Mariana BEŇOVÁ, Branislav DOBRUCKÝ

University of Žilina

Methodological approach to steady-state and transient investigation of electric circuits using numerical infinite series of two-phase system

***Abstract**. The paper deals with two-phase system investigation using theory of series for analysis and determination of root-mean-square values and harmonic distortion factor of the periodical non-harmonic. The investigated quantities are output voltages and currents of power electronic converter supplying resistive-inductive load. These voltages are strongly non-harmonic ones, so they must be pulse-modulated due to requested nearly sinusoidal currents with low requested total harmonic distortion. A new method of calculation based on theory of numerical series is presented in paper. Basic idea is based on knowledge of harmonic spectrum of the quantities of sources and/or appliances, respectively.*

Streszczenie.** Analizowano napięcie wyjściowe i prąd elektronicznego dwufazowego konwertera mocy zasilającego odbiornik rezystancyjno-indukcyjny. Przebiegi napięcia i prądu wyjściowego są niesinusoidalne. Modulacja powinna zapewnić przebieg prądu najbardzej zbliżony do sinusoidalnego. Zaproponowano metodę obliczania współczynników szeregu Fouriera opisujących okresowe napięcie i prąd wyjściowy. Do obliczenia harmonicznych prądu wykorzystuje się spectrum napięcia i charakterystykę częstotliwościową odbiornika. Wyprowadzono analityczną postać współczynnika THD. **(Metodologiczne podejście do badania stanów ustalonych i nieustalonych przy pomocy numerycznych szeregów nieskończonych systemów dwufazowych)

Keywords: periodical non-harmonic function, Fourier transform, total harmonic distortion, root-mean-square value (effective value)
Słowa kluczowe: periodyczne funkcje niesinusoidalne, transformata Fouriera, wartości skuteczne, zniekształcenia harmoniczne

Introduction

It is well known that for a 2π-periodic function $f(x)$ that is integrable on $[-\pi, \pi]$, the Fourier series of $f(x)$ can be derived together with Fourier coefficients of $f(x)$ on $[-\pi, \pi]$, [2]. Knowing properties of non-harmonic function it is possible to determine the total harmonic distortion of the investigated function. This is needed for quality evaluation of the electrical quantities. Investigation will be done using Fourier and other numerical infinite and finite series.

Harmonic analysis of output voltage of the voltage sourced inverter in two-phase system

Two-phase systems, Fig. 1 (on the contrary to three-phase transformed *α,β*-systems) have the same both amplitude- and phase-spectrum of both phases [1], [2]. Such system can be created by two single phase inverters shifted by 90 degrees.

Using Fourier analysis ([2], [3]) we obtain the relations for amplitudes of harmonic components A_ν of output voltage for both phases, Figs. 2b,c

(1a,b)
$$A_{\nu\alpha} = \frac{4}{\pi}\frac{1}{\nu}U_{\rm DC}\cos\left(\nu\frac{\gamma}{2}\right) \quad \text{or} \quad A_{\nu\alpha} = \frac{4}{\pi}\frac{1}{\nu}U_{\rm DC}\sin\left(\nu\frac{\delta}{2}\right)$$
$$A_{\nu\beta} = \frac{4}{\pi}\frac{1}{\nu}U_{\rm DC}\cos\left(\nu\frac{\gamma}{2}+\frac{\pi}{2}\right) \text{ or } A_{\nu\beta} = \frac{4}{\pi}\frac{1}{\nu}U_{\rm DC}\sin\left(\nu\frac{\delta}{2}+\frac{\pi}{2}\right)$$

where ν - odd number equal to $2n+1$, $n = 1, 2, 3, ...$, γ - control angle, δ - pulse width, and $U_{\rm DC}$ - supply voltage.

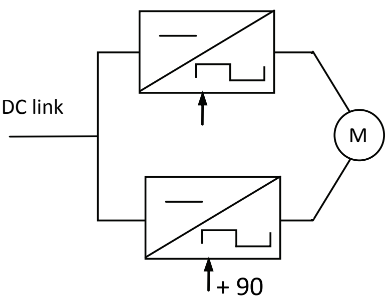


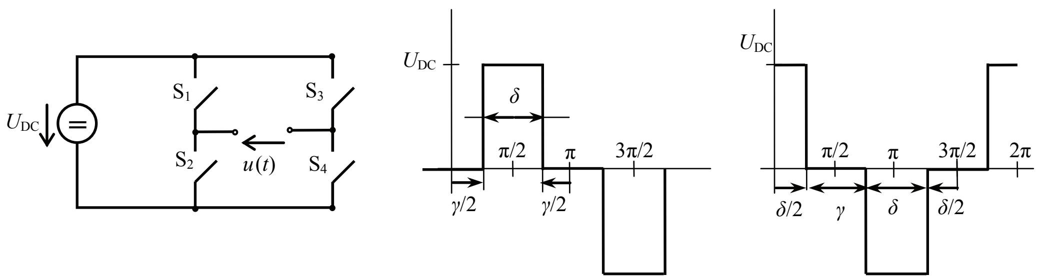
Fig. 1. Two-phase orthogonal power electronic system

So, Fourier series of the voltage $u_\alpha(t)$ in Fig.2b is [3]:

(2)
$$u_\alpha(t) = \sum_{\nu=1}^{\infty}\left[A_{\nu\alpha}\sin(\nu\omega t)\right] = \frac{4}{\pi}U_{\rm DC}\sum_{\nu=1}^{\infty}\left[\frac{\cos(\nu\gamma/2)}{\nu}\sin(\nu\omega t)\right]$$

where ω – the angular frequency, and t – the time.

Let us next analyze steady-state and transients of one phase of that electric circuit..



a) b) c)

Fig. 2. Circuit diagram of single-phase inverter (a) and output voltage waveforms of the system on Figure 1 b) $u_\alpha(t)$, c) $u_\beta(t)$

RMS value determination of output voltage using numerical infinitive series with numerical solution

Root-mean-square value (RMS) of each harmonic component will be

(3)
$$U_\nu = \frac{A_\nu}{\sqrt{2}} = \frac{2\sqrt{2}}{\pi}U_{\rm DC}\frac{1}{\nu}\cos\left(\nu\frac{\gamma}{2}\right)$$

and RMS value of total voltage waveform

$$U = \sqrt{\sum_{\nu=1}^{\infty} U_{\nu}{}^{2}} = \sqrt{\sum_{\nu=1}^{\infty}\left[\frac{2\sqrt{2}}{\pi}U_{\mathrm{DC}}\frac{1}{\nu}\cos\left(\nu\frac{\gamma}{2}\right)\right]^{2}} \quad (4)$$

$$= \frac{2\sqrt{2}}{\pi}U_{\mathrm{DC}}\sqrt{\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2}}\cos^{2}\left(\nu\frac{\gamma}{2}\right)}$$

For $\gamma = 0\ (\delta = \pi)$ we have

$$U = \frac{2\sqrt{2}}{\pi}U_{\mathrm{DC}}\sqrt{\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2}}} = \frac{2\sqrt{2}}{\pi}U_{\mathrm{DC}}\sqrt{\frac{\pi^{2}}{8}} = U_{\mathrm{DC}} \quad (5)$$

where [3] $\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2}} = 1 + \frac{1}{3^{2}} + \frac{1}{5^{2}} + \frac{1}{7^{2}} + \frac{1}{9^{2}} + \ldots = \frac{\pi^{2}}{8}$

For $\gamma = \frac{\pi}{3}$ $\left(\delta = \frac{2\pi}{3}\right)$ we have

$$U = \frac{2\sqrt{2}}{\pi}U_{\mathrm{DC}}\sqrt{\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2}}\cos^{2}\left(\nu\frac{\pi}{6}\right)} = \frac{2\sqrt{2}}{\pi}U_{\mathrm{DC}}\sqrt{\frac{\pi^{2}}{12}} = U_{\mathrm{DC}}\sqrt{\frac{2}{3}} \quad (6)$$

where [2] $\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2}}\cos^{2}\left(\nu\frac{\pi}{6}\right) = \frac{3/4}{1} + 0 + \frac{3/4}{5^{2}} + \frac{3/4}{7^{2}} + 0 + \ldots =$

$$= \frac{3}{4}\left(\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2}} - \frac{1}{3^{2}}\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2}}\right) = \frac{3}{4}\frac{\pi^{2}}{8}\left(1 - \frac{1}{3^{2}}\right) = \frac{\pi^{2}}{12}$$

Similarly, we can continue for other control angles γ [6].

Let us assume the orthogonal voltages with bipolar PWM control depicted in Figs 2a,b [2]. Using Fourier analysis we again obtain for both phases the relation for amplitudes of harmonic components [2], [3].

Methodology of voltage harmonic distortion determination for PWM pulse train

Contribution of the voltage pulse pair (Fig. 3) to total root-mean-square value will be

$$U = \sqrt{\sum_{\nu=1}^{\infty}\left[\frac{2\sqrt{2}}{\pi}U_{\mathrm{DC}}\frac{1}{\nu}\cos\left(\nu\frac{\gamma}{2}\right)\right]^{2} + \sum_{\nu=1}^{\infty}\left[\frac{2\sqrt{2}}{\pi}U_{\mathrm{DC}}\frac{1}{\nu}\cos\left(\nu\frac{\gamma}{2}\right)\right]^{2}} \quad (8)$$

$$= \sqrt{2\left[\frac{2\sqrt{2}}{\pi}U_{\mathrm{DC}}\frac{\pi}{\sqrt{24}}\right]^{2}} = U_{\mathrm{DC}}\sqrt{\frac{2}{3}}$$

Contribution of voltage pulse pair (Fig.3) to total root-mean-square value of fundamental harmonic will be

$$U_{1\mathrm{a}} = \frac{2\sqrt{2}}{\pi}U_{\mathrm{DC}}\sin\left(\frac{\pi}{6}\right)$$

so, for pulse pair $U_{1} = \frac{2}{\pi}U_{\mathrm{DC}}$

For pulse series of PWM control (see Fig. 4) one can generalize, Fig. 5.

For total root-mean-square value of the PWM train

$$U = \sqrt{2\sum_{k=1}^{N}U_{I}{}^{2} + U_{II}{}^{2} + \ldots + U_{N}{}^{2}} \quad (9)$$

and similarly for root-mean square value of the sum of fundamental harmonics

$$U_{1} = \sqrt{2\sum_{k=1}^{N}U_{1I}{}^{2} + U_{1II}{}^{2} + \ldots + U_{1N}{}^{2}} \quad (10)$$

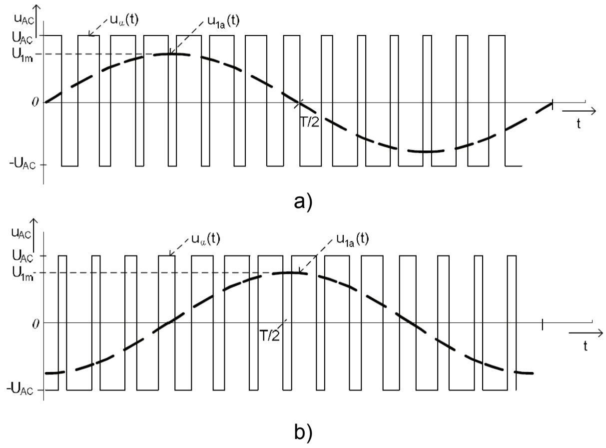


Fig.3. Output orthogonal voltages of the half-bridge matrix converter system with bipolar pulse-width-modulation: direct (a) and quadrature voltage (b)

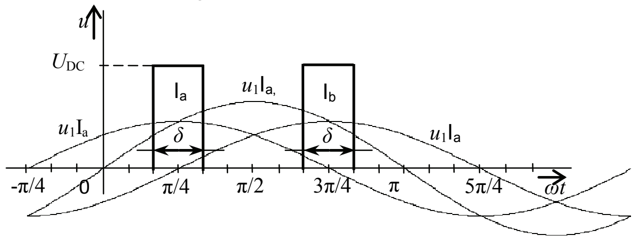


Fig.4. Symmetrical placement of each pulse pair of PWM

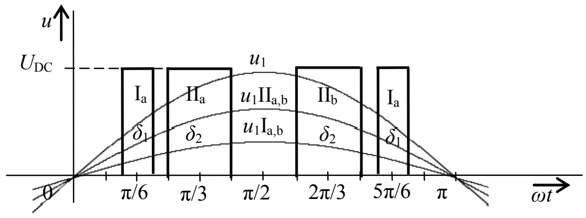


Fig.5. Decomposition of PWM series into symmetrical pulse pairs

Another way is to summarize the coherent sine-wave of the each pulse pairs

$$A_{1} = \sum_{k=1}^{N}\left\{\sqrt{2A_{1Ia,b}{}^{2}} + \sqrt{2A_{1IIa,b}{}^{2}} + \ldots + \sqrt{2A_{1Na,b}{}^{2}}\right\} \quad (11)$$

then $U_{1} = \frac{A_{1}}{\sqrt{2}}$

THD factor determination of output voltage using numerical infinitive series with numerical solution

The total harmonic distortion factor of the voltage THD_{U} will be

$$THD_{\mathrm{U}} = \sqrt{\left(\frac{U}{U_{1}}\right)^{2} - 1} = \sqrt{\frac{\sum_{\nu=1}^{\infty}\frac{1}{\nu^{2}}\cos^{2}\left(\nu\frac{\gamma}{2}\right)}{\cos^{2}\left(\frac{\gamma}{2}\right)} - 1} \quad (12)$$

Example #1 for $\gamma = 0\ (\delta = \pi)$

$$THD_{\mathrm{U}}/\pi = \sqrt{\frac{\pi^{2}}{8} - 1} = 0{,}48 \rightarrow 48\%$$

Example #2 for $\gamma = \frac{\pi}{3}\ \left(\delta = \frac{2\pi}{3}\right)$

$$THD_{\mathrm{U}}\Big/\frac{2\pi}{3} = \sqrt{\frac{\pi^2/12}{\left(\sqrt{3}/2\right)^2} - 1} = \sqrt{\frac{4}{3}\cdot\frac{\pi^2}{12} - 1} = 0{,}31\ \rightarrow 31\%$$

Harmonic analysis of output current of the voltage sourced inverter - the case of known steady-state current time waveforms

Supposing *R-L* load (Fig. 6) and $\gamma = 0\ (\delta = \pi)$ the current i_{RL} during positive half-period will be

$$\begin{aligned} i_{RL}(t) &= I_{RL\max}\left(1 - e^{-t/\tau}\right) + i_{RL}(0)\, e^{-t/\tau} \\ &= \frac{U_{\mathrm{DC}}}{R}\left(1 - e^{-t/\tau}\right) + i_{RL}(0)\, e^{-t/\tau} \\ &= \frac{U_{\mathrm{DC}}}{R}\left[1 - \left(1 + \tanh\frac{T}{4\tau}\right)\right] e^{-T/\tau} \end{aligned} \tag{13}$$

where $\tau = L/R$ and according waveforms on Fig.6

$$i_{RL}(0) = -\, i_{RL}(T/2) = \frac{U_{R\max}}{R}\frac{\left(1 - e^{-T/2\tau}\right)}{\left(1 + e^{-T/2\tau}\right)} = -\frac{U_{\mathrm{DC}}}{R}\tanh\frac{T}{4\tau}$$

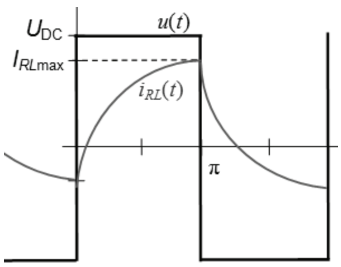


Fig. 6. Steady-state current time waveforms for *R-L* load

RMS value and THD factor determination of output current of the voltage sourced inverter - the case of known steady-state current time waveforms

RMS value of total current waveform

$$\begin{aligned} I &= \sqrt{\frac{2}{T}\int_0^{T/2} i^2(t)dt} = \frac{U_{\mathrm{DC}}}{R}\sqrt{\frac{2}{T}\int_0^{T}\left(1 - \left[1 + \tanh\frac{T}{4\tau}\right]\cdot e^{-t/\tau}\right)^2 dt} = \\ &= \frac{U_{\mathrm{DC}}}{R}\sqrt{\frac{2}{T}\int_0^{T/2}\left(1 - 2\left[1 + \tanh\frac{T}{4\tau}\right]\cdot e^{-t/\tau} + \left[1 + \tanh\frac{T}{4\tau}\right]^2\cdot e^{-2t/\tau}\right)dt} \end{aligned} \tag{14}$$

For simplification let us take $\tau = T/2$. Thus $T/4\tau = 1/2$ and $\tanh 1/2 \doteq 0{,}462$. Then

$$I \doteq \frac{U_{\mathrm{DC}}}{R}\sqrt{\frac{2}{T}\int_0^{T/2}\left(1 - 2{,}924\cdot e^{-2/T} + 1{,}462^2\cdot e^{-4/T}\right)dt} \tag{15}$$

and after discretization and using development of exponential function into geometrical series [2]

$$I_{\mathrm{RMS}}\Big/_{\substack{\delta=\pi \\ \tau=\mathrm{T}/2}} \doteq \frac{U_{\mathrm{DC}}}{R}\sqrt{\frac{2}{N}\sum_{n=1}^{N/2} 1 - 2{,}924\frac{2}{N}\sum_{n=1}^{N/2} r_1^{-n} + 1{,}462^2\,\frac{2}{N}\sum_{n=1}^{N/2} r_2^{-n}} \tag{16}$$

where

$$r_1^{-n} = e^{-(2/T)\Delta T\, n} = e^{-(1/180)n}, \quad r_2^{-n} = e^{-(4/T)\Delta T\, n} = e^{-(1/90)n}$$

Then for RMS value I_{RMS}

$$I_{\mathrm{RMS}}\Big/_{\substack{\delta=\pi \\ \tau=\mathrm{T}/2}} \doteq \frac{U_{\mathrm{DC}}}{R}\sqrt{\frac{2}{N}\frac{N}{2} - 2{,}924\frac{2}{N}\frac{1-e^{-1}}{1-e^{-1/180}} + 1{,}452^2\,\frac{2}{N}\frac{1-e^{-2}}{1-e^{-1/90}}} \tag{17}$$

Based on above equations, *THD* factor for *N* = 360 is

$$THD_{\mathrm{I}} \doteq \frac{U_{\mathrm{DC}}}{R}\sqrt{\frac{\frac{2}{N}\left(\frac{N}{2} - 2{,}924\frac{1-e^{-1}}{1-e^{-1/180}} + 1{,}452^2\,\frac{1-e^{-2}}{1-e^{-1/90}}\right)}{\frac{8}{\pi^2}\frac{1}{1+\pi^2}}} < 0{,}22 \tag{18}$$

i. e. < 22 %, where $I_{1RMS} = \frac{2\sqrt{2}}{\pi}\frac{U_{\mathrm{DC}}}{R}\frac{1}{\sqrt{1+\pi^2}}$.

Another cases of loads are shown in [2]. Actual examples are shown in [6]. Note that both currents and voltages of the phases have the same amplitude- and phase-spectra of both phases [1], [2].

Conclusion

A new method for two-phase system investigation using numerical series for determination and calculation has been presented. The investigated quantities have been input and output voltages and currents of power electronic converter supplying resistive-inductive load. These voltages are strongly non-harmonic ones, so they must be pulse-modulated due to requested nearly sinusoidal currents with low requested total harmonic distortion.

The solution given in the paper makes it possible to calculate the root-mean-square values and harmonic distortion factors more effectively and analyze effect of each harmonic component comprised in total waveform on resistive-inductive load more precisely.

ACKNOWLEDGEMENT

The authors wish to thank for the financial support to Slovak grant agency VEGA through the projects No. 1/0470/09 and R&D operational program Centre of excellence of power electronics systems and materials for their components No. OPVaV-2008/2.1/01-SORO, ITMS 26220120003 funded by European regional development fund (ERDF).

REFERENCES

[1] B. Dobrucky, M. Benova, M. Pokorny: Using Virtual Two-Phase Theory for Instantaneous Single-Phase Power System Demonstration. In *Electrical Review /Przeglad Elektrotechniczny/*, Vol. LXXXV (2009), Sigma-Not Publisher (PL), No. 1, pp. 174-178.

[2] M. Benova, B. Dobrucky, M. Marcokova: On Specific Utilization of Infinite Series for Periodical Non-Harmonic Functions in Electrical Engineering. In: *Journal of Applied Mathematics*, Vol. 3 (2010), STU Bratislava (SK), No. 3, pp. 1-14, Feb 2010.

[3] J. Skrasek, Z. Tichy: *Fundamentals of Applied Mathematics II* (in Czech). SNTL Publisher, Prague (CZ), 1986.

[4] P. P. Biringer, M. A. Slonim, "Determination of Harmonics of Converter Current and/or Voltage Waveforms (New Method for Fourier Coefficient Calculations), Part II: Fourier Coefficients of Non-Homogeneous Functions", in *IEEE Transactions on Industry Applications*, Vol. IA-16, No. 2, March/April (1980), 248-253.

[5] H.J. Bartsch: *Mathematic Formulae* (in Czech). Academia, Prague (CZ), 2006, 832 pgs., ISBN 80-200- 1448-9.

[6] Dobrucký, B., Beňová, M., Pokorný, M.: Using Theory of Series for Distortion Determination of Non-Harmonic Quantities of Two-Phase Systems. In *CD - Proceedings of the X. Conference-Seminar ISNCC 2010*, ISBN 978-1-4244-7894-1, pp. 37-42, Lagów, Poland, 15-18 June, 2010

***Authors**: dr. inż. Mariana Beňová, prof. dr. inż. Branislav Dobrucký, University of Žilina, Faculty of Electrical Engineering, Univerzitná 1, 01 026 Zilina, Slovakia,*
benova@fel.uniza.sk, branislav.dobrucky@fel.uniza.sk;