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Optimization of the unit circle approximation by a polygon

Abstract. *The paper presents two optimization criteria of the approximation of the unit circle by a polygon: minimization of maximum approximation errors and minimization of mean square approximation errors. It is shown that application of the unit circle approximation by a polygon requires to compromise between minimization of three types of errors. The most beneficial approximation parameters values range is obtain for optimal application of the presented unit circle approximation by polygon.*

Streszczenie. *Przedstawiono dwa kryteria optymalizacji aproksymacji okręgu jednostkowego przez wielokąt: minimalizacja błędów maksymalnych aproksymacji i minimalizacja błędów średniokwadratowych aproksymacji. Wykazano, że zastosowanie aproksymacji okręgu jednostkowego wielokątem wymaga kompromisu pomiędzy minimalizacją trzech rodzajów błędów. Dla optymalnego stosowania przedstawionej aproksymacji przedstawiono zakres najkorzystniejszych wartości parametrów aproksymacji (**Optymalizacja aproksymacji okręgu jednostkowego wielokątem**).*

Keywords: unit circle, approximation, polygon, LIDFT.
Słowa kluczowe: okrąg jednostkowy, aproksymacja, wielokąt, LIDFT.

Introduction

In modern signal processing in measuring, telecommunications, acoustics, controls, and many other systems, the digital signal processing (DSP) algorithms are used in more and more dominant range. Many sophisticated methods are developed for complex mathematical problems [1-6]. DSP algorithms are supplemented by the analog to digital (A/D) and digital to analog (D/A) conversion. In digital signal processing the theory of discrete signals is used and the description in frequency domain, based on the Fourier transform and Fourier series, is very important for such signals. The kernel function in these transformations is a complex exponential function, which has the geometric interpretation as the unit circle on the complex plane. For description of the signals and systems the Z transform is also applied, which is defined for all complex plane. However, it is especially important the calculating of the Z transform for its kernel function corresponding to unit circle, since in this case Z transform points out the spectrum of the signal. Moreover, taking into account the position of the poles of the transmittance, unit circle pointed out the area of stability of discrete system. This paper presents an approximation of the unit circle by a polygon and optimization of the parameters of this approximation, in the context of a very important area of use of Fourier analysis, which is the estimation of multifrequency signal parameters.

1. The estimation of multifrequency signal parameters

Multifrequency signal is called a signal which is the sum of many sinusoidal components, each of which is characterized by the independent amplitude, frequency and phase, and the estimation of these parameters is the purpose of the analysis of multifrequency signal. Methods for such an analysis can be classified, due to the achieved accuracy and computational effort required, into two groups.

The first group covers the exact parametric estimation methods, which include: Prony method and its modifications, the method transmittance modeling and subspace methods (based mainly on the properties of the signal autocorrelation matrix and its eigenvalues) [1-4, 7]. These methods require large computational effort for a signal consisting of a large number of sinusoidal components. Furthermore, they require non-linear procedures in determining the frequency of signal components, such as procedures for finding the zeros of a high degree polynomial.

The second group of methods is based on calculating Fourier spectrum of signal (most often DFT – discrete Fourier transform) and its subsequent interpretation to determine multifrequency signal parameters [8-40]. These methods are characterized by a smaller computational effort in comparison to the methods of the first group, mainly due to the use of fast FFT algorithm and the increasing availability of specialized systems for their implementation, such as digital signal processors. The most important limitation of methods based on the Fourier spectrum is the error caused by superposition of three effects: the spectral leakage, picket fence effect and the discrete nature of the resulting spectrum due to numerical sampling in the frequency domain. These effects mean that the estimation methods based on Fourier spectral analysis, have limited resolution and estimation accuracy. Despite this they are widely used and still improved, due to their high efficiency and simplicity of implementation. Their modern improvement is the development of the time windows to minimize spectral leakage effect, and development of methods for interpolation of the spectrum.

Approximation of the unit circle by a polygon is derived from one of the spectrum interpolation methods, i.e. the DFT linear interpolation method (LIDFT). In order to show its advantages the spectrum interpolation methods and the application of the presented circle approximation by polygon is briefly reviewed in the next section.

2. Spectrum interpolation methods

Calculation of the DFT spectrum using the FFT algorithm and determination parameters on the basis of local maxima is characterized by large errors of estimation amplitudes and frequencies due to the picket fence effect and discrete nature of resulting spectrum. The way to reduce them is to determine more precisely the spectrum by nonparametric methods, the so-called [8-9] nonparametric spectrum interpolation methods: a technique padding zeros [10], chirp-Z transform [11-12], warped DFT transform [13], the interpolation by decimation [8]. Another way to increase the accuracy of estimation is to remain in the determination of the DFT spectrum and application of appropriate interpolation formulas in the frequency domain to approximate local maxima of a continuous spectrum [14-38]. Most of these interpolation relationships neglects the effect of spectrum leakage by assumption that the applied time window reduces this effect to a negligible level [39]. Among non iterative DFT spectrum interpolation methods, only two of them take into account the leakage of the spectrum in their equations: the DFT linear interpolation (LIDFT) [25-27, 37, 40] and multipoint interpolated DFT (MWIDFT) [28, 30, 35]. However, the method MWIDFT is only defined for the selected type of data windows, i.e. class I of Rife-Vincent windows [14] (which belong to the cosine family windows), also called as the maximum sidelobe decay windows [30, 33, 35]. Some methods of spectrum interpolation are only defined for the selected time windows [14,16-22,31,36,38].

The LIDFT is non iterative method for DFT spectrum interpolation and an extended versions of this method [37, 40] uses also zero padding technique. It is proved that LIDFT uses a unit circle approximation by a polygon [40]. This approximation in the paper is treated here as a separate issue because of the possibility of its use in other methods than LIDFT.

3. Unit circle approximation by a polygon

Approximation of the unit circle:

(1) $$W_N^{n\lambda} = e^{-j2\pi n\lambda / N}$$

by a polygon is obtained on the base of approximation of the part of this circle:

(2) $$W_M^{n\gamma} = e^{-j2\pi n\gamma / M}, \quad \gamma \in [-1/2, 1/2], \quad M = NR$$

by a line segment $\hat{W}_M^{n\gamma}$ (Fig.1):

(3) $$\hat{W}_M^{n\gamma} = \alpha_n + j\gamma\beta_n, \quad \gamma \in [-1/2, 1/2]$$

An appropriate choice of functions α_n, β_n, allows to locate the approximating line segment between the extreme cases shown in Figure 2.

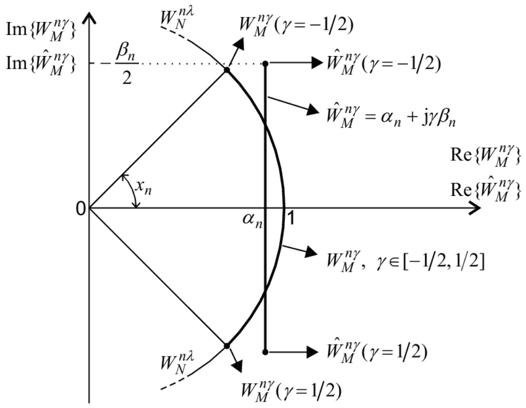

Fig.1. Approximation of the circle part $W_N^{n\lambda}$ by line segment $\hat{W}_M^{n\gamma}$

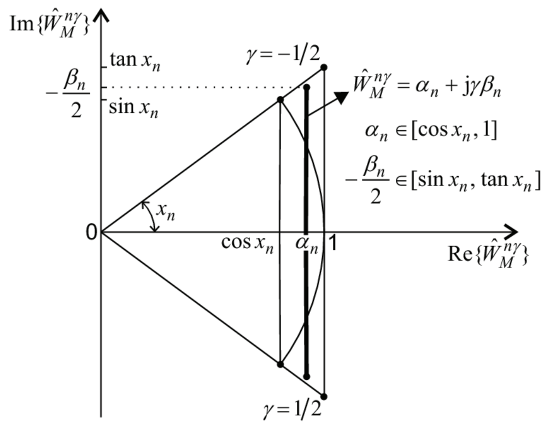

Fig.2. Approximation line segment $\hat{W}_M^{n\gamma}$ location

The functions α_n, β_n are defined as the parametric functions (with parameters η_1, η_2) on the base of trigonometric relations for the angle x_n in Figure 2:

(4) $$\alpha_n(\eta_1) = (1-\eta_1)\cos x_n + \eta_1, \quad \eta_1 \in [0, 1]$$

(5) $$\beta_n(\eta_2) = -2 \cdot [(1-\eta_2)\sin x_n + \eta_2 \tan x_n], \quad \eta_2 \le \eta_1$$

By an appropriate rotation of approximating line segment with an angle $-2kx_n$ a polygon is obtained that approximates the entire unit circle (Fig.3).

(6) $$e^{-j2\pi n\lambda_k / N} \approx e^{-j2\pi nk / M}[\alpha_n + j\gamma\beta_n]$$

where:

(7) $$\lambda_k = \frac{1}{R}(k+\gamma), \quad \gamma \in [-1/2, 1/2], \quad M = NR$$

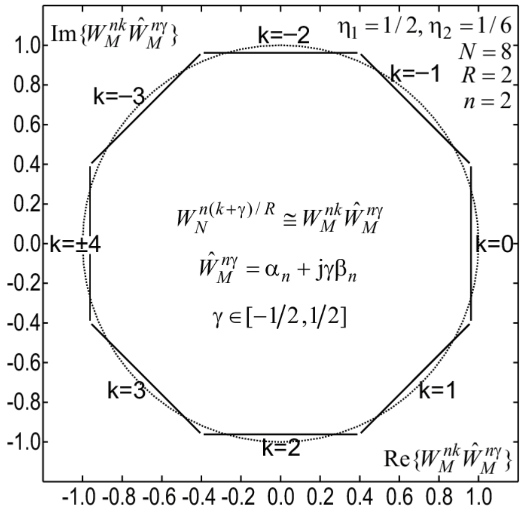

Fig.3. Unit circle approximation by a polygon

Based on the results obtained in the method LIDFT it can be concluded that the unit circle polygon approximation allows:

– linearization of relations, from which the multifrequency signal parameters are calculated,
– reduce the influence of spectrum leakage (through the window spectrum approximation by linear functions),
– independence of solutions for a wide class of windows, because the unit circle polygon approximation is independent of the applied time window.

4. Accuracy of the unit circle approximation by polygon

The accuracy of the unit circle approximation by a polygon results from the accuracy of approximation part (2) of this circle by the line segment (3) and can be defined by the real and imaginary part of the approximation error and the difference of arguments:

(8) $$\Delta_{\mathrm{r}}(\gamma) = \mathrm{Re}\{\hat{W}_M^{n\gamma} - W_M^{n\gamma}\} = \alpha_n - \cos 2\gamma x_n$$

(9) $$\Delta_{\mathrm{i}}(\gamma) = \mathrm{Im}\{\hat{W}_M^{n\gamma} - W_M^{n\gamma}\} = \gamma\beta_n + \sin 2\gamma x_n$$

(10) $$\Delta_{\mathrm{a}}(\gamma) = \arg\{\hat{W}_M^{n\gamma}\} - \arg\{W_M^{n\gamma}\} = 2\gamma x_n + \arctan\frac{\gamma\beta_n}{\alpha_n}$$

Optimal selection of approximation parameters η_1, η_2 allows to minimize the approximation errors (8)-(10) according to specified criteria optimization. Minimizing the maximum value of the modules (8)-(10) means minimizing (to a value of 1) three factors:

(11) $$k_{\mathrm{r}}(\eta_1) = \frac{\max_\gamma |\Delta_{\mathrm{r}}(\gamma)|}{\min_{\eta_1}\{\max_\gamma |\Delta_{\mathrm{r}}(\gamma)|\}}$$

(12) $$k_{\mathrm{i}}(\eta_2) = \frac{\max_\gamma |\Delta_{\mathrm{i}}(\gamma)|}{\min_{\eta_2}\{\max_\gamma |\Delta_{\mathrm{i}}(\gamma)|\}}$$

(13) $$k_{\mathrm{a}}(\eta_1, \eta_2) = \frac{\max_\gamma |\Delta_{\mathrm{a}}(\gamma)|}{\min_{\eta_1,\eta_2}\{\max_\gamma |\Delta_{\mathrm{a}}(\gamma)|\}}$$

Detailed analysis [40] shows, that k_r=1 for η_1=1/2, k_i=1 for η_2=1/12 and k_a=1 for the condition $\eta_1-\eta_2$=1/6. These conditions can not be fulfilled at the same time and the compromise values must be search within the triangle formed on the plane (η_1, η_2) by obtained optimizing conditions (Fig. 4). To evaluate this compromise one can use the contour plot (Fig. 5), which show, that for k_r=k_i=1 there is k_a=2.5 (which means 2.5-fold increase in the maximum module error (10) respect the optimal value), for k_r=k_a=1 there is k_i=4 and for k_i=k_a=1 there is k_r=1.5.

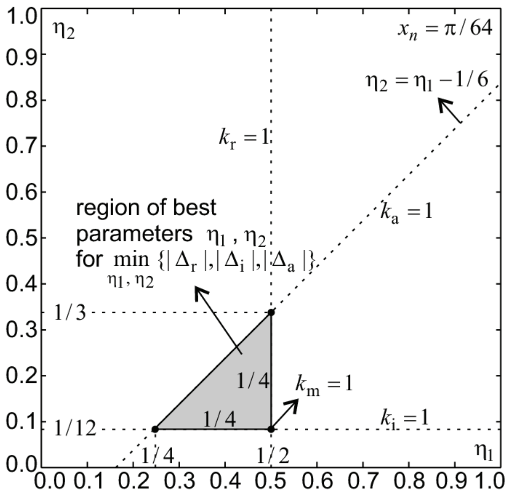

Fig.4. The triangle of the best parameters values η_1, η_2 for minimize the maximum errors of circle approximation by polygon

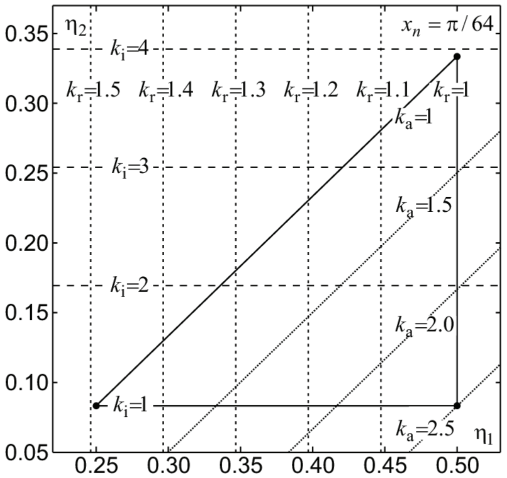

Fig.5. Contour plot k_r, k_i and k_a within the triangle from Figure 4

5. Mean square minimizing

Minimizing the maximum errors of approximation, that is minimizing factors (11)-(13) can be supplemented by minimizing mean square errors, which in some applications may be more advantageous. For this purpose the mean square errors q_r, q_i, q_a are possible to define on the base of errors (8)-(10):

$$q_r(\eta_1) = \int_{-1/2}^{1/2} |\varDelta_r(\gamma)|^2 \, d\gamma \tag{14}$$

$$q_i(\eta_2) = \int_{-1/2}^{1/2} |\varDelta_i(\gamma)|^2 \, d\gamma \tag{15}$$

$$q_a(\eta_1,\eta_2) = \int_{-1/2}^{1/2} |\varDelta_a(\gamma)|^2 \, d\gamma \tag{16}$$

and also coefficients k'_r, k'_i and k'_a in the same way as in (11)-(13), but taking into account the definitions (14)-(16):

$$k'_r(\eta_1) = \left(\frac{q_r(\eta_1)}{\min_{\eta_1} q_r(\eta_1)} \right)^{1/2} \tag{17}$$

$$k'_i(\eta_2) = \left(\frac{q_i(\eta_2)}{\min_{\eta_2} q_i(\eta_2)} \right)^{1/2} \tag{18}$$

$$k'_a(\eta_1,\eta_2) = \left(\frac{q_a(\eta_1,\eta_2)}{\min_{\eta_1,\eta_2} q_a(\eta_1,\eta_2)} \right)^{1/2} \tag{19}$$

Using in (14)-(16) definitions (8)-(10), taking into account the first terms of extensions (14)-(16) into the Maclaurin series respect x_n and calculating and comparing to zero the first derivative respect the parameters η_1, η_2 and [3(η_1–η_2)–2] result the conditions for minimizing (14)-(16). The value η_1=2/3 minimizes q_r (then $k'_r = 1$), η_2=2/15 minimizes q_i (then $k'_i = 1$) and condition $\eta_1-\eta_2$=4/15 minimizes q_a (then $k'_a = 1$). Analogously to (11)-(13) and figures 4, 5 the triangle of best parameters values of the mean square minimization can be defined, i.e. minimization of the errors (14)-(16) – Figures 6, 7.

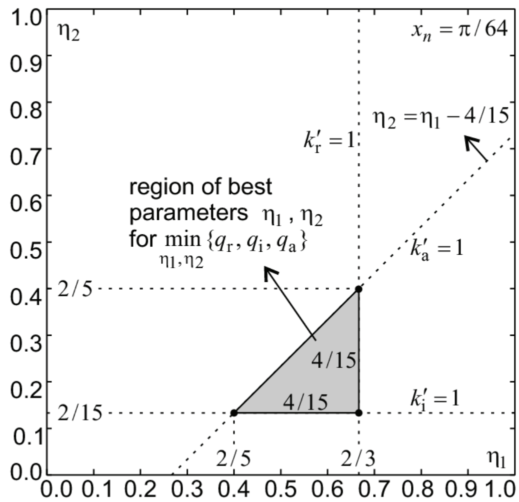

Fig.6. The triangle of best parameters values η_1, η_2 for mean square minimization of the unit circle approximation by polygon

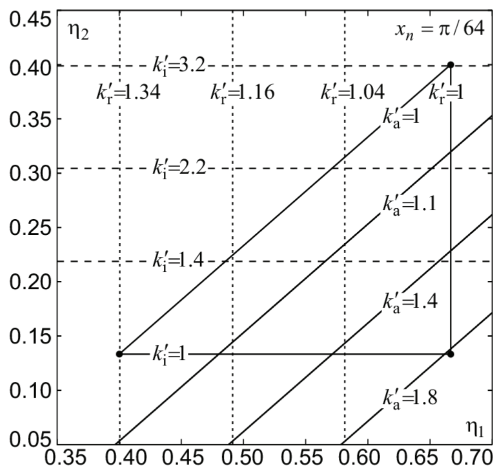

Fig.7. Contour plot k'_r, k'_i, k'_a within the triangle from Figure 6

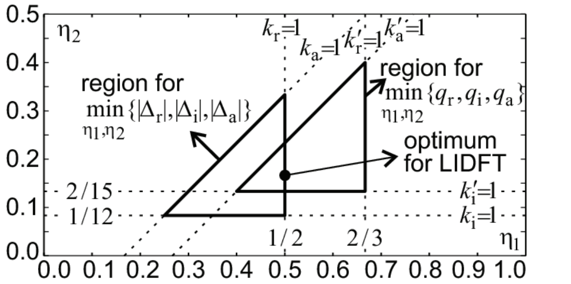

Fig.8. The sum of best regions for approximation parameters and the point for minimizing estimation errors in LIDFT method.

Conclusion

The approximation of the unit circle is defined by (6), (7) based on functions (4), (5), which depend on the parameters η_1, η_2. This approximation can be used in various signal processing methods, which use definition (1) of the unit circle. Parameter values η_1, η_2 are defined by a triangle of the best values by Figures 4, 5 for minimizing the maximum error values (11)-(13) and by Figures 6, 7 for minimizing the mean square values of these errors. These triangles are limited overlap (Fig. 8), and their sum defines the area of approximation values that allows one to search a compromise between minimizing the maximum estimation error values and the mean square errors values of estimation parameters. One of the applications for presented unit circle approximation by a polygon is linear interpolation DFT (LIDFT) method. Computer simulations show that the values of parameters η_1, η_2 that minimize the maximum estimation errors of multifrequency signal parameters for the LIDFT method are as follows: $\eta_1 \cong 1/2$, $\eta_2 \cong 1/6$ (Fig. 8). For these values there are: $k_r \cong 1$, $k_i \cong$ $k_a \cong 2$ and $k'_r \cong 1.15$, $k'_i \cong 1.08$ and $k'_a \cong 1.11$.

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