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# Monolithic Model of Induction Heating of Thin Conductive Plate with Respecting Thermoelasticity

***Abstract**. A new approach to modeling of induction heating of thin plates is presented. The model of magnetic field is based on the electric vector potential **T** and the distribution of temperature is described by a modified equation including heat sources and sinks. The thermoelastic displacements are also respected. The methodology is illustrated by an example whose results are discussed.*

***Streszczenie.** W artykule przedstawiono nowe podejście do modelowania grzania indukcyjnego w cienkiej płytce. Model pola magnetycznego wykorzystuje elektryczny potencjał wektorowy **T** a rozkład temperatury opisany jest zmodyfikowanym równaniem, zawierającym źródła ciepła i wydatki ciepła. Uwzględniono również przesunięcia termoelastyczne. Metodyka tego podejścia została zilustrowana przykładem obliczeniowym z dyskusją wyników. (**Model monolityczny grzania indukcyjnego cienkiej płytki przewodzącej z uwzględnieniem przesunięcia termoelastycznego**)*

**Keywords:** induction heating, thin plate, electric vector ***T***-potential, thermoelastic displacements, numerical modeling.
**Słowa kluczowe**: grzanie indukcyjne, cienka płytka, elektryczny potencjał wektorowy, przesunięcia termoelastyczne, modelowanie numeryczne

## Introduction

Induction heating is a technological process that is employed in numerous industrial, transport, and other applications. Its goal is to prepare the heated metal element for its further heat treatment (for instance forming, hardening, hot pressing, melting, etc.).

A typical heat treatment used in case of thin plates is their annealing performed after hot forming whose aim is to suppress internal mechanical strains and stresses. An example is depicted in Fig. 1 showing a set of plate springs made of phosphor bronze by a Swiss company Grewis [1].

Fig. 1. Thin plate springs manufactured by Swiss company Grewis

Usually, such thin plates are heated only locally, at the places where the internal mechanical strains and stresses occur. The process itself is realized mostly by induction, even when direct resistance heating may be preferred in case of heating smaller domains.

The most efficient way of local induction heating of thin plates employs appropriately arranged inductors with cores of ferromagnetic materials. One of the possible arrangements is indicated in Fig.1 in [2], where we performed the first study concerning the topic.

The process of heating is energy consuming and must be optimized. This requires a good knowledge of its particulars, which, however, is a quite complicated business. The process is not only evolutionary, but also involves various nonlinearities due to temperature dependencies of physical parameters of the system.

The complete description of the above process means mapping of three physical fields influencing one another. The primary of them is time variable magnetic field generated by an external source. This field produces Joule losses in the heated plate and its consequent heating. And the thermal gradients in the plate lead to its deformations that are manifested by corresponding changes of its shape. From both physical and mathematical viewpoints, this task represents a challenging coupled problem.

Let us shortly summarize the existing approaches to modeling of the mentioned fields.

In the tasks of induction heating, the distribution of magnetic field in metal bodies is mostly modeled by magnetic vector potential $\boldsymbol{A}$ [3–5]. Formulations based on combinations of magnetic vector and scalar potentials [6] or [7] are not so popular since the relevant tasks are often characterized by the presence of large domains containing ferromagnetics and electrically conductive materials with induced currents.

Nevertheless, the above can fail when the heated body is characterized by geometrically incommensurable dimensions. This is typical for various planar structures such as thin plates or thin-wall pipes. Here we must face a serious problem connected with discretizing the plate (or wall) and solving magnetic field distribution in a large 3D domain.

Solution of the temperature field in thin plates may also lead to complications. These are connected with uneasily determinable description of the boundary condition, particularly the term $\partial T / \partial n$ necessary for finding the convection and radiation losses from the upper and lower surfaces of the plate, whose numerical approximation might exhibit unacceptably high error.

The paper presents a simplified, but still sufficiently efficient and accurate solution to the problem. The distribution of magnetic field is described by the electric vector $\boldsymbol{T}$-potential. Another improvement of the model is the description of the temperature field in the plate by a modified heat-transfer equation that also includes sources and sinks of heat. Finally, the field of the thermoelastic displacements is modeled using the Lamé equations.

The problem is then solved in the monolithic formulation, which means that the above physical fields are calculated at each time level simultaneously. All principal nonlinearities are respected.

### Definition of the problem and goals of the paper

Consider a thin nonferromagnetic plate of a general shape in plane $x, y$ that is depicted in Fig. 2. Its surface is denoted by symbol $\Omega_1$ and its boundary is $\Gamma$. The plate of thickness $\delta \to 0$ is exposed by external magnetic field $\boldsymbol{B}_{\text{ext}}(r,t)$ in a domain $\Omega_2 \subset \Omega_1$.

The material of the thin plate is described by the following physical parameters:

- electrical conductivity $\gamma$ ,
- magnetic permeability $\mu$ (in our case $\mu = \mu_0$ ),
- thermal conductivity $\lambda$ ,
- heat capacity $\rho c_{\mathrm{p}}$ ( $\rho$ denoting the specific mass and $c_{\mathrm{p}}$ being the specific heat at a constant pressure),
- Young modulus of elasticity $E$ ,
- Poisson number $\nu$ , and
- coefficient of the thermal dilatability $\alpha_{\mathrm{T}}$ .

All these parameters (except for permeability $\mu_0$ ) are generally temperature-dependent functions.

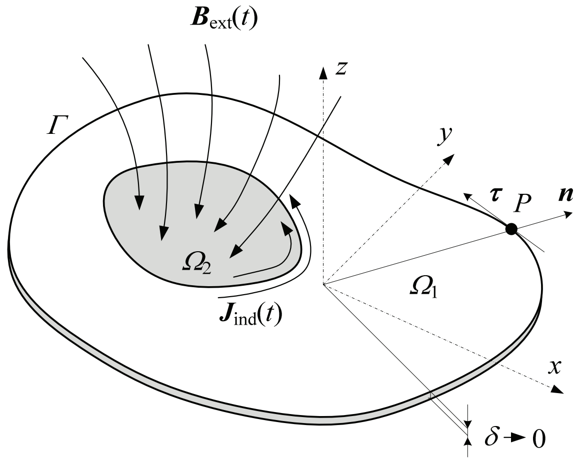


Fig. 2. Considered thin plate exposed by external magnetic field

The task is to map the distribution of currents induced in the plate, corresponding distribution of local Joule losses and time evolution of distributions of the temperature and thermoelastic displacements in the plate in the dependence of selected input parameters.

**Continuous mathematical model**

The assumption of a negligible thickness $\delta$ of the plate allows accepting another important assumption that no physical quantity changes along it. This leads to the possibility of significant simplification of the mathematical model of the problem.

The description of the magnetic field in the system based on using electric vector $\boldsymbol{T}$-potential was carried in the introductory study of the problem [2], so that we only confine ourselves to its short summarization.

The total magnetic flux density $\boldsymbol{B}$ in the system consists of two components: external magnetic flux density $\boldsymbol{B}_{\mathrm{ext}}$ and magnetic flux density $\boldsymbol{B}_{\mathrm{ind}}$ produced by currents of density $\boldsymbol{J}_{\mathrm{ind}}$ induced in the plate.

The total electric field strength $\boldsymbol{E}$ in the system follows from the second Maxwell equation

(1) $$\mathrm{curl}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{\partial \boldsymbol{B}_{\mathrm{ext}}}{\partial t} - \frac{\partial \boldsymbol{B}_{\mathrm{ind}}}{\partial t}$$

For the domain of the plate of electrical conductivity $\gamma$ we can write

(2) $$\mathrm{curl}\,\frac{\boldsymbol{J}_{\mathrm{ind}}}{\gamma} = -\frac{\partial \boldsymbol{B}_{\mathrm{ext}}}{\partial t} - \frac{\partial \boldsymbol{B}_{\mathrm{ind}}}{\partial t} .$$

Introduce now the electric vector $\boldsymbol{T}$-potential by the relation [6]

(3) $$\boldsymbol{J}_{\mathrm{ind}} = -\,\mathrm{curl}\,\boldsymbol{T} .$$

Hence,

(4) $$\mathrm{curl}\left(\frac{1}{\gamma}\mathrm{curl}\,\boldsymbol{T}\right) = \frac{\partial \boldsymbol{B}_{\mathrm{ext}}}{\partial t} + \frac{\partial \boldsymbol{B}_{\mathrm{ind}}}{\partial t} .$$

Neglecting the displacement currents in the first Maxwell equation (values of applied frequencies typical for induction heating are supposed to be smaller than $10^5$ Hz) we can write

(5) $$\mathrm{curl}\,\boldsymbol{H}_{\mathrm{ind}} = \mathrm{curl}\,\frac{\boldsymbol{B}_{\mathrm{ind}}}{\mu_0} = \boldsymbol{J}_{\mathrm{ind}}$$

and using (3) we obtain

(6) $$-\mathrm{curl}\,\boldsymbol{T} = \mathrm{curl}\,\frac{\boldsymbol{B}_{\mathrm{ind}}}{\mu_0} .$$

Hence, generally

(7) $$-\boldsymbol{T} = \frac{\boldsymbol{B}_{\mathrm{ind}}}{\mu_0} - \mathrm{grad}\,\psi ,$$

where $\psi$ is any scalar function. But in our case, the quantity $\boldsymbol{T}$ represents the electric vector potential produced only by the induced magnetic flux density $\boldsymbol{B}_{\mathrm{ind}}$ . That is why $\mathrm{grad}\,\psi = 0$ .

On this assumption (4) may be rewritten into the form

(8) $$\mathrm{curl}\left(\frac{1}{\gamma}\mathrm{curl}\,\boldsymbol{T}\right) + \mu_0\frac{\partial \boldsymbol{T}}{\partial t} = \frac{\partial \boldsymbol{B}_{\mathrm{ext}}}{\partial t} .$$

which holds everywhere in region $\Omega_1$. As in region $\Omega_1 - \Omega_2$ the external field $\boldsymbol{B}_{\mathrm{ext}}$ does not exist, the term $\partial B_{z,\mathrm{ext}} / \partial t$ vanishes there.

The principal advantage of this approach consists in the fact that that the $\boldsymbol{T}$-potential is only defined in the plate (while the $\boldsymbol{A}$-potential everywhere in the system, which would lead to the solution of a task characterized by one more dimension).

The boundary conditions for the $\boldsymbol{T}$-potential are very simple, because the normal component of the induced current densities $\boldsymbol{J}_{\mathrm{ind}}$ along the boundary $\Gamma$ of the plate vanishes. In other words

(9) $$\left[\frac{\partial \boldsymbol{T}}{\partial \tau}\right]_{\Gamma} = \boldsymbol{0} ,$$

where $\tau$ is the tangential direction to boundary $\Gamma$ at its arbitrary point. In other words, the value of $\boldsymbol{T}$ along the boundary $\Gamma$ is constant, and as before the heating process there is no magnetic field in the system, we can reasonably put $\boldsymbol{T}(\Gamma) = \boldsymbol{0}$ .

The volumetric Joule losses $w_{\mathrm{J}}$ produced by the induced currents and representing the heat sources are given by the relation

(10) $$w_{\mathrm{J}}=\frac{\left|\boldsymbol{J}_{\mathrm{ind}}\right|^2}{\gamma}.$$

The evolutionary temperature field $T(r,t)$ in the plate is described by equation [8]

(11) $$\operatorname{div}(\lambda\cdot\operatorname{grad}T)=\rho c\frac{\partial T}{\partial t}-w_{\mathrm{J}}$$

with the boundary condition

(12) $$-\lambda\frac{\partial T}{\partial n}=\alpha\left(T-T_{\mathrm{ext,c}}\right)+\varepsilon_{\mathrm{SB}}C_{\mathrm{r}}\left(T^4-T_{\mathrm{ext,r}}^4\right).$$

Here $\alpha$ is the coefficient of the convective heat transfer, $\varepsilon_{\mathrm{SB}}$ is the Stefan-Boltzmann constant, $C_{\mathrm{r}}$ is the coefficient of emissivity, and $T_{\mathrm{ext,c}}$, $T_{\mathrm{ext,r}}$ are the distant temperatures for simulation of convection and radiation.

For very thin plates, however, it is uneasy to accurately determine the derivative of temperature $T$ with respect to the normal $n$ that occurs on the left-hand side of (12). Therefore, it is better to insert the thermal fluxes in the plate in terms of heat sources (given by the volumetric Joule losses in it) and heat sinks (convection and radiation from both its sides) directly to (11), see Fig. 3.

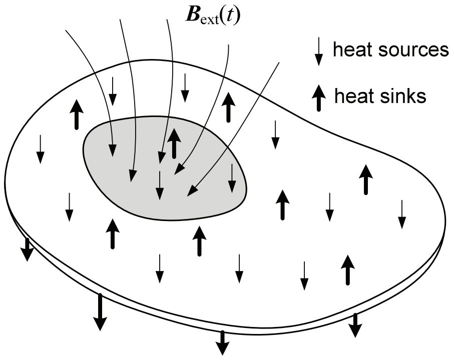


Fig. 3. Heat sources and sinks in the disk

Using this approach the heat-transfer equation (11) may be transformed into the form

(13) $$\begin{aligned}&\operatorname{div}(\lambda\operatorname{grad}T)=\rho c\frac{\partial T}{\partial t}-w_{\mathrm{J}}+\\&+\frac{1}{\delta}\left[\left(\alpha_{\mathrm{c,up}}+\alpha_{\mathrm{c,dn}}\right)\left(T-T_{\mathrm{ext,c}}\right)+2\varepsilon_{\mathrm{SB}}C_{\mathrm{r}}\left(T^4-T_{\mathrm{ext,r}}^4\right)\right],\end{aligned}$$

where $\alpha_{\mathrm{c,up}}$ and $\alpha_{\mathrm{c,dn}}$ are the coefficients of the convective heat transfer along the upper and lower sides of the spring (that can considerably differ from one another). Now the boundary condition is given by zero thermal flux from the plate.

The distribution of the thermoelastic displacements in the plate may advantageously be described by the Lamé equation in the form [9]

(14) $$\begin{aligned}&(\varphi+\psi)\cdot\operatorname{grad}(\operatorname{div}\boldsymbol{u})+\psi\cdot\Delta\boldsymbol{u}-\\&-(3\varphi+2\psi)\cdot\alpha_{\mathrm{T}}\cdot\operatorname{grad}T+\boldsymbol{f}=\mathbf{0},\end{aligned}$$

where $\varphi$ and $\psi$ are coefficients given by the relations

(15) $$\varphi=\frac{\nu\cdot E}{(1+\nu)(1-2\nu)},\ \psi=\frac{E}{2\cdot(1+\nu)}.$$

Here $\boldsymbol{u}$ represents the vector of the displacement, and $\boldsymbol{f}$ the vector of internal volumetric forces (these include the gravitational and Lorentz forces, but in comparison with the thermoelastic stresses they are very small and may be neglected without any significant error).

**Numerical solution**

The authors developed their own code written in Free Pascal. At present, the code solves the $(r,t)$ and $(r,z,t)$ arrangements. The coupled partial differential equations describing the electromagnetic, temperature and displacement fields are solved by the finite difference method using explicit approximations of accuracy $O\left((\Delta r)^2+(\Delta z)^2+\Delta t\right)$ [10], where $\Delta r$ is the step in the $r$-direction, $\Delta z$ is the step in the $z$-direction and $\Delta t$ is the time step. The computations were carefully checked with respect to both convergence and stability. Some results were with a very good accordance checked by professional code COMSOL Multiphysics.

**Illustrative example**

The algorithm was tested (see Fig. 4) on a circular aluminum disk of radius $r_2=0.1$ m that is exposed in a circular region of radius $r_1=0.02$ m by external harmonic magnetic field $\boldsymbol{B}_{\mathrm{ext}}(r,t)=\mathbf{z}_0B_{z,\mathrm{ext}}(r,t)$. Its thickness $\delta=1$ mm.

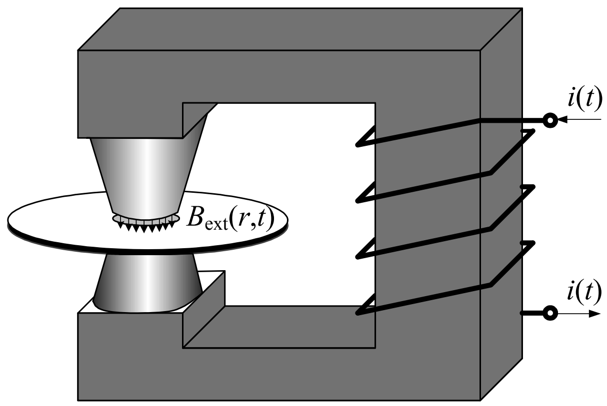


Fig. 4. Heated aluminum disk

The external magnetic flux density is supposed to vary harmonically ($B_{z,\mathrm{ext}}(t)=B_0\sin(2\pi ft)$). All material parameters of Al were considered as temperature dependent functions and these dependencies were taken over from the database [11].

The computations provided a lot of results. We will present the most illustrative of them.

Figure 5 shows the distribution of the current density $\boldsymbol{J}_{\mathrm{ind}}$ along the radius of the plate within the tenth period. In this case their generation may already be considered steady state. The distribution holds for $B_0=1$ T, $f=50$ Hz and is depicted for selected time levels of the period.

Figure 6 shows the distribution of temperature $T$ along the radius $r$ of the plate (at the same parameters $B_0$ and $f$ as above) for different time levels up to $180$ s. It is clear that the domain of the highest temperature is at the place of its exposition by the external magnetic field (the field is supposed to be produced by a time-variable-current-excited magnetic circuit and inside its gap the process of heating may be considered practically adiabatic).

Finally, Fig. 7 depicts the dependence of the radial displacement $u_r$ on the amplitude $B_0$ of the external magnetic field. The thin lines show its time evolution at radius $r_1$ (for $B_0 = 0.50, 0.75, 1.00$ T, respectively); the thick lines depict the same quantity at radius $r_2$.

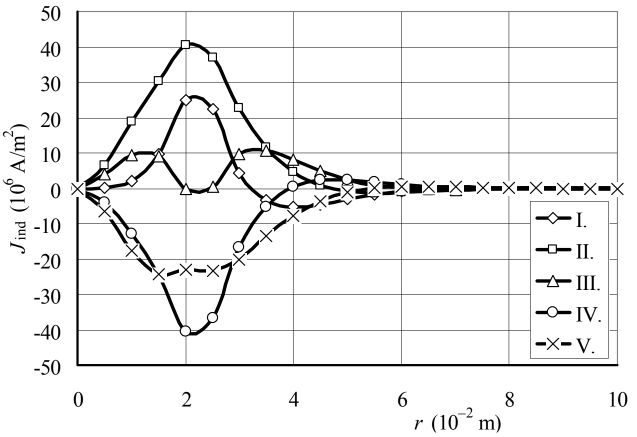


Fig. 5. Distribution of the current density $J_{ind}$ along the radius of the disk for the selected time levels of the 10th period: I.–0%, II.–20%, III.–40%, IV.–60%, V.–80% ( $r_1 = 0.02$ m, $\delta = 0.001$ m)

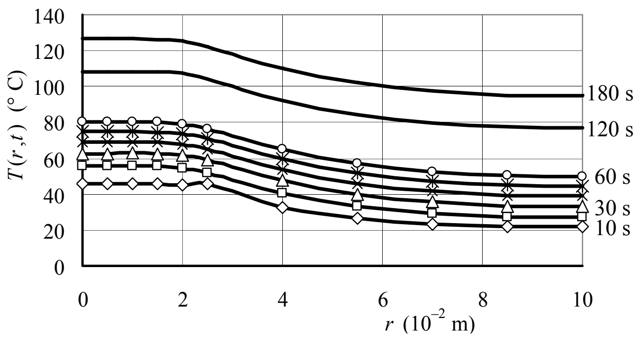


Fig. 6. Distribution of the temperature $T$ along the radius $r$ of the disk for different time levels ( $r_1 = 0.02$ m, $\delta = 0.001$ m)

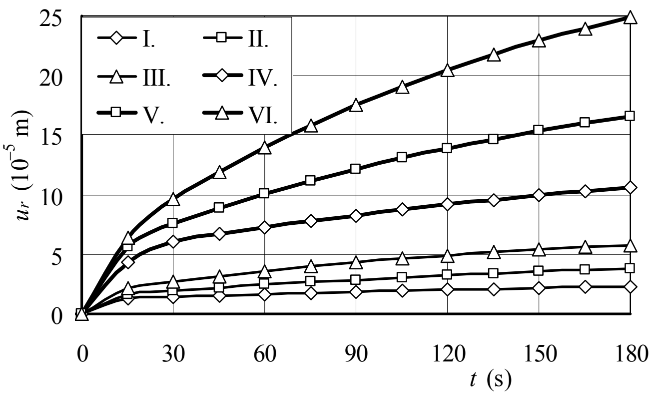


Fig. 7. Dependencies of radial displacements $u_r$ on amplitude $B_0$ of the external magnetic field for $r_1 = 0.02$ m, $\delta = 0.001$ m, $f = 50$ Hz):
I. $B_0 = 0.50$ T, $r = r_1$, II. $B_0 = 0.75$ T, $r = r_1$,
III. $B_0 = 1.00$ T, $r = r_1$, IV. $B_0 = 0.50$ T, $r = r_2$,
V. $B_0 = 0.75$ T, $r = r_2$, VI. $B_0 = 1.00$ T, $r = r_2$.

**Conclusion**

Electric vector potential seems to be a powerful tool for modeling induction heating in systems characterized by geometrically incommensurable elements. The reason is that in specific cases it allows handling specific 3D arrangements as 2D problems.

Next work in the field will be aimed at the back influence of thermoelastic displacements in the plate (variations of the discretization mesh) on the distribution of magnetic and temperature fields and extension of the methodology to geometrically more general problems. This way of modeling seems to be prospective, for example, for induction heating of rotating thin pipes in stationary magnetic fields and other similar tasks or for local heating of planar plates of any shapes.


**Acknowledgement**

*This work was supported by the European Regional Development Fund and Ministry of Education, Youth and Sports of the Czech Republic under the project No. CZ.1.05/2.1.00/03.0094 – RICE, by the Grant project IAA100760702 (GAASCR), Grant project 102/09/1305 (GACR) and Research Plan MSM 6840770017.*


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