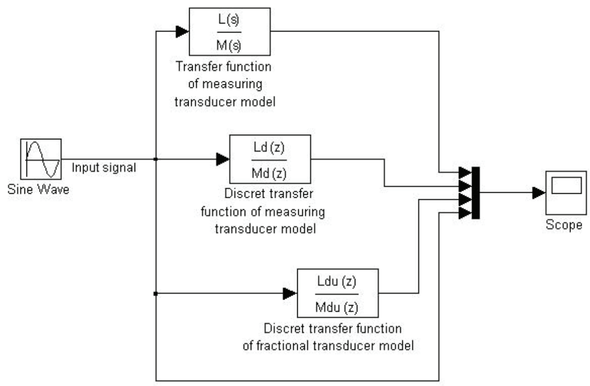

Fig.2. Block diagram of the measurement system for the measurement transducer [8], [9]

Figure 3 depicts responses of all models of the measurement transducer to the sinusoidal input signal having the frequency of 100 rad/s.

Following the comparison of model responses to the sinusoidal input signal (Figure 3) it can be concluded that: the derivative-integration model of the measurement transducer (12) from the very beginning of the simulation reproduces the input signal amplitude correctly. The models determined in the classical way - (10) and (11) – reproduce the input signal amplitude correctly after leaving the transient state.

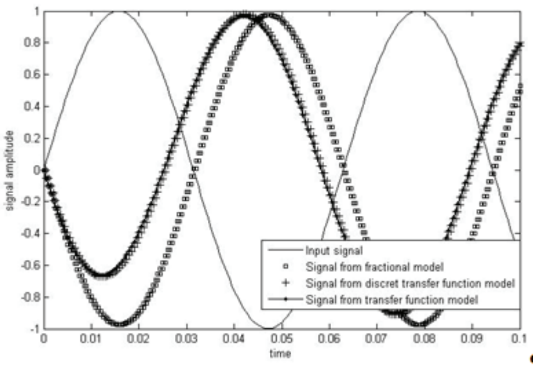

Fig.3. Comparison of responses of the measurement transducer models [8], [9]

Figure 4 compares responses of the presented models to the impulse input. The classical model response reaches the steady state after 0.08 s from the moment the signal occurs. In the case of the derivative-integration model this time is reduced to 0.001 s.

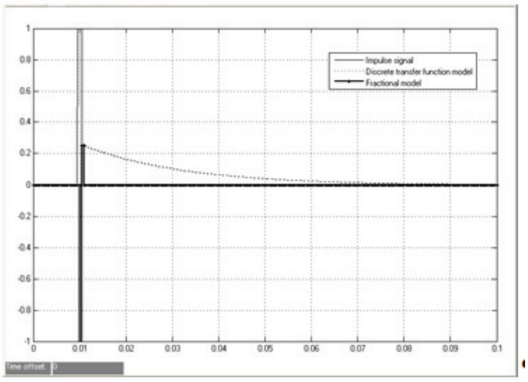

Fig.4. Comparison of responses of the measurement transducer models to the impulse input [9]

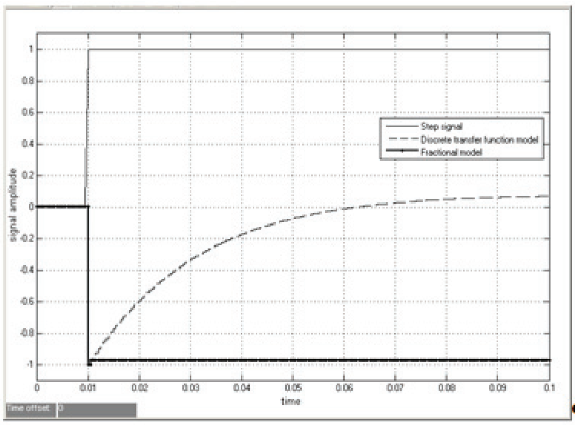

Fig.5. Comparison of the measurement transducer model responses to the step function [9]

Figure 5 depicts a comparison of the presented model responses to the step function. The classical model response passes into the steady state after 0.06 s from the moment the signal occurs. In the case of the derivative-integration model the steady state occurs after 0.005 s and assumes the value approximating that of the input function amplitude with an opposite sign.

Figure 6 depicts a comparison of Bode frequency plots for the discrete model (11) and discrete model of the measurement transducer determined by the derivative-integral notation (12).

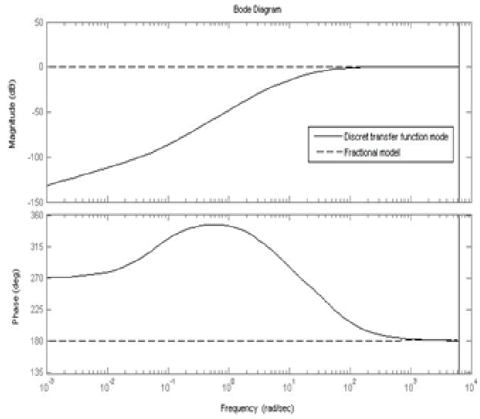

Fig.6. Comparison of Bode plots for models of the measurement transducer

The Bode plots (Figure 6) indicate that for the measurement transducer model determined by the derivative-integral method when compared with the model determined in the classical way, the range of the input signal processing is extended by low frequencies. For the presented characteristics, the amplification of the derivative-integration model amplitude equal 0 dB is reached for the frequencies from 0.001 rad/s, and for the “classical” model - from 100 Hz, at a stable phase shift of 180°.

3. Identification algorithm of measurement system using fractional calculus

In our work a laboratory system for examining the characteristics of dynamic accelerometers as depicted in Figure 7 was used. The system consisting of an accelerometer, conditioner and a measuring card was modelled. An equation describing the system dynamics was determined by means of the self-regression method with an external ARX input. The voltage signal from the end of the examined measurement chain is an identified signal, whereas the comparative signal is the one from the model accelerometer being a response to the sinusoidal input of a generator having the frequency of 200 Hz.

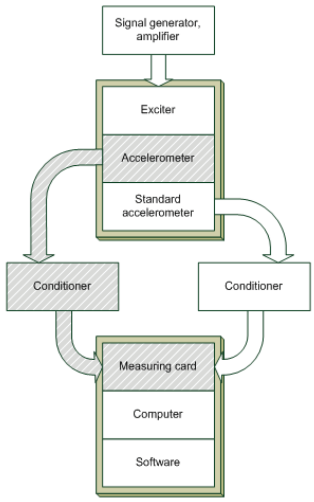

Fig.7. Block diagram of the laboratory measurement system examining measurement transducer

As a result of the application of the ARX identification method transfer function describing the system dynamics was obtained:

$$G(s)=\frac{0.03215s^2+1319.6s+1.338\cdot 10^6}{s^2+4.678\cdot 10^4 s+2.309\cdot 10^7} \tag{13}$$

Discrete transfer function (14) for the sampling time $T_p = 10^{-5}$ s was determined on the basis of transfer function (13).

$$G(z)=\frac{0.03215z^2-0.05368z+0.02163}{z^2-1.625z+0.6264} \tag{14}$$

Discrete transfer function (14) can be written down in the form of a general differential equation (5) or as a matrix equation (6). Differential equation (7) becomes (9) – the same as for the measurement transducer. Th equation coefficients (7) are:

$$\begin{aligned} &A_0 = 0.0001, A_1 = -0.0106, A_2 = 0.0322 \\ &B_0 = 0.0018, B_1 = -0.3755, B_2 = 1 \end{aligned} \tag{15}$$

A discrete model determined by the derivative-integral notation takes the following form:

$$G(z)=\frac{z^2-0.3755z+0.001844}{0.03215z^2-0.01062z+0.0001068} \tag{16}$$

The model of the actual measurement system in the form of discrete transfer function (14) was compared to the model in the form of a derivative-integral notation (16) having coefficients (15).

Both models were determined on the basis of a continuous model (13) obtained by means of the ARX identification method.

Figure 8 depicts a block diagram of the measurement simulation executed in the Matlab-Simulink package. The simulation was carried out with the use of the ode3 integration method and the sampling time of 10^{-5} s for the sinusoidal input signal having the frequency of 200 Hz.

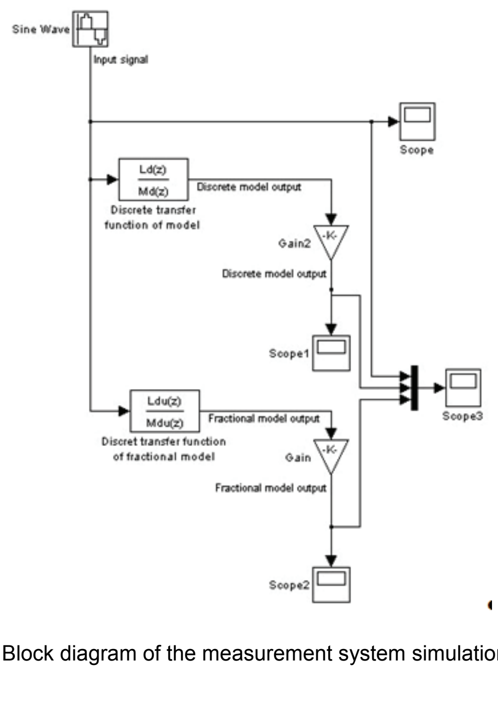

Fig.8. Block diagram of the measurement system simulation

An accelerometer having sensitivity of 10.18 mV/ms^{-2} was used in the examined system. The model signal was obtained from the reference accelerometer having sensitivity of 317 mV/ms^{-2}.

High amplification visible in the amplitude characteristics of the discrete model (Figure 9) results from the application of voltage signals in this examination. The reference amplification level determined as a ratio of the measurement transducer sensitivity to the model one in the logarithmic scale is W_r = - 29.87 dB. For this value the ratio of acceleration determined on the basis of the measuring sensor voltage signal to acceleration determined by means of a model sensor equals 1.

For the determined discrete model a significant phase shift dependent on the input signal frequency can be observed. In the case when the derivative of the integral model was applied, constancy of the phase shift was achieved.

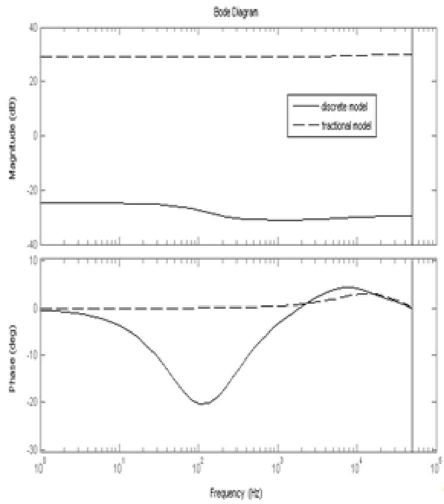

Fig.9. Bode plots for the discrete transfer function model and derivative-integration model

The obtained characteristics (Figure 10) confirm the results obtained from the examination of the amplitude-phase characteristics of derivative-integration model: a reduced phase shift and signal amplitude lower than the model signal.

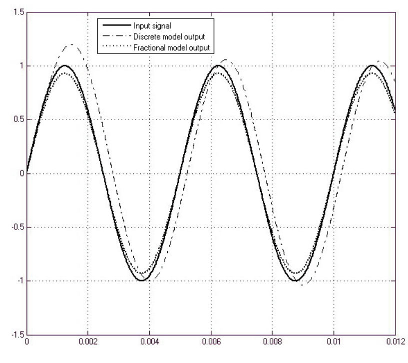

Fig.10. Comparison of the response of the discrete model and the derivative-integration model with an input signal

Attention must be drawn to the behavior of the derivative-integration model at the beginning of the simulation. Unlike in the case of the discrete model, the response of this model does not reveal a transitory state resulting from the startup.

4. Conclusions

A derivative and an integral of arbitrary order open unimaginable possibilities in the field of dynamic system identification, and creation of new, earlier inaccessible, algorithms of feedback system control.

While applying the derivative-integration model for the creation of the measurement transducer models one obtains models of ideal, in the case of amplitude reproduction, input signal processing.

In the case of the measurement system model, the phase shift obtained was reduced and signal amplitude lower than the model signal.

Moreover, at the beginning of the simulation process, unlike in the case of the discrete model, the derivative-integration model does not have a transient state resulting from the start-up.

It must be emphasized here that the commonly known derivatives are special cases of the calculus presented in this paper. The integer order integral is understood as an equivalent of the definite integral multiplication factor. Therefore we should speak about differential and integral calculus of the integer and fractional order, that is of arbitrary order.

It seems necessary to continue the examinations in order to check how the presented models determined by the derivative-integral method reflect the actual models and whether they reflect the dynamics of the input signal processing more accurately than the models described by the "classical" differential equations.

REFERENCES

[1] Cioć R.: Korekcja charakterystyk dynamicznych przetworników pomiarowych w diagnostyce wibracyjnej wagonu kolejowego, Doctoral dissertation, Biblioteka Główna Politechniki Radomskiej, Radom 2007, (in Polish)

[2] Cioć R., Luft M.: Valuation of software method of increase of accuracy measurement data on example of accelerometer, Advances in Transport Systems Telematics, Monograph, Faculty of Transport, Silesian University of Technology, Katowice 2006

[3] Cioć R., Luft M.: Correction of transducers dynamic characteristics in vibration research of means of transport – part 1 – simulations and laboratory research, 10th International Conference "Computer Systems Aided Science, Industry and Transport", Transcomp 2006, vol.1, Zakopane 2006

[4] Cioć R., Luft M.: Metoda programowej korekcji dynamicznych błędów przetwarzania przetworników pomiarowych, Pomiary Automatyka Komputery w gospodarce i ochronie środowiska, Kwartalnik Naukowo-Techniczny nr 2/2009, str. 22-25, ISSN 1889-6981, Fundacja Nauka dla Przemysłu i Środowiska. Rzeszów, 2009, (in Polish)

[5] Kaczorek T.: Wybrane zagadnienia teorii układów niecałkowitego rzędu, Oficyna Wydawnicza Politechniki Białostockiej, Białystok 2009, (in Polish)

[6] Luft M., Cioć R.: Increase of accuracy of measurement signalsreading from analog measuring transducers, Zeszyty Naukowe Politechniki Śląskiej 2005, Transport z. 59, nr kol. 1691, Gliwice 2005.

[7] Luft M., Cioć R., Pietruszczak D.: Porównanie klasycznego i różniczkowo-całkowego modelu przetwornika pomiarowego, Logistyka nr 6/2010, (in Polish)

[8] Luft M., Cioć R., Pietruszczak D.: Measurement transducer modeled by means of classical integer-order differential equation and fractional calculus, Proceedings of the 8th International Conference ELEKTRO 2010, pp TA5_88-TA5_91, (ISBN 978-80-554-0196-6), Slovak Republic, Zilina 2010

[9] Luft M., Szychta E., Cioć R., Pietruszczak, D.: Measurement transducer modelled by means of fractional calculus, Communications in Computer and Information Science 104, pp.286-295, ISBN 978-3-642-16471-2. Springer-Verlag Berlin Heidelberg 2010

[10] Matlab®&Simulink®7. ThesMathWorks™, 2008.

[11] Ostalczyk P.: Epitome of the fractional calculus. Theory and its applications in automatics, Wydawnictwo Politechniki Łódzkiej, ISBN 978-83-7283-245-0, Łódź 2008, (in Polish)

[12] Szychta E.: Zero Voltage Switched Multiresonant Converters - Analysis and Design, Monograph 2008, (ISBN 978-80-8070-889-4), University of Zilina, Slovak Republic, Zilina 2008

***Authors**: prof. Mirosław Luft, Kazimierz Pułaski Technical University of Radom, Faculty of Transport and Electrical Engineering, Radom, Malczewskiego, E-mail: m.luft@pr.radom.pl;*
Ph.D. Radosław Cioć, Kazimierz Pułaski Technical University of Radom, Faculty of Transport and Electrical Engineering, Radom, Malczewskiego, E-mail: r.cioc@pr.radom.pl;
Daniel Pietruszczak, M. Sc., Kazimierz Pułaski Technical University of Radom, Faculty of Transport and Electrical Engineering, Radom, Malczewskiego, E-mai: d.pietruszczak@pr.radom.pl;