Tomasz JANICZEK[1], **Janusz JANICZEK**[2]

Wroclaw University of Technology, Institute of Computer Engineering, Control and Robotics (1)
Wroclaw University of Technology, Chair of Electronic and Photonic Metrology (2)

Methods of parameters identification of fractional systems

Abstract. *This paper presents the parameters of identification method generalized for systems that for obtaining estimates of parameters of models use fractional differential equations i.e. systems containing elements of fractional degree. These elements introduce fractional derivatives to differential equations and in the operator equations they are presented as operators of fractional degree. The discussed method based on the operators which affect the defined/ measured incoming and outgoing signals allows to obtain equations or a set of equations with unknown parameters. This method was generalized to systems in which there are elements of the fractional degree. The way how to transform the algorithm of the model as to avoid large scale mathematical complications and with a little loss of generality is demonstrated.*

Streszczenie. *W niniejszym artykule przedstawiono uogólnienie metody różniczkowej i zbadano ją pod kątem przydatności w identyfikacji systemów zawierających elementy stopnia ułamkowego. Elementy te w równaniach różniczkowych wprowadzają pochodne ułamkowe, zaś w równaniach operatorowych pojawiają się w postaci operatorów z potęgami ułamkowymi. Rozpatrywana poniżej metoda wykorzystująca operatory działające na zadane/ mierzone sygnały wejściowe i wyjściowe umożliwia takie przekształcenie sygnałów, aby uzyskać równania bądź układy równań z niewiadomymi parametrami. Metodę tą uogólniono do układów, w których występują elementy stopnia ułamkowego, obrazując sposób przekształceń i poszukiwanie parametrów wybranego modelu procesu. (**Metody identyfikacji parametrów systemów ułamkowych)***

Keywords: fractional calculus, differential method of identification.
Słowa kluczowe: ułamkowy rachunek różniczkowy, różniczkowa metoda identyfikacji.

Introduction

Fractional calculus is a term known since the early nineteenth century. However, the main development falls on its end, when the fundamental definitions and theorems were formulated [1]. Up to the 1970s and 1980s the theorem of fractional calculus functioned almost exclusively as a term used in mathematics [2–5]. Not earlier than that the fractional calculus found its way to engineering applications [6–10].

The existing methods of identification use the models which are based on integer degree elements, i.e. those which introduce a differential equation, derivatives or integrals of the integer degree of order. However, there are systems (physical objects) for which traditional mathematical approach is not sufficiently accurate or too complicated. In this case, you may find that the method of identification extended to include fractional components will be more accurate and will reflect the physical phenomena more closely.

Although fractional calculus is complicated and the process of mathematical transformation is time consuming, it introduces innovation in quality computation.[11,12]. Approximating functions with fractional differential equations enhance the accuracy of models and introduce a new dimension of the phenomena that could not be taken into account using the classical differential calculus. Therefore, it can be applied to identification of systems' parameters and thus can be used to assess their quality, better description of phenomena occurring in materials and for attaining better accuracy in measurements, to name some of its potential applications [13,14]. The increase in computing performance of modern computers allows the use of increasingly accurate and sophisticated data processing techniques, the ones using new mathematical tools such as fractional calculus, will find their way to practical applications soon.

Differential method

In this method the process and model are described with the use of differential equations [15]. It uses the least squares fit to find the parameters of the equations. It is considered to be a continuator and extended form of correlation methods.

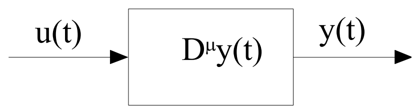

Fig. 1. Ideal system with a stimulation input u(t) and y(t) output signals

Let the process be described by the following differential equation, in which there is a fractional derivative [11, 12]

(1) $$D^{\mu} y = f(y,u,b)$$

where $D^{\mu} y$ symbolizes the fractional differential equation.

For simplicity the noise in the system is disregarded. The assumed model of the process, will be expressed by the following equation

(2) $$D^{\mu} y - f(y,u,\beta) = \varepsilon \ .$$

The functions stimulating both incoming and outgoing y signals are known. Solution of the problem is thus reduced to minimization of the ε error resulting from fitting the model to the process. Using the least squares estimation, the following equation is obtained

(3) $$\varepsilon(\beta) = {}_0D_T^{-1}\left| D^{\mu} y - f(y,u,\beta)\right|^2 .$$

Expression ${}_0D_T^{-1}$ is a symbolic denotation of a definite integral $\int_0^T$.

The precondition for the minimum in equation (3)

(4) $$\nabla_{\beta}\varepsilon(\beta)\,|_{\beta=\hat{\beta}} = 0$$

reduces to the equation

(5) $${}_0D_T^{-1}\left(\frac{\partial f'}{\partial \beta} D^{\mu} y\right) = {}_0D_T^{-1}\left(\frac{\partial f'}{\partial \beta} f(y,u,\hat{\beta})\right)$$

where the partial derivative is described as follows

(6) $$\frac{\partial f^{'}}{\partial \beta}=\left[\frac{\partial f_i}{\partial \beta_j}\right]=\begin{bmatrix}\frac{\partial f^{'}}{\partial \beta_0}\\ \vdots \\ \frac{\partial f^{'}}{\partial \beta_m}\end{bmatrix}=\begin{bmatrix}\frac{\partial f_1^{'}}{\partial \beta_0} & \cdots & \frac{\partial f_n^{'}}{\partial \beta_0}\\ \vdots & & \vdots \\ \frac{\partial f_1^{'}}{\partial \beta_m} & \cdots & \frac{\partial f_n^{'}}{\partial \beta_m}\end{bmatrix}.$$

Therefore, solving equations (5) will allow to determine the parameters of the model.

The existence of fractional derivatives in the algorithm of model does not complicate the differential method of finding parameters significantly. However, the existence of fractional derivatives impedes the conversion of the resulting set of equations (6).

Identification algorithm

The generalized algorithm for system parameters identification is as follows:

1. Stimulation with a selected signal.
2. Recording the system response and conversion to the y(t) function.
3. Determination or assumption of the algorithm order for the tested system model, including the degree of fractional part (if any), e.g. using the method utilizing the frequency characteristics [14].
4. Differentiation of the assumed model in respect to sought parameters.
5. Using the least square fit, adjusting the parameters of a model function to minimize the difference between the model and the real system.
6. Basing on calculations made in steps 4 and 5 creating the set of equations with unknown parameters.
7. The empirical verification of the assumed model. In case of unsatisfactory results you have to to go back to step 3 to correct the adopted model.

Example

To illustrate the problem one can consider the simple process in which there is a derivative of fractional order $\mu = n + 0.5$, where n is an integer.

Let the process will be described by the following equation

(7) $$Dy + a_1 D^{n+0.5} y + a_0 y = bu \ .$$

It can be assumed that the model for this process will be as follows

(8) $$Dy + a_1 D^{n+0.5} y + a_0 y - \beta u = \varepsilon$$

Using the differential method, the following dependencies are achieved

(9) $$\begin{cases}\frac{\partial \varepsilon}{\partial a_1} = D^{n+0.5} y\\ \frac{\partial \varepsilon}{\partial a_0} = y\\ \frac{\partial \varepsilon}{\partial \beta} = -u\end{cases} .$$

After applying the least squares estimation we obtain

(10) $$\begin{cases}{}_0D_T^{-1}\left(\varepsilon D^{n+0.5} y\right) = 0\\ {}_0D_T^{-1}(\varepsilon y) = 0\\ {}_0D_T^{-1}(\varepsilon u) = 0\end{cases} .$$

After substituting ε from equation (8), a set of equations with three unknown parameters a_1, a_0, β is created

(11)
$$\begin{cases}\mathbf{a_1}{}_0D_T^{-1}\left(D^{n+0.5} y\right)^2 + \mathbf{a_0}{}_0D_T^{-1}\left(yD^{n+0.5} y\right) + \\ -\beta_0 D_T^{-1}\left(uD^{n+0.5} y\right) = -{}_0D_T^{-1}\left(DyD^{n+0.5} y\right)\\ \mathbf{a_1}{}_0D_T^{-1}\left(yD^{n+0.5} y\right) + \mathbf{a_0}{}_0D_T^{-1}(y)^2 - \beta_0 D_T^{-1}(yu) = -{}_0D_T^{-1}(yDy)\\ \mathbf{a_1}{}_0D_T^{-1}\left(uD^{n+0.5} y\right) + \mathbf{a_0}{}_0D_T^{-1}(yu) - \beta_0 D_T^{-1}(u)^2 = -{}_0D_T^{-1}(uDy)\end{cases}$$

The problem is thus reduced to solving this set of equations. Because in some way the functions y and u are given as input and output signals in the process, therefore the main problem is the transformation of integrals of functions embraced by the fractional derivatives, in this case of $\mu = n + 0.5$ order.

In the example considered herein, the integrals are a product of two functions, in which at least one of them is a $D^{n+0.5}$. It is assumed here that the functions y and $D^{n+0.5} y$ meet the requirements resulting from the definition of integrals and fractional derivatives presented in [11, 12].

Property

Let the function f(t) be continuous on the $[0, T]$ interval and let g(t) be an analytic function on the interval $[0, T]$ and $t \in (0, T]$. Then for $z > 0$, the dependence can be described by Leibniz's formula:

(12) $$D^{-z}[f(t)g(t)] = \sum_{k=0}^{\infty}\binom{-z}{k}[D^k g(t)][D^{-z-k} f(t)] .$$

Making the use of the above dependencies [11], the integral of the product of functions can be expressed as

(13) $${}_0D_T^{-1}[f(t)g(t)] = \left[\sum_{k=0}^{\infty}\binom{-1}{k}[D^k g(t)][D^{-1-k} f(t)]\right]\Big|_0^T$$

or basing on the definition, the fractional derivative shortened to [11]

(14) $$D^{n+0.5} f(t) = D^{n+1}[D^{-0.5} f(t)] .$$

In this case we receive conversion into equation of integral derivative n+1 order of the fractional integral of -0.5 order.

This results in the transition of derivative of the integral of fractional order -0.5 to the derivative equation of the integer degree of order, $n+1$ in this case. Fractional integral for simple function f(t) can be calculated directly from the definition [11] using the formula

(15) $$D^{-0.5} f(t) = \frac{1}{\Gamma(0.5)}\int_0^t (t-\varepsilon)^{-0.5} f(\varepsilon) d\varepsilon$$

Other processes/ models in which the fractional derivative takes place will behave in similar way. Such case can complicate when noise is introduced. However, the general technique will remain the same.

The differential method due to the nature of transitions, which include the least square fit method of estimation is rather complicated and time consuming, so sometimes it is better to use other methods like the method of Fourier or

Laplace transformations. This method was tested for completeness of the issue, as it is known that even in the absence of fractional derivatives it can give very large errors in identification.

Conclusions

The presented herein identification method works well for simple systems. In the case of more complicated systems there is a problem with identification of the equation order – especially when there are fractional parts presented. They produce a variety of abnormalities in the system characteristics, which require certain attention.

It was shown that the presented method is capable to distinguish the elements of a fractional degree. These algorithms were generalized for a wider group of systems. The generalized method was demonstrated in the identification of a simple model.

The results obtained allow to better understand the necessity of utilising the fractional calculus in the identification. This work is essential to the further development of identification methods based on fractional derivatives and integrals.

The authors believe that the use of fractional derivatives and integrals in the identification can contribute to improving the quality of the assessment of systems such as image processing or model identification for materials used in technology.

Further work on algorithms increasing the computing ability of fractional derivatives and integrals while retaining the accuracy of the transformation should follow.

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Authors: *dr inż. Tomasz Janiczek, Wroclaw University of Technology, Department of Electronics. The Institute of Computer Engineering, Control and Robotics. ul. Z. Janiszewskiego 11/17, 50-372 Wroclaw, E-mail: tomasz.janiczek@pwr.wroc.pl; dr hab. inż. Janusz Janiczek, Wroclaw University of Technology, Department of Electronics Chair of Electronic and Photonic Metrology, ul. B. Prusa 53/55, 51-317 Wroclaw, E-mail: janusz.janiczek@pwr.wroc.pl*