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Numerical integration for symmetric Galerkin Boundary Element Method

Streszczenie.. *Metoda elementów brzegowych Galerkina (GBEM), w przeciwieństwie do metody klasycznej, przy pewnych warunkach generuje symetryczny układ równań algebraicznych. Najtrudniejszym elementem obliczeń numerycznych w standardowej metodzie elementów brzegowych są całki osobliwe. Trudność ta jeszcze wzrasta w przypadku podejścia Galerkina. W niniejszym artykule przedstawiono propozycje numerycznego wyznaczania całek w symetrycznej metodzie GBEM. (**Całkowanie numeryczne w symetrycznej metodzie elementów brzegowych Galerkina**).*

Abstract. *The classical BEM produces fully populated coefficients matrix. With Galerkin Boundary Element Method (GBEM) is possible to produce the symmetric coefficients matrix. Generally the Galerkin boundary integral equations lead to the algebraic system where known and unknown boundary values are defined by one or two dimensional integrals. The main problems are related to the integrals evaluation and treatment of the singularities. These paper presents problems associated with integration for GBEM.*

Słowa kluczowe: metoda elementów brzegowych, podejście Galerkina, całkowanie numeryczne, całki osobliwe.
Keywords: Boundary Element Method, Galerkin approach, numerical integration, singular integrals

Introduction

Galerkin approach (GBEM) [1,2] is more difficult than the standard Boundary Element Method [3,4,5,6] mainly due to a very complicated procedure of the singular integrals calculation. Galerkin method leads to a double surface integration. GBEM may, under some conditions, generate a symmetric system of algebraic equations. That is a major advantage of Galerkin approach [2].

The concept of GBEM formulation is presented for the boundary integral equations limited to constant elements and 2D space.

Symmetric system of Galerkin BEM

Basic integral equation (BIE) for the BEM is constructed by the convolution with the fundamental solution [3,4,5,6].

Figure 1 presents domain $\Omega \subset R^n$ with Dirichlet and Neuman boundary.

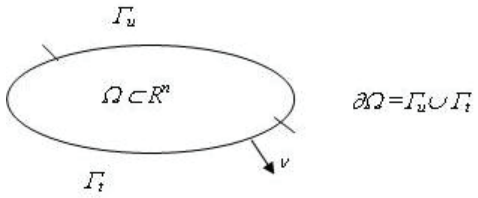

Fig.1. The domain Ω

The basic principles of traditional BEM are presented for the paradigmatic example of the n-dimensional stationary heat conduction described by :

$$\Delta u(x) = -f(x), x \in \Omega \subset R^n, \Delta = \sum_{k=1}^{n} \partial^2 / \partial x_k^2 \quad (1)$$

$$u(x) = u_\Gamma(x), \quad x \in \Gamma_u \subset \Omega,$$

$$t(x) = t_\Gamma(x), \quad x \in \Gamma_t \subset \Omega,$$

where: Δ - Laplace operator, u – the unknown quantity, f - the known volume sources in Ω.
The flux on the boundary is:

$$t = \mathrm{A}_t u = -\partial \nu\, u = -\nu \cdot \nabla u \quad (2)$$

where: ∇, ν - the gradient and the outer unit normal, $\mathrm{A}_t = -\nu \cdot \nabla$ - the boundary operator, $\partial / \partial x_k$ - the partial derivatives denotes ∂_k, x – n-dimensional vector, dx – the short form for $dx_1 dx_2$ (or $dx_1 dx_2 dx_3$).

To obtain a well posed problem, half of the boundary data (either u on Γ_u or t on Γ_t should be defined by boundary conditions, i.e. $\Gamma_u \cup \Gamma_t = \partial\Omega$.

Most of the numerical methods are based on a weak form of the differential equation. The basic weak form for BEM is:

$$\int_\Omega u \Delta \upsilon \, d\Omega = \int_\Omega \Delta u \upsilon \, d\Omega - \int_\Gamma \left(\frac{\partial u}{\partial \nu} \upsilon - u \frac{\partial \upsilon}{\partial \nu} \right) d\Gamma \quad (3)$$

and is equivalent to Green's formula.

Basic integral equation for the BEM is constructed by the convolution with the fundamental solution $U(x)$. The ν in (2) is replaced by $U(x-y)$, where $U(x-y)$ is the response of the infinite medium to a single source $f(x) = \delta(y)$ (Dirac distribution):

$$\Delta U(x-y) = -\delta(x-y), x, y \in R^n. \quad (4)$$

The fundamental solutions inherit their singular character from the Dirac distribution δ. Unfortunately analytic formulas for the fundamental solution can only be found for simple differential operators. Nevertheless, as long as the coefficients of the differential operator are constant, the existence of the fundamental solution can always be assured [1].
The known and unknown boundary quantities u, t are approximated by a sum of polynomial trial functions ϕ_u^i, ϕ_t^i with the coefficients u^i, t^i:

$$u(x) \approx \sum_i^{N_u} u^i \phi_u^i(x), \quad t(x) \approx \sum_i^{N_t} t^i \phi_u^i(x) \quad . \quad (5)$$

For convergence reasons, the trial functions for the u should be at least linear, for the t it is sufficient to take constant trial functions.

Galerkin BIE lead to the algebraic system of BIEs [1,4]:

$$\sum_i K_u^{ji} u^i = F_u^j + \sum_i H_u^{ji} t^i - \sum_i G_u^{ji} u^i \quad (6)$$

$$\sum_i K_t^{ji} t^i = F_t^t + \sum_i H_t^{ji} t^i - \sum_i G_t^{ji} u^i \quad (7)$$

where the vector and matrices are defined as follows [1]:

$$F_u^j := \int_{\Gamma_x} \phi_t^j(x) \int_{\Omega} f(y) U(x-y) d\Omega_y d\Gamma_x ,$$

$$H_u^{ji} := \int_{\Gamma_x} \phi_t^j(x) \int_{\Gamma_y} \phi_t^i(y) U(x-y) d\Gamma_y d\Gamma_x ,$$

$$G_u^{ji} := \int_{\Gamma_x} \phi_t^j(x) \int_{\Gamma_y} \phi_u^i(y) A_t^i U(x-y) d\Gamma_y d\Gamma_x ,$$

$$K_u^{ji} := \int_{\Gamma_x} \phi_t^j(x) \kappa(x) \phi_u^i(x) d\Gamma_x ,$$

$$F_t^j := \int_{\Gamma_x} \phi_u^j(x) \int_{\Omega} f(y) A_t^j U(x-y) d\Omega_y d\Gamma_x ,$$

$$H_t^{ji} := \int_{\Gamma_x} \phi_u^j(x) \int_{\Gamma_y} \phi_t^i(y) A_t^j U(x-y) d\Gamma_y d\Gamma_x ,$$

$$G_t^{ji} := \int_{\Gamma_x} \phi_u^j(x) \int_{\Gamma_y} \phi_u^i(y) A_t^j A_t^i U(x-y) d\Gamma_y d\Gamma_x ,$$

$$K_t^{ji} := \int_{\Gamma_x} \phi_u^j(x) (\kappa \phi_t^i + \phi_u^i A_t^j \kappa) d\Gamma_x .$$

In general case the $\kappa(x)$ definition may be find in [1]. For smooth part of a boundary $\kappa(x) = 1/2$.

The Galerkin BIE lead to the final matrix system $\sum_i A^{ji} X^i = Y^j$, where the matrix A is fully populated and symmetric [1,4].

Numerical integration

Computer implementation of GBEM [1, 4] requires the same skills as classical BEM [3,8]. Integration with respect to two unknowns in the Galerkin approach (from a mathematical point of view) is equivalent to the surface integral:

$$H_u^{ji} := \int_{\Gamma_x} \phi_t^j(\mathbf{x}) \int_{\Gamma_y} \phi_t^i(\mathbf{y}) U(\mathbf{x}-\mathbf{y}) d\Gamma_y d\Gamma_x , \tag{8}$$

where: i, j – number of elements, $\mathbf{x} = [x_1, x_2]$, $\mathbf{y} = [y_1, y_2]$ (two dimensional case).

For the numerical calculation, the equation (8) can be written as:

$$H^{ji} = \int_{d1}^{g1} \int_{d2}^{g2} U(\mathbf{x}-\mathbf{y}) dy dx = \int_{d1}^{g1} \int_{d2}^{g2} U(x_1, y_1, x_2, y_2) dy dx , \tag{9}$$

where $d1$, $g1$ and $d2$, $g2$ denotes the integration limits.

Determination of double integrals is usually carried out using Gaussian quadrature with the -1, 1 integration limits. The nodes coordinates are transformed to a local coordinate system [5,6] using the transformation (the same for **x** and **y**):

$$x(\xi) = \frac{1}{2}(1-\xi) x_1 + \frac{1}{2}(1+\xi) x_2 , \tag{10}$$

where ξ is the local coordinate.

After dividing the boundary curve Γ_x into elements Γ_i and Γ_y into elements Γ_j the numerical integration in local coordinate over each element is equals:

$$\int_{\Gamma_i} \int_{\Gamma_j} f(x,y) d\Gamma_x d\Gamma_y = \int_{-1}^{1} \int_{-1}^{1} f(x(\xi_1), y(\xi_2)) J(\xi_1) J(\xi_2) d\xi_1 d\xi_2 , \tag{11}$$

where: f means any function, $J(\xi) = \frac{d\Gamma}{d\xi}$ is the Jacobian of transformation .

Double integrals in local coordinates corresponds to the integration under square surface. The singularity appears along the diagonal. Non-singular integrals can be calculated in a similar way as for the conventional BEM using twice Gauss-Legendre integration rules. In order to get correct results for singular integrals the different numbers of integration points in each direction should be used [9] (table 1):

$$I = \int_{-1}^{+1+1} \int f(\xi_1, \xi_2) d\xi_1 \xi_2 = \sum_{i=0}^{n-1} \left(\sum_{j=0}^{m-1} f(\xi_{1j}, \xi_{2i}) \varpi_j \right) \omega_i , \tag{12}$$

where: n – number of integration points in ξ_1 direction, m – number of integration points in ξ_2 direction, ξ_{1j}, ξ_{2i} - the Gauss-Legendre integration points, ω_j, ϖ_i - the weights for the integration.

Table 1. The Gaussian points and weights

4 integration points	
The points	The weights
-0,3400	0,6521
-0,8611	0,3479
0,8611	0,3479
0,3400	0,6521
5 integration points	
The points	The weights
-0,9062	0,2369
-0,5385	0,4786
0	0,5689
0,5385	0,4786
0,9062	0,2369

The complete list of the weight coefficients and Gaussian integration points can be found in [9].

To improve efficiency for the singular integrals, the regularization method need to be implemented [5,6].

For the GBEM the regularisation method rely on subdividing the square region into two sub-triangles and next those sub-triangles are mapped into two squares as it is shown in figure 2.

That transformation for first triangle can be defined as:

$$\eta_1 = -\xi_1, \quad \eta_2 = \frac{1 + \xi_1 - 2\xi_2}{1 - \xi_1} , \tag{13}$$

and for the second triangle as:

$$\eta_1 = -\xi_2, \quad \eta_2 = \frac{-1 + 2\xi_1 - \xi_2}{1 - \xi_2} . \tag{14}$$

Next the standard Gauss-Legendre rule for numerical integration can be used, The Jacobian of transformation is the same in both cases and is equal to:

$$J_{T1} = J_{T2} = \frac{1 + \eta_1}{2} . \tag{15}$$

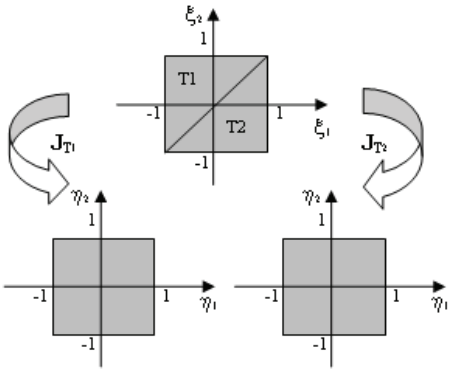

Fig.2. Regularization method of integration of double singular integrals

Numerical example

As the test example, the Dirichlet problem of the Poisson equation:

$$\Delta u(x) = -f(x), x \in \Omega,\ u(x) = u_\Gamma = 0, \quad x \in \Gamma,$$

is considered in a quadratic two-dimensional domain $\Omega = [0,1] \times [0,1]$ at the boundaries $u=0$. The interior is subjected to stationary heat source f. The boundary $\partial\Omega$ is divided into eight elements (Fig. 3) with constant trial function $\phi_t(x) = 1$.

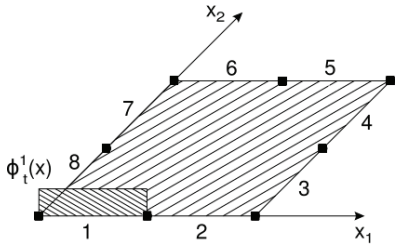

Fig. 3. Quadratic domain with eight boundary elements and constant trial function

The fundamental solution is:

$$U(x) = U(x_1, x_2) = \frac{1}{2\pi} \ln \sqrt{x_1^2 + x_2^2}, \tag{16}$$

where:

$$U(x-y) = U(x_1, y_1, x_2, y_2) = \frac{1}{2\pi} \ln \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

Taking into account that u = 0 at the boundaries, the general system of BIE (equation 6) can be reduced to:

$$0 = F_u^j + \sum_i H_u^{ji} t^i \tag{17}$$

where:

$$H_u^{ji} = \int_{\Gamma_x} \phi_t^j \int_{\Gamma_y} \phi_t^i U(x-y) d\Gamma_y d\Gamma_x,$$

$$F_u^j = \int_{\Gamma_x} \phi_t^j(x) \int_{\Omega} f(y) U(x-y) d\Omega_y d\Gamma_x.$$

The calculation of the H (8x8) matrix coefficients leads to the example integration:

$$H^{11} = \int_0^{1/2} \int_0^{1/2} U(x_1, y_1, 0, 0) dx_1 dy_1,$$

$$H^{52} = \int_{1/2}^{1} \int_{1}^{1/2} U(x_1, y_1, 1, 0) dx_1 dy_1,$$

$$H^{45} = \int_{1/2}^{1} \int_{1}^{1/2} U(1, y_1, x_2, 1) dx_2 dy_1.$$

For the constant boundary elements, the Jacobian value (equations 10, 11) is constant and proportional to the element length L: $J(\xi) = L/2$.

The 64 matrix H coefficients were calculated (using Matlab symbolic toolbox) symbolically [7], numerically and numerically with regularization method.

Table 2 presents the values of the first rows of matrix H calculated numerically using Gauss rule with 4 x 5 and 20 x 21 points of integration.

Table 2. Results of calculations for the first row of the matrix H

Symbolically	Numerically (4 x 5)	Numerically (20x21)
-0,08726255367854	0,08241078918265	-0,08609376042378
-0,03210365364037	-0,03207395713560	-0,03210357534770
-0,00934285053778	-0,00934285068230	-0,00934285053778
0,00233501514488	0,00233501540380	0,00233501514488
0,00079080546173	0,00079080565786	0,00079080546173
0,00484298721934	0,00484298712768	0,00484298721934
-0,04222282866900	-0,04224040701302	-0,04222291373550
-0,00934285053778	-0,00934285678014	-0,00934285053778

Using the symbolical results the relative error is calculated for 4x5 (Fig. 4), 20x21 (Fig. 5) and 60x61 (Fig. 6) Gaussian points.

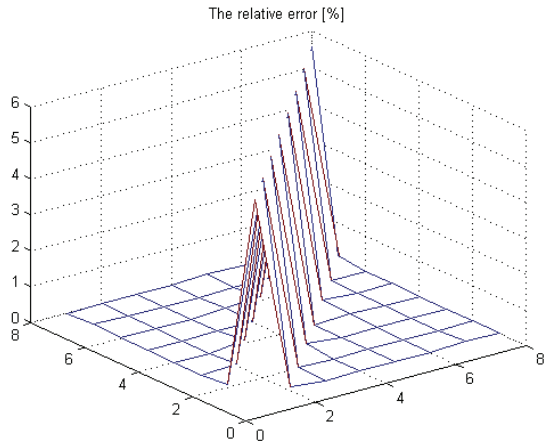

Fig.4. The relative error for 20 (4x5) Gaussian points of direct integration method

For singular integrals (the diagonal elements of matrix H), inspecting the figures 4-6, it can be seen that for 20 (4x5) Gaussian points the error is about 6%, for the 420 (20 x 21) points the error is 1,5% and for 3660 (60 x 61) the error is $1{,}4 \cdot 10^{-5}\%$. This satisfactory results was achieved using huge numbers of integration points but such number of integration points is not justified. That is why new, more effective regularization method with better precision and less time consuming need to be used.

Figure 7 and 8 presents the analogical results for the regularization method.

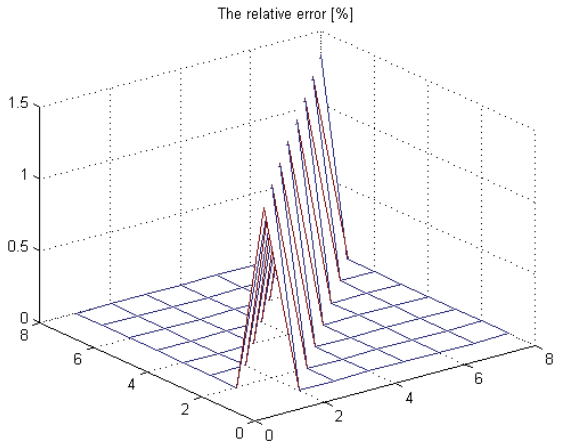

Fig.5. The relative error for 420 (20x21) Gaussian points of direct integration method

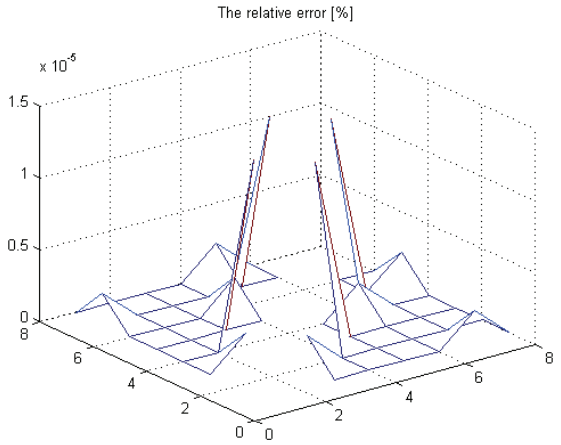

Fig.6. The relative error for 3660 (60x61) Gaussian points of direct integration method

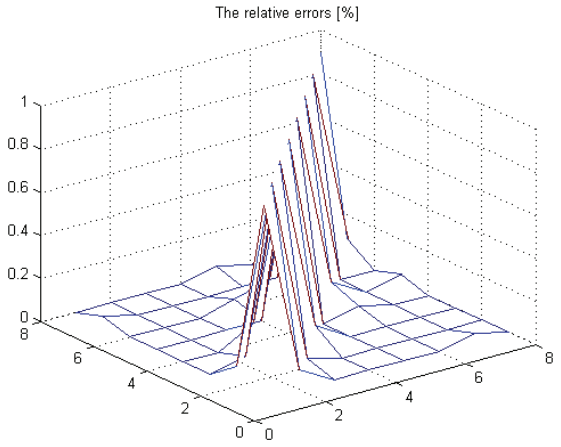

Fig.7. The relative error for 20 (4x5) Gaussian points of regularization method

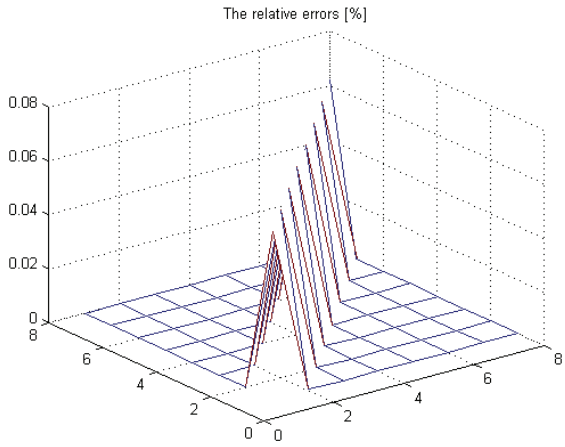

Fig.8. The relative error for 420 (20x21) Gaussian points of regularization method

Conclusions

The numerical integration in the Galerkin approach is more difficult than the standard BEM method. The singular integrals calculation procedure is quite complicated. The simple Gauss integration gives good results for minimum 420 (20 x 21) points of integration. More effective calculation and the same results are achieved using regularization method for 20 (4x5) Gaussian points.

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