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Analysis of borderline cases of electromagnetic field variational problem

Abstract. *The paper offers analysis of expression of energetic functional for formulation of electromagnetic field variational problem in nonlinear anisotropic media. Additional conditions applied to the expression result in receiving borderline cases in the form of separate stationary problems. The results are compared with analogical linear tasks of mathematical physic.*

Streszczenie. *W artykule przedstawiono rozwinięcie funkcjonału energetycznego dla wariacyjnego sformułowania zadania analizy pola elektromagnetycznego w nieliniowym środowisku anizotropowym. Dodatkowe ograniczenia narzucone na otrzymane rozwiązanie pozwalają uzyskać sformułowanie przypadków szczególnych w formie oddzielnych zadań stacjonarnych. Wyniki porównano z analogicznymi zadaniami fizyki matematycznej. (**Rozwinięcie funkcjonału energetycznego dla wariacyjnego sformułowania zadania analizy pola elektromagnetycznego w nieliniowym środowisku anizotropowym**)*

Keywords: variation methodology, energetic functional, action function, electromagnetic field.
Słowa kluczowe: metoda wariacyjna, energetyczny funkcjonał, działanie po Hamiltonu, pole elektromagnetyczne.

Introduction

There are phenomena of different nature that may take place in a physical system. They occur as a result of transformation of energies of various types (electromagnetic, mechanical, thermal etc) in dielerctric, semiconducting, and conductive media. According to the principle of Hamilton-Ostrogradsky, the real motion of a system between two given points differs from all other motions in the fact that for the real motion the variation of action function is equal to zero. The measurement unit of the action function is the measurement unit of energy multiplied by the measurement unit of time because the action function is the time integral of the Lagrangian which is defined as the kinetic energy of the system minus its potential energy.

In our paper we will show that the minimization of other expression (namely the expression of full energy of the system where the sum of all kinds of energies is used) gives us the same results. It allows us to avoid the time integration of Lagrange function.

Initial assumptions

Let us write down the full energy expression for an anisotrophic nonlinear unhomogenous hysteresis medium of the volume V on the condition that the following assumptions are acceptable:

- No mechanic work on displacement of charged conductive bodies, conductive contours with currents or parts of the medium has being performed;
- The values of differential characteristics of electromagnetic field in any point located on the boundary of the medium (i. e. on the surface S that bounds the volume V) are given;
- The values of differential characteristics of electromagnetic field in any point located in the medium at the initial time point $t = t_0$ are given;
- The value of the energy at the initial moment equals to zero.

Such problem is called mixed (or initial-boundary value problem) because both boundary and initial values are given.

Variation of energetic functional

When we accept the formulated assumptions, the energy expression consists of the three parts: the energy of electric field, the energy of magnetic field, and the work done to displace charged particles (due to the parts with finite electrical conductivity and existence of free particles) [2]

(1) $$W = \int_V dV \int_{\overline{D}_0}^{\overline{D}} \overline{E} d\overline{D} + \int_V dV \int_{\overline{B}_0}^{\overline{B}} \overline{H} d\overline{B} + \int_{t_0}^{t} dt \int_V \overline{E}\overline{J} dV,$$

where we must remember that the differentials $d\overline{D}$ та $d\overline{B}$ in the equation are not full differentials; they characterize only the time change of the vectors $\overline{D}$ and $\overline{B}$.

As a result of mathematical transformations the following energy expression has been obtained [2]:

(2) $$W = \int_V dV \int_{\overline{B}_0}^{\overline{B}} \overline{H} d\overline{B} - \int_V dV \int_{\overline{A}_0}^{\overline{A}} (\partial \overline{D} / \partial t + \overline{J}) \cdot \overline{dA} +,$$

$$+ \int_V dV \int_{grad\varphi_0}^{grad\varphi} (\overline{D} + \int_{t_0}^{t} \overline{J} dt) \cdot d(grad\varphi) + \int_V \varphi\rho_0 dV + C,$$

where C is a constant that depends only on initial and boundary conditions.

The variation of the functional (2) in the case of scalar electric potential variation can be obtained in the form

(3) $$\delta_\varphi W = \int_V (\overline{D} + \int_{t_0}^{t} \overline{J} dt) grad(\delta\varphi) dV + \int_V \rho_0 \delta\varphi \cdot dV$$

Let us apply the formulas of vector analysis $div(\chi\overline{X}) = \chi div\overline{X} + \overline{X} div\chi$ and $\int_V div\overline{X} dV = \oint_S \overline{X}\overline{dS}$ and transform the expression (3) into

(4) $$\delta_\varphi W = \int_V div\{(\overline{D} + \int_{t_0}^{t} \overline{J} dt)\delta\varphi) dV - \int_V div(\overline{D} +$$

$$= \int_{T_0}^{T} \overline{J} dt)\delta\varphi \cdot dV + \int_V \rho_0 \delta\varphi \cdot dV = \oint_S (\overline{D} + \int_{t_0}^{t} \overline{J} dt)\delta\varphi \cdot \overline{dS} -$$

$$- \int_V div\overline{D}\,\delta\varphi \cdot dV - \int_V div(\int_{t_0}^{t} \overline{J} dt)\delta\varphi \cdot dV + \int_V \rho_0 \delta\varphi \cdot dV =$$

$$= -\int_V div\overline{D}\,\delta\varphi \cdot dV - \int_V (\rho - \rho_0)\delta\varphi \cdot dV + \int_V \rho_0 \delta\varphi \cdot dV =$$

$$= -\int_V div\overline{D}\,\delta\varphi \cdot dV - \int_V (\rho - \rho_0)\delta\varphi \cdot dV + \int_V \rho_0 \delta\varphi \cdot dV =$$

$$= -\int_V (-div\overline{D} + \rho)\delta\varphi \cdot dV.$$

In these transformations we hve applied the condition that $\delta\varphi \equiv 0$ on the boundary S as well as the known formula $div\overline{J} = -\partial\rho / \partial t$.

The variation of the functional (2) in the case of vector magnetic potential variation can be obtained in the form

(5) $\delta_{\overline{A}} W = \int\limits_V \overline{H} \cdot \delta\overline{B} \cdot dV - \int\limits_V (\partial\overline{D} / \partial t + \overline{J}) \delta\overline{A} \cdot dV$

Let us apply the formulas of vector analysis $div(\overline{X} \times \overline{Y}) = = \overline{Y} rot \overline{X} - \overline{X} rot \overline{Y}$ and $\int\limits_V div \overline{X} dV = \oint\limits_S \overline{X} d\overline{S}$ and transform the expression (5) into

(6) $\delta_{\overline{A}} W = \int\limits_V div(\overline{H} \times \delta\overline{A}) dV + \int\limits_V rot\overline{H} \delta\overline{A} \cdot dV -$

$- \int\limits_V (\partial\overline{D} / \partial t + \overline{J}) \delta\overline{A} \cdot dV = \oint\limits_S (\overline{H} \times \delta\overline{A}) d\overline{S} + \int\limits_V (rot\overline{H} - \partial\overline{D} / \partial t - \overline{J}) \cdot$

$\delta\overline{A} \cdot dV = \int\limits_V (rot\overline{H} - \partial\overline{D} / \partial t - \overline{J}) \delta\overline{A} \cdot dV.$

In these transformations we hve applied the condition that $\delta\overline{A} \equiv 0$ on the boundary S .

The minimum of the functional (2) can be found when we make its varations (4) and (6) equal to zero, i. e.

(7) $\delta_{\varphi} W = 0; \quad \delta_{\overline{A}} W = 0;$

$\int\limits_V (-div\overline{D} + \rho) \delta\varphi \cdot dV = 0;. \int\limits_V (rot\overline{H} - \partial\overline{D} / \partial t - \overline{J}) \delta\overline{A} \cdot dV = 0.$

Since the variations $\delta\varphi$ and $\delta\overline{A}$ are arbitrary, the minimum can be achieved only when integrands be equal to zero:

(8) $div\overline{D} = \rho; \quad rot\overline{H} = \overline{J} + \partial\overline{D} / \partial t.$

It can be concluded that functions which supply a minimal value to the energetic functional (2) are functions that satisfy the first and third Maxwell's equations.

Let us obtain the separate expressions of the functional for borderline cases.

Formulation of electrostatics variational problem

The conditions which determine the existence of an electrostatic field are following:

(9) $\rho = \rho_0 = const; \quad \overline{J} = 0.$

It may be admitted that before the moment $t = t_0$ no electromagnetic field existed in the medium of volume V . Then at the moment $t = t_0$ the electrical charge of density ρ was placed into the medium causing the emerging of elctromagnetic field. This dictates the following expression of the functional (2)

(10) $W_{es} = \int\limits_V dV \int\limits_0^{grad\varphi} \overline{D} d(grad\varphi) + \int\limits_V \varphi\rho \cdot dV + const$

Variation of the functional (10) can be written down in the form

(11) $\delta W_{es} = \int\limits_V \overline{D} \cdot grad\delta\varphi \cdot dV + \int\limits_V \varphi\rho \cdot dV =$

$= \int\limits_V div(\overline{D} \cdot \delta\varphi) dV - \int\limits_V div\overline{D} \cdot \delta\varphi \cdot dV + \int\limits_V \delta\varphi \cdot \rho \cdot dV =$

$= \int\limits_V (-div\overline{D} + \rho) \delta\varphi \cdot dV.$

In the transformations (11) we hve applied the condition that $\delta\varphi \equiv 0$ on the boundary S .

The minimum of the functional (10) can be found when we make its varation (11) equal to zero, i. e.

(12) $\delta W_{es} = 0; \quad \int\limits_V (-div\overline{D} + \rho) \delta\varphi \cdot dV = 0.$

Since the variation $\delta\varphi$ is arbitrary, the minimum can be achieved only when the integrand be equal to zero:

(13) $div\overline{D} = \rho.$

It can be concluded that function which supplies a minimal value to the energetic functional (10) is a function that satisfies the first Maxwell's equation.

Formulation of Magnetostatics Variational Problem

The conditions which determine the existence of a magnetostatic field are following:

(14) $\rho = \rho_0; \quad \overline{J} = const.$

It may be admitted that before the moment $t = t_0$ no electromagnetic field (or only electrostatic field) existed in the medium of volume V . Then at the moment $t = t_0$ the electrical current of density $\overline{J}$ was caused to emerge what called to existance of a magnetostatic field (as well as a stationary electric field). This dictates the following expression of the functional (2)

(15) $W_{ms} = \int\limits_V dV \int\limits_0^{\overline{B}} \overline{H} d\overline{B} - \int\limits_V dV \int\limits_0^{\overline{A}} \overline{J} d\overline{A} + C =$

$= \int\limits_V dV \int\limits_0^{\overline{B}} \overline{H} d\overline{B} - \int\limits_V \overline{J} \cdot \overline{A} \cdot dV + const$

Variation of the functional (15) can be written down in the form

(16) $\delta W_{ms} = \int\limits_V \overline{H} \cdot \delta\overline{B} \cdot dV \int\limits_V \overline{J} \cdot \delta\overline{A} \cdot dV =$

$= \int\limits_V div(\overline{H} \times \delta\overline{A}) dV + \int\limits_V rot\overline{H} \cdot \delta\overline{A} \cdot dV - \int\limits_V \overline{J} \cdot \delta\overline{A} \cdot dV =$

$= \int\limits_V (rot\overline{H} - \overline{J}) \delta\overline{A} \cdot dV.$

In the transformations (16) we hve applied the condition that $\delta\overline{A} \equiv 0$ on the boundary S .

The minimum of the functional (15) can be found when we make its varation (15) equal to zero, i. e.

(17) $\delta W_{ms} = 0; \quad \int\limits_V (rot\overline{H} - \overline{J}) \delta\overline{A} \cdot dV.$

Since the variation $\delta\overline{A}$ is arbitrary, the minimum can be achieved only when the integrand be equal to zero:

(18) $rot\overline{H} = \overline{J}.$

It can be concluded that function which supplies a minimal value to the energetic functional (15) is a function that satisfies the first Maxwell's equation.

Formulation of stationary field variational problem

The conditions which determine the existence of a magnetostatic field are following:

(14) $\rho = \rho_0; \quad \overline{J} = const.$

It may be admitted that before the moment $t = t_0$ no electromagnetic field (or only electrostatic field) existed in the medium of volume V . Then at the moment $t = t_0$ the electrical current of density $\overline{J}$ was caused to emerge what called to existance of a stationary electric field (as well as a magnetostatic field). This dictates the following expression of the functional (2)

(19) $W_{st} = \int\limits_V dV \int\limits_0^{grad\varphi} (\overline{D} + \overline{J}(t - t_0)) d(grad\varphi) + \int\limits_V \varphi\rho \cdot dV + C$

Variation of the functional (19) can be written down in the form

(20) $\delta W_{st} = \int_V (\overline{D} + \overline{J}(t-t_0)) grad\delta\varphi \cdot dV + \int_V \delta\varphi \cdot \rho \cdot dV =$

$= \int_V div((\overline{D} + \overline{J}(t-t_0))\delta\varphi)dV - \int_V div(\overline{D} + \overline{J}(t-t_0))\delta\varphi \cdot dV +$

$+ \int_V \delta\varphi \cdot \rho \cdot dV = \int_V (-div(\overline{D} + \overline{J}(t-t_0))\rho)\delta\varphi \cdot dV.$

In the transformations (20) we hve applied the condition that $\delta\varphi \equiv 0$ on the boundary S.

The minimum of the functional (19) can be found when we make its varation (20) equal to zero, i. e.

(21) $\delta W_{st} = 0; \quad \int_V (-div(\overline{D} - \overline{J}(t-t_0) + \rho)\delta\varphi \cdot dV = 0.$

Since the variation $\delta\varphi$ is arbitrary, the minimum can be achieved only when the integrand be equal to zero:

(22) $-div\overline{D} + \rho - (t-t_0)div\overline{J} = 0.$

Since the equity (22) must be correct for any time instant, its both parts may be considered separately:

(23) $-div\overline{D} + \rho = 0; \quad div\overline{J} = 0.$

It can be concluded that function which supplies a minimal value to the energetic functional (19) is a function that satisfies the first Maxwell's equation and the condition of electric charge preservation.

Discrete model of the problem

Let us introduce the following symbols for denotation of the parts of the integrand of the functional (2)

$w = \int_{\overline{B}_0}^{\overline{B}} \overline{H} d\overline{B}$; $g = \int_{\overline{A}_0}^{\overline{A}} (\partial\overline{D}/\partial t + \overline{J}) d\overline{A}$;

$e = \int_{grad\varphi_0}^{grad\varphi} (\overline{D} + \int_{t_0}^{t} \overline{J}dt) \cdot d(grad\varphi)$; $v = \varphi\rho_0$,

The functional (2) can be written down in the form

(24) $W = \int_V (w - g + e + v)dV + const$

To find its minimum we should make its derivatives after the components of vector magnetic potential and after electric scalar potential equal to zero. The formulas of the derivatives are as follows:

$\partial w/\partial A_\xi = \overline{H}\,\partial\overline{B}/\partial A_\xi$; $\xi = x, y, z$ $\quad \partial w/\partial\varphi = 0$;

(25) $\partial g/\partial A_\xi = \overline{\varsigma}(\partial\overline{D}/\partial t + \overline{J}) = \partial D_\xi/\partial t + J_\xi$; $\xi = x, y, z$;

$\varsigma = i, j, k$; $\partial g/\partial\varphi = 0$;

$\partial e/\partial A_x = \partial e/\partial A_y = \partial e/\partial A_z = 0$;

$\partial e/\partial\varphi = (\overline{D} + \int_{t_0}^{t} \overline{J}dt)\,\partial(grad\varphi)/\partial\varphi$;

$\partial v/\partial A_x = \partial v/\partial A_y = \partial v/\partial A_z = 0$; $\partial v/\partial\varphi = \rho_0$.

The expression of time derivative can be written using a back differentiation formula:

(26) $\partial\overline{D}/\partial t = a_0\overline{D} + \overline{C}_D$,

where a_0 is the coefficient of the back differentiation formula that depends on its order and the step of time integration; $\vec{C}_D = \bar{i}C_{Dx} + \bar{j}C_{Dy} + \bar{k}C_{Dz}$ is an element that depends on the values of $\overline{D}$ on previous integration steps as well as on the order of the back integration formula and the step of time integration. The expression of time integral can be written using a quadrature integration formula:

(27) $\int_{t_0}^{t} \overline{J}dt = b_0\overline{J} + \overline{C}_J$,

where b_0 is the coefficient of the quadrature formula that depends on its order and the step of time integration; $\overline{C}_J = \bar{i}C_{Jx} + \bar{j}C_{Jy} + \bar{k}C_{Jz}$ in the element that depends on the values of $\overline{J}$ on previous integration steps as well as on the order of the quadrature formula and the step of time integration.

We will conduct the minimization of the functional (24) by means of invariant approximations methodology [1].

The distribution of any variable (either scalar or vector) inside a finite element is approximated by an appropriate Taylor's polynomial [1] of n-th order in the form

(26) $X_\xi[x,y,z] = \vec{T}T_m^{-1}\vec{X}_{\xi m} = \vec{K}[x,y,z]\vec{X}_{\xi m}$;

$X = A, B, H, J, D$; $\xi = x, y, z$;

$\varphi[x,y,z] = \vec{T}T_m^{-1}\vec{\varphi}_m = \vec{K}[x,y,z]\vec{\varphi}_m$;

$grad\varphi[x,y,z] = \vec{T}\overline{N}T_m^{-1}\vec{\varphi}_m = \vec{\overline{R}}_{\nabla m}[x,y,z] \cdot \vec{\varphi}_m$,

$div\overline{D}[x,y,z] = \vec{T}\overline{N}T_m^{-1}\vec{\overline{D}}_m = \vec{\overline{R}}_{\nabla m}[x,y,z] \cdot \vec{\overline{D}}_m$,

$rot\vec{\overline{H}}[x,y,z] = \vec{T}\overline{N}T_m^{-1} \times \vec{\overline{H}}_m = \vec{\overline{R}}_{\nabla m}[x,y,z] \times \vec{\overline{H}}_m$,

$B_x = -\partial A_y/\partial z + \partial A_z/\partial y = -\vec{T}\boldsymbol{N}_z\boldsymbol{T}^{-1}\vec{A}_{ym*} + \vec{T}\boldsymbol{N}_y\boldsymbol{T}^{-1}\vec{A}_{zm*}$;

$B_y = -\partial A_z/\partial x + \partial A_x/\partial z = -\vec{T}\boldsymbol{N}_x\boldsymbol{T}^{-1}\vec{A}_{zm*} + \vec{T}\boldsymbol{N}_z\boldsymbol{T}^{-1}\vec{A}_{xm*}$;;

$B_z = -\partial A_x/\partial y + \partial A_y/\partial x = -\vec{T}\boldsymbol{N}_y\boldsymbol{T}^{-1}\vec{A}_{xm*} + \vec{T}\boldsymbol{N}_x\boldsymbol{T}^{-1}\vec{A}_{ym*}$;

where $\vec{T}$ - Taylor's vector of a point located inside the m-th finite element.; T_m^{-1} (49) - inverse Taylor's matrix for m-th finite element; $\vec{X}_{\xi m}$ - nodal columns of m-th finite element; (50) $\overline{N} = \bar{i}N_x + \bar{j}N_y + \bar{k}N_z$ - Hamilton's matrix; $\vec{\overline{R}}_{\nabla m} = = \bar{i}\vec{R}_{xm} + \bar{j}\vec{R}_{ym} + \bar{k}\vec{R}_{zm}$ - difference analogy of Hamilton's operator in the linear space where functions are uniquely determined by their nodal columns.

Conclusion

The expression of energetic functional for variational problem of electromagnetic field analysis has been proposed as the total of the energy of electric field, the energy of magnetic field and the work done to displace charged particles. The application of different specific conditions leads to borderline cases of the task of electromagnetic field analysis. It is shown how to converse this expression into its algebraic counterpart on the basis of invariant approximations methodology. The minimization of the functional gives us solutions of the first and the third Maxwell's equations.

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